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COINCIDENCE THEOREMS ON NONCONVEX SETS AND ITS APPLICATIONS

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Abstract. In this paper, we establish some coincidence theorems, generalized variational inequality theorems and minimax inequality theorems for the family $KKM^*(X,Y)$ and the generalized Φ-mapping on a nonconvex set.

1. Introduction and Preliminaries

In 1929, Knaster, Kurnatoaski and Mazurkiewicz [11] had proved the well-known KKM theorem on n-simplex. In 1961, Ky Fan [7] had generalized the KKM theorem in the infinite dimensional topological vector space. Later, the KKM theorem and related topics, for example, matching theorem, fixed point theorem, coincidence theorem, variational inequalities, minimax inequalities and so on had been presented a grand occasions. Rcecntly, Chang and Yen [4] introduced the family $KKM(X,Y)$, and got some results about fixed point theorems, coincidence theorems and some applications on this family. In this paper, we establish some coincidence theorems, generalized variational inequality theorems and minimax inequality theorems for the family $KKM^*(X,Y)$ and the generalized Φ-mapping.

Let X and Y be two sets, 2^X denotes the class of all nonempty subsets of X, and let $T : X \to 2^Y$ be a set-valued mapping. We shall use the following notations in the sequel.

(i) $T(x) = \{y \in Y : y \in T(x)\}$,

(ii) $T(A) = \cup_{x \in A} T(x)$,

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(iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$,

(iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \phi\}$,

(v) $T^{*}(y) = \{x \in X : y \notin T(x)\}$, and

(vi) if D is a nonempty subset of X, then $\langle D \rangle$ denotes the class of all nonempty finite subset of D.

For the case that X and Y are two topological spaces. Then $T : X \to 2^{Y}$ is said to be closed if its graph $\mathcal{G}_T = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed. T is said to be compact if the image $T(X)$ of X under T is contained in a compact subset of Y.

A convex space X is a convex set (in a linear space) with any topology that induces the Euclidean topology on the convex hull of its finite subset.

A nonempty subset X of a Hausdorff topological vector space E is said to be almost-convex [14] if for every finite subset $A = \{x_1, x_2, ..., x_n\}$ of X and every neighborhood V of the origin 0 of E, there is a mapping $h_{A,V} : A \to X$ such that $h_{A,V}(x_i) \in x_i + V$ for each $i \in \{1, 2, ..., n\}$ and $co(h_{A,V}(A)) \subset X$. We call $h_{A,V}$ a convex-inducing mapping.

We now introduce some properties of the almost-convex sets of a Hausdorff topological vector space E, as follows:

(i) In general, the convex-inducing mapping $h_{A,V}$ is not unique. If $U \subset V$, then it is clear that any $h_{A,U}$ can be regarded as an $h_{A,V}$.

(ii) It is clear that the convex set is almost-convex, but the converse is not true, for an counterexample,

Let $E = l^2(R^{\infty})$. Then the set $B(1) = \{x \in E : 0 < \|x\| < 1\}$ is an almost-convex subset of E, not a convex set.

Lemma 1. *If E is a Hausdorff topopogical vector space, X an almost-convex subset of E, and Y an open convex subset of E, then $X \cap Y$ is also almost-convex.*

Proof. Let $A = \{x_1, x_2, ..., x_n\} \subset X \cap Y$. Since Y is open, there exists an open neighborhood U of the origin 0 of E such that $A + U \subset Y$. For any neighborhood V of the origin 0 of E with $V \subset U$, since $A \subset X$ and X is almost-convex, there exists a convex-inducing mapping $h_{A,V} : A \to X$ such that $h_{A,V}(x_i) \in x_i + V$ for all $i = 1, 2, ..., n$ and $co(h_{A,V}(A)) \subset X$. Since $h_{A,V}(A) \subset co(h_{A,V}(A)) \subset X$ and $h_{A,V}(A) \subset A + V$, we get $h_{A,V}(A) \subset (A + V) \cap X \subset (A + U) \cap X \subset X \cap Y$, and so $co(h_{A,V}(A)) \subset Y$, since Y is convex. Therefore, we conclude that $X \cap Y$ is almost-convex. ∎

Remark 1. Let us note that the open condition of the above Lemma1 is really needed. For instance, if we consider the Euclidean topology in $\Re^2$, $X =$

$int(co(\{(1,1),(-1,1),(-1,-1),(1,-1)\}))\cup\{(1,1),(-1,1),(-1,-1),(1,-1)\}$, and $Y=co(\{(-1,1),(-2,1),(-2,-1),(-1,-1)\})$, then $X\cap Y=\{(-1,1),(-1,-1)\}$ is not almost-convex.

In [4], Chang and Yen had introduced the class $KKM(X,Y)$, we now extended this class to be the class $KKM^*(X,Y)$ for the almost-convex set X.

Definition 1. Let X be a nonempty almost-convex subset of a topological vector space E, and Y a topological space. If $T,F:X\to 2^Y$ are two set-valued mappings such that for each finite subset A of X and every neighborhood V of the origin 0 of E, there exists a convex-inducing mapping $h_{A,V}:A\to X$ such that $T(co(h_{A,V}(A)))\subset F(A)$, then we call F a generalized KKM^* mapping with respect to T.

If the set-valued mapping $T:X\to 2^Y$ satisfies the requirement that for any generalized KKM^* mapping $F:X\to 2^Y$ with respect to T, the family $\{\overline{Fx}:x\in X\}$ has the finite intersection property, then T is said to have the KKM^* property. Denote

$$KKM^*(X,Y)=\{T:X\to 2^Y\ \mid\ T\ \text{ has the }\ KKM^*\ \text{ property}\}.$$

Definition 2. Let Y be a topological space and X be a convex space. A set-valued mapping $T:Y\to 2^X$ is called a Φ-mapping if there exists a set-valued mapping $F:Y\to 2^X$ such that

(i) for each $y\in Y$, $A\in\langle F(y)\rangle$ implies $co(A)\subset T(y)$, and
(ii) $Y=\cup_{x\in X}intF^{-1}(x)$.

Moreover, the mapping F is called a companion mapping of T.

Definition 3. Let Y be a topological space, X a nonempty almost-convex subset of a topological vector space E, and V be a neighborhood of the origin 0 of E. A set-valued mapping $T:Y\to 2^X$ is called a generalized Φ-mapping if there exists a set-valued mapping $F:Y\to 2^X$ such that

(i) for each $y\in Y$, $A\in\langle F(y)\rangle$ implies $co(h_{A,V}(A))\subset T(y)$, where $h_{A,V}$ is a convex-inducing mapping , and
(ii) $Y=\cup_{x\in X}intF^{-1}(x)$.

Moreover, the mapping F is called a generalized companion mapping of T.

Remark 2.

(i) A Φ-mapping is also a generalized Φ-mapping, but the converse is not true.

(ii) If $T : Y \to 2^X$ is a generalized Φ-mapping (Φ-mapping), then for each nonempty subset Y_1 of Y, $T|_{Y_1} : Y_1 \to 2^X$ is also a generalized Φ-mapping (Φ-mapping).

Let X be a convex space, and Y a topological space. A real-valued function $f : X \times Y \to \Re$ is said to be quasiconvex in the first variable if for each $y \in Y$ and for each $\xi \in \Re$, the set $\{x \in X : f(x,y) \leq \xi\}$ is convex, and f is said to be quasiconcave if $-f$ is quasiconvex.

Definition 4. Let X be a nonempty almost-convex subset of a topological vector space, and Y a topological space. A real-valued function $f : X \times Y \to \Re$ is said to be almost quasiconvex in the first variable if for each $y \in Y$ and for each $\xi \in \Re$, the set $\{x \in X : f(x,y) \leq \xi\}$ is almost-convex, and f is said to be almost quasiconcave if $-f$ is almost quasiconvex.

Definition 5. Let X be a convex space, Y a nonempty set, and let $f, g : X \times Y \to \Re$ be two real-valued functions. For any $y \in Y$, g is said to be f-quasiconcave in the first variable if for each $A = \{x_1, x_2, ..., x_n\} \in \langle X \rangle$, we have

$$\min_{1 \leq i \leq n} f(x_i, y) \leq g(x, y), \text{ for all } x \in co(A).$$

Definition 6. Let X be a convex space, Y a nonempty set, and let $f, g : X \times Y \to \Re$ be two real-valued functions. For any $y \in Y$, g is said to be f-quasiconvex in the first variable if for each $A = \{x_1, x_2, ..., x_n\} \in \langle X \rangle$, we have

$$\max_{1 \leq i \leq n} f(x_i, y) \geq g(x, y), \text{ for all } x \in co(A).$$

Definition 7. Let X be a nonempty almost-convex subset of a topological vector space E, Y a nonempty set, and let $f, g : X \times Y \to \Re$ be two real-valued functions. For any $y \in Y$, g is said to be almost f-quasiconcave in the first variable if for each $A = \{x_1, x_2, ..., x_n\} \in \langle X \rangle$ and for every neighborhood V of the origin 0 of E, there exists a convex-inducing mapping $h_{A,V} : A \to X$ such that

$$\min_{1 \leq i \leq n} f(x_i, y) \leq g(x, y), \text{ for all } x \in co(h_{A,V}(A)).$$

Remark 3. It is clear that if $f(x,y) \leq g(x,y)$ for each $(x,y) \in X \times Y$, and if for each $y \in Y$, the mapping $x \mapsto f(x,y)$ is almost quasiconcave(quasiconcave), then g is almost f-quasiconcave(f-quasiconcave) in the first variable.

2. Coincidence Theorems

The following lemma will plays an important role for this section, in order to establish some coincidence theorems.

Lemma 2. *Let X be a compact set, and Y a nonempty almost-convex subset of a Hausdorff topological vector space E. If $T : X \to 2^Y$ is a generalized Φ-mapping with a companion mapping $F : X \to 2^Y$, then there exists a continuous function $f : X \to Y$ such that for each $x \in X$, $f(x) \in T(x)$, that is, T has a continuous selection.*

Proof. Let V be a neighborhood of the origin 0 of E. Since X is compact, there exists $A = \{y_1, y_2, ..., y_n\} \subset Y$ such that $X = \cup_{i=1}^n intF^{-1}(y_i)$. Since Y is almost-convex and $A \in \langle Y \rangle$, there exists a convex-inducing mapping $h_{A,V} : A \to Y$ such that $co(h_{A,V}(A)) \subset Y$.

Let $\{\lambda_i\}_{i=1}^n$ be a partition of the unity subordinated to the cover $\{intF^{-1}(y_i)\}_{i=1}^n$ of X. Define a continuous mapping $f : X \to co(h_{A,V}(A))$ by

$$f(x) = \sum_{i=1}^{n} \lambda_i(x) h_{A,V}(y_i) = \sum_{i \in I(x)} \lambda_i(x) h_{A,V}(y_i), \quad \text{for each } x \in X.$$

where $I(x) = \{i \in \{1, 2, ..., n\} : \lambda_i \neq 0\}$. Noting that $i \in I(x)$ if and only if $x \in F^{-1}(y_i)$; that is, $y_i \in F(x)$. Since T is a generalized Φ-mapping, we conclude that $f(x) = \sum_{i=1}^n \lambda_i(x) h_{A,V}(y_i) \in co(h_{A,V}(A)) \subset T(x)$, for each $x \in X$. This completes the proof. ■

By above Lemma 2, we immediately get the following corollary.

Corollary 1. *Let X be a compact set, Y a convex space, and let $T : X \to 2^Y$ be a Φ-mapping. Then T has a continuous selection.*

A polytope in X is denoted by $\Delta = co(A)$ for each $A \in \langle X \rangle$.

Theorem 1. *Let X be a convex space, Y a nonempty almost-convex subset of a Hausdorff topological vector space, and let $T : X \to 2^Y$ be a generalized Φ-mapping. Then $T \in KKM(X, Y)$.*

Proof. Since $T : X \to 2^Y$ is a generalized Φ-mapping, $T|_\Delta$ is also a generalized Φ-mapping. Since Δ is compact, T has a continuous selection, and so $T|_\Delta \in KKM(\Delta, Y)$. Applying Proposition $3(i)$ [4], we conclude that $T \in KKM$ (X, Y) ■

Lemma 3. *Let X be a nonempty almost-convex subset of a Hausdorff topological vector space E, and let Y, Z be two topological spaces. Then*

(i) *if $T \in KKM^*(X,Y)$ and $f \in C(Y,Z)$, then $fT \in KKM^*(X,Z)$;*

(ii) *if $T \in KKM^*(X,Y)$ and D is a nonempty almost-convex subset of X, then $T|_D \in KKM^*(D,Y)$.*

Proof. The proof is analyogous to the proof of Lemma 2 of Chang and Yen [4]. ∎

The following theorem and corollary are well-known, cf [4] and [8].

Theorem 2. *Let X be a nonempty almost-convex subset of a locally convex space E. If $T \in KKM^*(X,X)$ is compact and closed, then T has a fixed point in X.*

Corollary 2. *Let X be a nonempty convex subset of a locally convex space E. If $T \in KKM(X,X)$ is compact and closed, then T has a fixed point in X.*

By Lemma 2, we have the following coincidence theorem.

Theorem 3. *Let X be a nonempty almost-convex subset of a locally convex space E, and let Y be a topological space. Assume that*

(i) *$T \in KKM^*(X,Y)$ is compact and closed, and*

(ii) *$F : Y \to 2^X$ is a generalized Φ-mapping.*

Then there exists $(\overline{x},\overline{y}) \in X \times Y$ such that $\overline{y} \in T(\overline{x})$ and $\overline{x} \in F(\overline{y})$.

Proof. Since T is compact, we have $K = \overline{T(X)}$ is compact in Y. By (ii), $F|_K$ is also a generalized Φ-mapping. By Lemma 2, $F|_K$ has a continuous selection $f : K \to X$. So, by Lemma 3, we have $fT \in KKM^*(X,X)$, and so, by Theorem 2, there exists $x \in X$ such that $x \in fT(x) \subset FT(x)$; that is, there exists $y \in T(x)$ such that $x \in F(y)$. ∎

Corollary 3. [1] *Let X be a nonempty convex subset of a locally convex space E, and let Y be a topological space. Assume that*

(i) *$T \in KKM(X,Y)$ is compact and closed, and*

(ii) *$F : Y \to 2^X$ is a Φ-mapping.*

Then there exists $(\overline{x},\overline{y}) \in X \times Y$ such that $\overline{y} \in T(\overline{x})$ and $\overline{x} \in F(\overline{y})$.

Applying Theorem 1 and Corollary 3, we also have the following theorem.

Theorem 4. *Let X be a nonempty convex subset of a locally convex space E_1, and Y a nonempty almost-convex subset of a Hausdorff topological vector space*

E_2. If $T : X \to 2^Y$ is a generalized Φ-mapping, $F : Y \to 2^X$ is a Φ-mapping, and if T is compact and closed, then exists $(x, y) \in X \times Y$ such that $y \in T(x)$ and $x \in F(y)$.

We next establish the another coincidence theorem, as follows:

Theorem 5. *Let X be a nonempty almost-convex subset of a topological vector space E, and let Y be a topological space. Suppose that $T, F : X \to 2^Y$ are two mutifunctions satisfying*

(i) $T \in KKM^*(X, Y)$ *is compact, and*

(ii) $F^{-1} : Y \to 2^X$ *is a generalized Φ-mapping.*

Then there exists $x_0 \in X$ such that $T(x_0) \cap F(x_0) \neq \phi$.

Proof. Since T is compact, $\overline{T(X)}$ is compact. By (ii), there exists a companion mapping $G : Y \to 2^X$ such that $Y = \cup_{x \in X} intG^{-1}(x)$. Hence, there exists $A \in \langle X \rangle$ such that $\overline{T(X)} \subset \cup_{x \in A} intG^{-1}(x)$.

Case 1. If $\overline{T(X)} \subset intG^{-1}(x_0)$ for some $x_0 \in X$, then $T(x_0) \subset intG^{-1}(x_0)$. Take $y_0 \in T(x_0)$. Then $y_0 \in intG^{-1}(x_0)$, which implies $x_0 \in G(y_0)$. And, by (ii), we have $x_0 \in F^{-1}(y_0)$, $y_0 \in F(x_0)$. This shows $T(x_0) \cap F(x_0) \neq \phi$.

Case 2. If $\overline{T(X)} \nsubseteq intG^{-1}(x)$ for all $x \in X$, then $\overline{T(X)} \setminus intG^{-1}(x) \neq \phi$ for all $x \in X$. Define $S : X \to 2^Y$ by

$$S(x) = \overline{T(x)} \setminus intG^{-1}(x) \;\text{ for }\; x \in X.$$

Then $S(x)$ is nonempty and closed for all $x \in X$. Let $A = \{x_1, x_2, ..., x_n\} \in \langle X \rangle$. We claim that S is not a generalized KKM^* mapping with respect to T. Suppose, on the contrary, S is a generalized KKM^* mapping with respect to T. Since $T \in KKM^*(X, Y)$, $\{S(x) : x \in X\}$ has finite intersection property. Thus, $\cap_{x \in N} S(x) \neq \phi$ for each $N \in \langle X \rangle$, which implies $\overline{T(X)} \nsubseteq \cup_{x \in N} intG^{-1}(x)$ for each $N \in \langle X \rangle$, and so we get a contradiction. Therefore, there exists a neighborhood V of the origin 0 of E such that for any convex-inducing mapping $h_{A,V} : A \to X$ one has $T(co(h_{A,V}(A))) \nsubseteq S(A)$. Choose $x_0 \in co(h_{A,V}(A))$ and $y_0 \in T(x_0) \subset Y$ such that $y_0 \notin S(A)$. By the definition of S, $y_0 \in intG^{-1}(x_i)$ for all $i = 1, 2, .., n$. This implies $x_i \in G(y_0)$ for all $i = 1, 2, .., n$. By (ii), we have $co(h_{A,V}(A)) \subset F^{-1}(y_0)$, and so $y_0 \in F(x_0)$. This shows $T(x_0) \cap F(x_0) \neq \phi$. We complete the proof. ∎

Corollary 4. *Let X be a convex space, and let Y be a topological space. Suppose that $T, F : X \to 2^Y$ are two mutifunctions satisfying*

(i) $T \in KKM(X,Y)$ *is compact, and*

(ii) $F^{-1} : Y \to 2^X$ *is a* Φ*-mapping.*

Then there exists $x_0 \in X$ *such that* $T(x_0) \cap F(x_0) \neq \phi$.

Corollary 5. *Let* X *and* Y *be two convex spaces. Suppose that* $T, F : X \to 2^Y$ *are two mutifunctions satisfying*

(i) T *is a* Φ*-mapping and compact, and*

(ii) $F^{-1} : Y \to 2^X$ *is a* Φ*-mapping.*

Then there exists $x_0 \in X$ *such that* $T(x_0) \cap F(x_0) \neq \phi$.

A subset X of a topological vector space E is said to be admissible (in the sense of Klee [10]) provided that, for any nonempty compact subset A of X and every neighborhood V of 0 of E, there exists a continuous mapping $h_{A,V} : A \to X$ such thta $h(x) \in x + V$ for all $x \in A$ and $h(A)$ is contained in a finite-dimensional subspace L of E.

In [3], Chang et al. had introduced the class $S - KKM(D,X,Y)$ on an admissible onvex set X, we now apply the Corollary 1 and Theorem 3.1[3], we also have the following coincidence theorem for the Φ-mapping and the class $S - KKM(D,X,Y)$.

Theorem 6. *Let* X *be an admissible convex subset of a topological vector space,* D *be a nonempty subset of* X*, and let* Y *be a topological space. Suppose that*

(i) $s : D \to X$ *is surjective,*

(ii) $T \in s - KKM(D,X,Y)$ *is comapct and closed, and*

(iii) $F : Y \to 2^X$ *is a* Φ*-mapping.*

Then there exists $(x_0, y_0) \in X \times Y$ *such that* $y_0 \in T(x_0)$ *and* $x_0 \in F(y_0)$.

Proof. Since the proof is analyogous to the proof of Theorem 3, we omit it. ∎

Applying Theorem 1 and Theorem 6, we also have the following coincidence theorem.

Theorem 7. *Let* X *be an admissible convex subset of a Hausdorff topological vector space* E_1*, and* Y *an almost-convex subset of a Hausdorff topological vector space* E_2*. Suppose that* $T : X \to 2^Y$ *and* $F : Y \to 2^X$ *satisfy*

(i) T *is comapct and closed,*

(ii) T *is a generalized* Φ*-mapping, and*

(iii) *F is a Φ-mapping.*

Then there exists $(x_0, y_0) \in X \times Y$ such that $y_0 \in T(x_0)$ and $x_0 \in F(y_0)$.

Proof. Since T is a generalized Φ-mapping, by Theorem 1, we get $T \in KKM(X,Y)$. Let $S = i_X$, then $T \in s - KKM(X,X,Y)$. Thus all of the assumptions of Theorem 6 are satisfied. So, there exists $(x_0, y_0) \in X \times Y$ such that $y_0 \in T(x_0)$ and $x_0 \in F(y_0)$. ■

3. Generalized Variational Theorems and Minimax Inequality Theorems

Definition 8. Let X and Y be two topological spaces, and let $F : X \to 2^Y$.

(i) F is said to be tranfer open if for any $x \in X$ and $y \in F(x)$, there exists an $x' \in X$ such that $y \in intF(x')$, and

(ii) F is said to be tranfer closed if for any $x \in X$ and $y \notin F(x)$, there exists an $x' \in X$ such that $y \notin intF(x')$.

Definition 9. Let X and Y be two topological spaces. A function $f : X \times Y \to \overline{\Re}$ is said to be transfer upper semicontinuous(resp. transfer lower semicontinuous) in the first variable if for each $\lambda \in \Re$ and all $x \in X$, $y \in Y$ with $f(x,y) < \lambda$ (resp. $f(x,y) > \lambda$), there exists a $y' \in Y$ and a neighborhood N_x of x such that $f(u,y') < \lambda$ (resp. $f(u,y') > \lambda$) for all $u \in N_x$.

Remark 4. It is easy to prove (see [12], Lemma 2.2) that f is transfer upper semicontinuous(resp. transfer lower semicontinuous) in the first variable if and only if the set-valued mapping $F : Y \to 2^X$, $F(y) = \{x \in X : f(x,y) < \lambda\}$ (resp. $F(y) = \{x \in X : f(x,y) > \lambda\}$) is transfer open valued.

Lemma 4. [13]. *Let X and Y be two topological spaces, and let $F : X \to 2^Y$ be a set-valued mapping. Then the following conditions are equivalent:*

(i) *F^{-1} is tranfer open valued on Y, and*

(ii) *$X = \cup_{y \in Y} intF^{-1}(y)$.*

Lemma 5. [2]. *Let X and Y be two topological spaces, and let $F : X \to 2^Y$ be a set-valued mapping. Then F is transfer closed if and only if $\cap_{x \in X} F(x) = \cap_{x \in X} \overline{F(x)}$*

Applying the above Lemma 5, we immediately obtain the following variational inequalities and minimax inequalities.

Theorem 8. *Let X be a nonempty almost-convex subset of a Hausdorff topological vector space E, Y a topological space, and let $F \in KKM^*(X,Y)$ be compact. If $f, g : X \times Y \to \Re$ are two real-valued functions satisfying the following conditions:*

(i) *for each* $x \in X$*, the mapping* $y \mapsto f(x, y)$ *is transfer lower sem-continuous on* Y*, and*

(ii) *for each* $y \in Y$*,* g *is almost* f*-quasiconcave,*

then for each $\xi \in \Re$*, one of the following properties holds:*

(1) *there exists* $(\overline{x}, \overline{y}) \in \mathcal{G}_F$ *such that*

$$g(\overline{x}, \overline{y}) > \xi, \quad \textit{or}$$

(2) *there exists* $y' \in Y$ *such that*

$$f(x, y') \leq \xi, \quad \textit{for all} \quad x \in X.$$

Proof. Let $\xi \in \Re$. Since F is compact, $\overline{F(X)}$ is compact in Y. Define $T, S : X \to 2^Y$ by

$$T(x) = \{y \in \overline{F(X)} : g(x, y) \leq \xi\}, \quad \text{for all} \quad x \in X, \quad \text{and}$$
$$S(x) = \{y \in \overline{F(X)} : f(x, y) \leq \xi\}, \quad \text{for all} \quad x \in X.$$

Suppose the conclusion (1) is false. Then for each $(x, y) \in \mathcal{G}_F$, $g(x, y) \leq \xi$. This implies that $\mathcal{G}_F \subset \mathcal{G}_T$.

Let $A = \{x_1, x_2, ...x_n\} \in \langle X \rangle$. By the condition (ii), we claim that S is a generalized KKM^* mapping with respect to T. If the above statement is not true, then there exists a neighborhood V of the orgin 0 of E such that for any convex-inducing mapping $h_{A,V} : A \to X$ one has $T(co(h_{A,V}(A))) \not\subseteq S(A)$. So there exist $x_0 \in co(h_{A,V}(A))$ and $y_0 \in T(x_0)$ such that $y_0 \notin S(A)$. From the definitions of T and S, it follows that $g(x_0, y_0) \leq \xi$ and $f(x_i, y_0) > \xi$ for all $i = 1, 2, ..., n$. This contradicts the condition (ii). Therefore, S is a generalized KKM^* mapping with respect to T, and so we get S is a generalized KKM^* mapping with respect to F. Since $F \in KKM^*(X, Y)$, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property, and since $\overline{S(x)}$ is compact for each $x \in X$, so we have $\cap_{x \in X} \overline{S(x)} \neq \phi$. From Lemma 5 and the condition (i), we have that $\cap_{x \in X} S(x) \neq \phi$. Take $y_0 \in \cap_{x \in X} S(x)$, then $f(x, y_0) \leq \xi$ for all $x \in X$ ■

Theorem 9. *If all of the assumptions of Theorem 8 hold, then we immediately conclude the following inequality.*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{(x,y) \in \mathcal{G}_F} g(x, y).$$

Proof. Let $\xi = \sup_{(x,y) \in \mathcal{G}_F} g(x, y)$. Then the conclusion (1) of Theorem 8 is false. So there exist $y_0 \in Y$ such that $f(x, y_0) \leq \xi$ for all $x \in X$.

This implies $\sup_{x\in X} f(x,y_0) \leq \xi$, and so we have $\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{(x,y)\in\mathcal{G}_F} g(x,y)$. ■

Proposition 1. *Let X be a nonempty almost-convex subset of a Hausdorff topological vector space E, Y a topological space, V a neighborhood of the origin o of E, and let $T, F : X \to 2^Y$ be two set-valued mappings. Then the following two statements are equivalent.*

(i) *for each $y \in Y$, $A \in \langle T^*(y)\rangle$ implies $co(h_{A,V}(A)) \subset F^*(y)$, where $h_{A,V} : A \to X$ is a convex-inducing mapping.*

(ii) *T is a generalized KKM^* mapping with respect to F.*

Theorem 10. *Let X be a nonempty almost-convex subset of a Hausdorff topological vector space E, Y a compact topological space, and let $S, T : X \to 2^Y$ be two set-valued mappings satisfying the following conditions:*

(i) *$T \in KKM^*(X,Y)$,*

(ii) *S is transfer closed valued on X,*

(iii) *for each $y \in Y$, $T^*(y)$ is almost-convex, and*

(iv) *for each $x \in X$, $T(x) \subset S(x)$.*

Then there exists $\overline{y} \in Y$ such that $S^*(\overline{y}) = \phi$.

Proof. Let V be a neighborhood of the origin 0 of E. By the conditions (iii) and (iv), we have that for each $y \in Y$ and any $A \in \langle S^*(y)\rangle$, $co(h_{A,V}(A)) \subset T^*(y)$, where $h_{A,V} : A \to X$ is a convex-inducing mapping. So, by Proposition 1, S is a generalized KKM^* mapping with respect to T. Therefore, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. Since Y is compact, $\cap_{x\in X}\overline{S(x)} \neq \phi$. By Lemma 5, we have $\cap_{x\in X}S(x) \neq \phi$. Let $\overline{y} \in \cap_{x\in X}S(x)$. Then $S^*(\overline{y}) = \phi$. ■

Corollary 6. *Let X be a convex space, Y a compact topological space, and let $S, T : X \to 2^Y$ be two set-valued mappings satisfying the following conditions:*

(i) *$T \in KKM(X,Y)$,*

(ii) *S is transfer closed valued on X,*

(iii) *for each $y \in Y$, $T^*(y)$ is convex, and*

(iv) *for each $x \in X$, $T(x) \subset S(x)$.*

Then there exists $\overline{y} \in Y$ such that $S^*(\overline{y}) = \phi$.

Theorem 11. *Let X be a convex space, Y a nonempty set, and Z a compact convex space. Let $s, g, t : X \times Z \to \Re$ and $f : X \times Y \to \Re$ be four functions such that*

(i) t *is* s*-quasiconvex in the second variable,*

(ii) t *is* g*-quasiconcave in the first variable,*

(iii) f *is transfer upper semicontinuous in the first variable,*

(iv) g *is upper semi-continuous on* $X \times Y$*, and*

(v) *for each* $y \in Y$*, there exists* $z \in Z$ *such that* $s(\cdot, z) \leq f(\cdot, y)$.

Then

$$\inf_{z \in Z} \sup_{x \in X} g(x,z) \leq \sup_{(x,z) \in X \times Z} t(x,z).$$

Proof. Let $\lambda = sup_{(x,z) \in X \times Z} t(x,z)$ and define the set-valued mappings $T, S, G : X \to 2^Z$ and $F : X \to 2^Y$ by

$$\begin{aligned} T(x) &= \{u \in Z : t(x,u) \leq \lambda\}, \\ G(x) &= \{v \in Z : g(x,v) \leq \lambda\}, \\ S(x) &= \{z \in Z : s(x,z) < \lambda\}, \text{ and} \\ F(x) &= \{y \in Y : f(x,y) < \lambda\}. \end{aligned}$$

By the condition (iii), F^{-1} is transfer open valued on Y, and then by Lemma 4, we have $X = \cup_{y \in Y} int F^{-1}(y)$. By the condition (v), for each $y \in Y$, there exists $z \in Z$ such that $F^{-1}(y) \subset S^{-1}(z)$. So we conclude that $X = \cup_{z \in Z} int S^{-1}(z)$.

We claim that for each $x \in X$, $N \in \langle S(x) \rangle$ implies $co(N) \subset T(x)$. Let $x \in X$, $N = \{z_1, z_2, ..., z_n\} \in \langle S(x) \rangle$ and $u \in co\{z_1, z_2, ..., z_n\}$. Since $z_i \in S(x)$ and t is s-quasiconvex in the second variable, we have

$$t(x,u) \leq max_{1 \leq i \leq n} s(x, z_i) < \lambda,$$

and hence $u \in T(x)$. These imply T is a Φ-mapping with a companion mapping S. Thus, by Theorem 1, $T \in KKM(X, Z)$.

By the condition (iv), $G(x)$ is closed for each $x \in X$, and by the condition (ii), we have $T(co(N)) \subset G(N)$ for each $N \in \langle X \rangle$. So G is a generalized KKM mapping with respect to T. Since $T \in KKM(X, Z)$, hence the family $\{G(x) : x \in X\}$ has the finite intersection property. Since Z is compact, we get $\cap_{x \in X} G(x) \neq \phi$. Let $\overline{z} \in \cap_{x \in X} G(x)$. Then $g(x, \overline{z}) \leq \lambda$ for all $x \in X$, which implies $sup_{x \in X} g(x, \overline{z}) \leq \lambda$. So we have $inf_{z \in Z} sup_{x \in X} g(x,z) \leq \lambda = sup_{(x,z) \in X \times Z} t(x,z)$. ■

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