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THE CHARACTERIZATIONS OF WEIGHTED SOBOLEV SPACES BY WAVELETS AND SCALING FUNCTIONS

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Abstract. We prove that suitable wavelets and scaling functions give characterizations and unconditional bases of the weighted Sobolev space $L^{p,s}(w)$ with A_p or A_p^{loc} weights. In the case of $w \in A_p$, we use only wavelets with proper regularity. Meanwhile, if we assume $w \in A_p^{\mathrm{loc}}$, not only compactly supported C^{s+1}-wavelets but also compactly supported C^{s+1}-scaling functions come into play. We also establish that our bases are greedy for $L^{p,s}(w)$ after normalization.

1. Introduction

We can characterize the L^2-norm of $f \in L^2(\mathbb{R}^n)$ in terms of the wavelet coefficients appearing in the wavelet expansion of f with the wavelet basis. In particular, if we use the wavelets with proper decay, proper smoothness or compact support, then they give characterizations and unconditional bases of various function spaces (cf. [1, 8, 9, 14, 17, 22]).

Now we make a brief view of the study on weighted L^p spaces $L^p(w) := L^p(\mathbb{R}^n, w(x)dx)$ $(1 < p < \infty)$. Lemarié-Rieusset showed that the Daubechies wavelets give a characterization and an unconditional basis of $L^p(w)$ with $w \in A_p$. Here A_p means the Muckenhoupt A_p class. He also considered for the case of A_p^{loc}, which is an extension of A_p. As a result, he proved that a characterization and an unconditional basis of $L^p(w)$ with $w \in A_p^{\mathrm{loc}}$ were given by means of the Daubechies wavelets and the Daubechies scaling functions ([14]). After that, Aimar, Bernardis and Mart´n-Reyes showed that the result similar to [14] was valid for 1-regular wavelets in the case of A_p ([1]).

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In this paper we study the weighted Sobolev spaces $L^{p,s}(w) := L^{p,s}(\mathbb{R}^n, w(x)dx)$ ($1 < p < \infty$, $s \in \mathbb{N}$) with $w \in A_p$ or $w \in A_p^{\mathrm{loc}}$. We shall need smoother wavelets and scaling functions in order to get the characterizations and the unconditional bases of $L^{p,s}(w)$. As a consequence, we have the similar results to the studies on $L^p(w)$ shown by [1] and [14].

Additionally we would like to comment on the construction of greedy bases. As is noted in [9], if we characterize $L^p(w)$ by wavelets and scaling functions and if we obtain unconditional bases in terms of wavelets and scaling functions, then we can construct the greedy bases in $L^p(w)$. The same method is applicable to $L^{p,s}(w)$, that is, we can construct the greedy bases in $L^{p,s}(w)$ using wavelets and scaling functions.

Let us explain the outline of this article. Section 2 consists of preliminaries. We describe the fundamental theory on wavelets, two classes of weights, some bases, weighted function spaces, and some known results on $L^p(w)$. Our results are contained in Sections 3, 4 and 5. We characterize $L^{p,s}(w)$ with $w \in A_p$ by wavelets in Section 3. On the other hand, we characterize $L^{p,s}(w)$ with $w \in A_p^{\mathrm{loc}}$ in terms of wavelets and scaling functions in Section 4. Lastly, in Section 5, we construct the unconditional bases and the greedy bases in $L^{p,s}(w)$ by applying the results in Sections 3 and 4.

Throughout this paper, s means a positive integer. We let $1 < p < \infty$ and denote by p' the conjugate exponent of p, i.e., p' satisfies $1/p + 1/p' = 1$. χ_F means the characteristic function of a measurable set $F \subset \mathbb{R}^n$. $\mathbb{Z}_+$ denotes the set of all non-negative integers. We shall also note that the Fourier transform of a function f is defined by $\mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx$.

2. Preliminaries

2.1. Wavelets and scaling functions

First let us recall the definition of wavelet ([17, 22]).

2.1. Notation

1. Given a function f defined on $\mathbb{R}^n$, we denote $f_{j,k}(x) := 2^{jn/2}f(2^jx - k)$ for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.
2. We define the index set E by $E := \{1, 2, \dots, 2^n - 1\}$.

Definition 2.2. A set of functions $\{\psi^e\}_{e\in E} \subset L^2(\mathbb{R}^n)$ is called a wavelet set if $\{\psi^e_{j,k} : e \in E,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$. Then $\{\psi^e_{j,k} : e \in E,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^n\}$ is said to be a wavelet basis in $L^2(\mathbb{R}^n)$ and each ψ^e is said to be a wavelet.

By way of multiresolution analysis ([17, 22]), we can construct a wavelet set $\{\psi^e\}_{e\in E}$ and a function $\varphi \in L^2(\mathbb{R}^n)$ such that the sequence $\{\varphi_{m,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} : e \in E,\ j \ge m,\ k \in \mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$ for each $m \in \mathbb{Z}$. The function φ is called a scaling function.

We give remarkable examples of scaling functions and wavelets obtained by tensor products (cf. [17, 22]).

Example 2.3.

1. The first example introduced here was constructed by Y. Meyer. There exist a scaling function φ and a wavelet set $\{\psi^e\}_{e\in E}$ such that φ and each ψ^e are in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, real-valued and band-limited with $\operatorname{supp}\mathcal{F}[\varphi] \subset \left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]^n$ and $\operatorname{supp}\mathcal{F}[\psi^e] \subset \left(\left[-\frac{8}{3}\pi, -\frac{2}{3}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{8}{3}\pi\right]\right)^n$. We call $\{\psi^e\}_{e\in E}$ the Meyer wavelet set (cf. [17, 22]).
2. The next one is I. Daubechies'. For each positive integers $N \ge 2$, we can construct a scaling function φ and a wavelet set $\{\psi^e\}_{e\in E}$ such that φ and each ψ^e are in $C^{r(N)}(\mathbb{R}^n)$, real-valued and compactly supported with $\operatorname{supp}\varphi = \operatorname{supp}\psi^e = [0, 2N-1]^n$. In our actual construction, $r(N) > 0$ is an increasing function of N. We say that φ is the Daubechies scaling function, $\{\psi^e\}_{e\in E}$ is the Daubechies wavelet set associated with φ, and each ψ^e is the Daubechies wavelet (cf. [6, 15]).

2.2. A_p weights and A_p^{loc} weights

By "weight" we mean a non-negative and locally integrable function.

2.4. Notation For a weight w and a measurable set $F \subset \mathbb{R}^n$, we denote $w(F) := \int_F w(x)\,dx$, while $|F|$ means the Lebesgue measure of F.

We consider the following two classes of weights in this paper.

Definition 2.5. Let w be a weight such that $w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

1. The class of weights A_p consists of all w satisfying
$$A_p(w) := \sup_{Q:\mathrm{cube}} \frac{1}{|Q|} w(Q) \left(\frac{1}{|Q|} w^{-1/(p-1)}(Q)\right)^{p-1} < \infty,$$
and each $w \in A_p$ is called an A_p weight.
2. The class of weights A_p^{loc} consists of all w satisfying
$$A_p^{\mathrm{loc}}(w) := \sup_{\substack{|Q|\le 1,\\ Q:\mathrm{cube}}} \frac{1}{|Q|} w(Q) \left(\frac{1}{|Q|} w^{-1/(p-1)}(Q)\right)^{p-1} < \infty, \qquad (0.1)$$
and each $w \in A_p^{\mathrm{loc}}$ is called an A_p^{loc} weight.

Several helpful remarks may be in order.

Remark 2.6.

1. For example, $|x|^a \in A_p$ for $-n < a < n(p-1)$ (cf. [21, Section IX. 4]).
2. The class of A_p^{loc} weights is independent of the upper bound for the cube size used in its definitions. Namely we can replace $|Q| \le 1$ by $|Q| \le r$ in (1) for any $0 < r < \infty$. In fact, if we define

$$A_p^{\mathrm{loc},r}(w) := \sup_{\substack{|Q|\le r,\\ Q:\mathrm{cube}}} \frac{1}{|Q|} w(Q) \left(\frac{1}{|Q|} w^{-1/(p-1)}(Q) \right)^{p-1}$$

for each $r > 0$, then it clearly follows that $A_p^{\mathrm{loc},r}(w) \le A_p^{\mathrm{loc}}(w)$ if $0 < r \le 1$. On the other hand, Rychkov gave the estimation that $A_p^{\mathrm{loc},r}(w) \le r^{-p} e^{cr} A_p^{\mathrm{loc}}(w)$ if $r > 1$, where $c > 0$ is a constant depending only on n, p and $A_p^{\mathrm{loc}}(w)$ (cf. [19]).
3. We shall also remark that $A_p \subsetneqq A_p^{\mathrm{loc}}$. In fact, $\exp(b|x|) \in A_p^{\mathrm{loc}} \setminus A_p$ for $b \in \mathbb{R} \setminus \{0\}$
4. We have that $w \in A_p$ if and only if $w^{-1/(p-1)} \in A_{p'}$. In fact, it clearly follows that $A_p(w) = A_{p'}(w^{-1/(p-1)})^{p-1}$. The same result is true for the case of A_p^{loc}.

The next lemma serves as a tool for reducing the matter to the case when $w \in A_p$.

Lemma 2.7. ([19, Proof of Lemma 1.1]). *Let $a \in \mathbb{R}$, $r, t > 0$ and $w \in A_p^{\mathrm{loc}}$. We define*

$$\tau_m(u) := \begin{cases} u & \text{if} \quad u \in [t(m+a), t(m+a+r)) \\ 2t(m+a+r) - u & \text{if} \quad u \in [t(m+a+r), t(m+a+2r)) \end{cases}$$

for $m \in \mathbb{Z}$ and $u \in [t(m+a), t(m+a+2r))$. We also define $\{w_l\}_{l \in \mathbb{Z}^n}$ to fulfill that

$$w_l(x) = w(\tau_{l_1}(x_1), \ldots, \tau_{l_n}(x_n)) \quad if \quad x \in \prod_{\nu=1}^{n} [t(l_\nu + a), t(l_\nu + a + 2r)),$$

and that each w_l is a $2tr\mathbb{Z}^n$-periodic function on $\mathbb{R}^n$ for all $l \in \mathbb{Z}^n$. Then it follows that $\{w_l\}_{l \in \mathbb{Z}^n} \subset A_p$ with $A_p(w_l) \le 3^{np} A_p^{\mathrm{loc},t^n r^n}(w)$ for every $l \in \mathbb{Z}^n$.

2.3. Bases

Let X be a Banach space. Let us make a view of Schauder basis and unconditional basis first.

Definition 2.8.

1. $\{x_k\}_{k=1}^{\infty} \subset X$ is said to be a Schauder basis if there exists a unique sequence $\{c_k(x)\}_{k=1}^{\infty} \subset \mathbb{C}$ such that for all $x \in X$,
$$x = \sum_{k=1}^{\infty} c_k(x) x_k \quad in \quad X. \tag{0.2}$$
2. A Schauder basis $\{x_k\}_{k=1}^{\infty} \subset X$ is said to be an unconditional basis if the convergence (2) is always unconditional.

It is known that there are several equivalent definitions of unconditional basis in Banach spaces ([10, 16, 22]). Next we introduce two kinds of bases defined by Konyagin and Temlyakov ([12]).

Definition 2.9. Let $\{x_k\}_{k=1}^{\infty}$ be a normalized Schauder basis in X. We call $\{x_k\}_{k=1}^{\infty}$ a greedy basis for X if there exists a constant $C > 0$ such that for every $x \in X$ there exists a permutation ρ of $\mathbb{N}$ which satisfies $\left|c_{\rho(1)}(x)\right| \geq \left|c_{\rho(2)}(x)\right| \geq \ldots \geq \left|c_{\rho(N)}(x)\right|$ and
$$\left\| x - \sum_{k=1}^{N} c_{\rho(k)}(x) x_{\rho(k)} \right\|_X \leq C \inf_{y \in \Sigma_N} \|x - y\|_X ,$$
for every $N \in \mathbb{N}$, where $\Sigma_N := \left\{ \sum_{\nu \in \Lambda} \alpha_\nu x_\nu : \alpha_\nu \in \mathbb{C},\ \sharp\Lambda \leq N,\ \Lambda \subset \mathbb{N} \right\}$.

Definition 2.10. Let $\{x_k\}_{k=1}^{\infty}$ be a normalized Schauder basis in X. We say that $\{x_k\}_{k=1}^{\infty}$ is a democratic basis for X if there exists a constant $D > 0$ independent of P and Q such that $\left\| \sum_{k \in P} x_k \right\|_X \leq D \left\| \sum_{k \in Q} x_k \right\|_X$ for any finite subsets $P, Q \subset \mathbb{N}$ with the same cardinality $\sharp P = \sharp Q$.

Theorem 2.11 we describe next becomes the key in Section 5 later.

Theorem 2.11. $[[12,\ \mathrm{Theorem}\ 1]]$ *Let $\{x_k\}_{k=1}^{\infty}$ be a normalized Schauder basis in X. Then $\{x_k\}_{k=1}^{\infty}$ is a greedy basis if and only if it is an unconditional and democratic basis.*

Remark 2.12. $[[\ 12,\ \mathrm{Section}\ 3]]$. Konyagin and Temlyakov give some examples of bases, showing that “democratic” and “unconditional” are independent notions.

2.4. Weighted L^p spaces and weighted Sobolev spaces

Definition 2.13. Let w be a weight.

1. The weighted L^p space $L^p(w)$ is the space of all measurable functions f with

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right)^{1/p} < \infty.$$

2. Suppose $w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The weighted Sobolev space $L^{p,s}(w)$ is the space of all measurable functions f satisfying that $f \in L^p(w)$ and weak derivatives $D^\alpha f \in L^p(w)$ for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ with $|\alpha| \le s$.

If a weight w satisfies $w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $L^p(w) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We also remark that $L^{p,s}(w)$ is a Banach space with the norm $\|f\|_{L^{p,s}(w)} := \sum_{|\alpha|\le s} \|D^\alpha f\|_{L^p(w)}$.

Remark 2.14. [cf. [21, Section IX. 4]]. For any $w \in A_p$, we have $(1 + |x|)^{-np} w(x) \in L^1(\mathbb{R}^n)$. Thus we see that $\mathcal{S}(\mathbb{R}^n) \subset L^{p,s}(w)$.

In the case of $w \in A_p$, we can replace $\|\cdot\|_{L^{p,s}(w)}$ as follows.

Theorem 2.15. *Let $w \in A_p$. Then $\|\cdot\|_{L^p(w)} + \sum_{|\beta|=s} \|D^\beta(\,\cdot\,)\|_{L^p(w)}$ is equivalent to $\|\cdot\|_{L^{p,s}(w)}$, where the embedding constants depend only on n, p, $A_p(w)$ and s.*

We can obtain Theorem 2.15 above by the same arguments as [8, Theorem 6.4 in Chapter 6] applying the next result given by Kurtz ([13, Theorem 4]).

Proposition 2.16. *Let $w \in A_p$ and $m \in C^n(\mathbb{R}^n \setminus \{(0, \ldots, 0)\})$. Suppose that*

$$\sup_{R>0} R^{2|\alpha|-n} \int_{R\le|x|\le 2R} |D^\alpha m(x)|^2 dx < \infty$$

for all $\alpha \in \mathbb{Z}_+{}^n$ with $|\alpha| \le n$. Then the operator T defined by $\mathcal{F}[Tf] = m\,\mathcal{F}[f]$ is bounded on $L^p(w)$.

Now we consider two maximal functions and the boundedness on $L^p(w)$.

Proposition 2.17. [cf. [2]] *Let $1 < q < \infty$ and $w \in A_p$. Then there exists a constant $C > 0$ depending only on n, p, q and $A_p(w)$ such that*

$$\|\|(Mf_\nu)_{\nu=1}^\infty\|_{l^q}\|_{L^p(w)} \le C\, \|\|(f_\nu)_{\nu=1}^\infty\|_{l^q}\|_{L^p(w)}$$

for all $(f_\nu)_{\nu=1}^\infty$ *with*

$$\| \| (f_\nu)_{\nu=1}^\infty \|_{l^q} \|_{L^p(w)} := \left(\int_{\mathbb{R}^n} \left(\sum_{\nu=1}^\infty |f_\nu(x)|^q \right)^{p/q} w(x)\, dx \right)^{1/p} < \infty.$$

Here M *is the Hardy-Littlewood maximal function defined by*

$$Mf(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\, dy \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over the cubes Q *centered at* x*.*

Next we state for the local case. Rychkov proved the following boundedness for the local Hardy-Littlewood maximal function ([19]). Remark that he gave the result for vector-valued case. In this paper we have only to apply it for scalar-valued case.

Proposition 2.18. *Let* $r > 0$ *and* $w \in A_p^{\mathrm{loc}}$*. Then there exists a constant* $C > 0$ *depending only on* r*,* n*,* p *and* $A_p^{\mathrm{loc}}(w)$ *such that* $\left\| M^{\mathrm{loc},r} f \right\|_{L^p(w)} \le C \|f\|_{L^p(w)}$ *for all* $f \in L^p(w)$*. Here* $M^{\mathrm{loc},r}$ *is the local Hardy-Littlewood maximal function defined by*

$$M^{\mathrm{loc},r} f(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\, dy \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over the cubes Q *centered at* x *and satisfying* $|Q| \le r$*.*

2.5. Density of $C_c^\infty(\mathbb{R}^n)$ in $L^{p,s}(w)$

We will need the following density to obtain characterizations of $L^{p,s}(w)$.

Theorem 2.19. $C_c^\infty(\mathbb{R}^n)$ *is dense in* $L^{p,s}(w)$ *whenever* $w \in A_p^{\mathrm{loc}}$*.*

E. Nakai, N. Tomita and K. Yabuta proved Theorem 2.19 for $w \in A_p$ ([18, Theorem 1.1]). We can easily prove Theorem 2.19 by the same arguments as the proof of [18, Theorem 1.1] with the following uniformly boundedness stated in Lemma 2.20.

Lemma 2.20. *Let* $w \in A_p^{\mathrm{loc}}$ *and* η *be a function on* $\mathbb{R}^n$ *which is bounded, compactly supported, non-negative, and radial decreasing as a function on* $(0, \infty)$*. Define* $\eta_t(x) := t^{-n} \eta(x/t)$ *for* $t > 0$*. Then there exists a constant* $C > 0$ *depending on* n*,* p*,* $A_p^{\mathrm{loc}}(w)$ *and* η *such that* $\|\eta_t * f\|_{L^p(w)} \le C \|f\|_{L^p(w)}$ *for all* $0 < t \le 1$ *and* $f \in L^p(w)$*.*

Proof of Lemma 2.20. Let us take $J \in \mathbb{N}$ so that $\operatorname{supp} \eta \subset [-J, J]^n$. Following the same calculations as [21, Proof of Proposition 2.3 in Chapter IV], we have

$$|\eta_t * f(x)| \le 2^{2n} \|\eta\|_{L^1(\mathbb{R}^n)} M^{\mathrm{loc},(2tJ)^n} f(x) \le 2^{2n} \|\eta\|_{L^1(\mathbb{R}^n)} M^{\mathrm{loc},(2J)^n} f(x)$$

for all $0 < t \le 1$ and $f \in L^p(w)$. By Proposition 2.18, there exists a constant $C > 0$ depending on J, n, p and $A_p^{\mathrm{loc}}(w)$ such that $\|M^{\mathrm{loc},(2J)^n} f\|_{L^p(w)} \le C\|f\|_{L^p(w)}$. Hence we get $\|\eta_t * f\|_{L^p(w)} \le C2^{2n}\|\eta\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(w)}$.

2.6. Wavelets, scaling functions and $L^p(w)$

We recall known results on the characterizations and the constructions of bases of $L^p(w)$.

2.21. Notation For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we define a dyadic cube $Q_{j,k} := \prod_{\nu=1}^{n} \left[2^{-j}k_\nu, 2^{-j}(k_\nu+1)\right)$ and denote $\chi_{j,k} := 2^{jn/2}\chi_{Q_{j,k}}$.

Definition 2.22. Let $r \in \mathbb{N}$. A function f on $\mathbb{R}^n$ is r-regular if for all $m \in \mathbb{N}$ there exists a constant $C_m > 0$ such that $|D^\alpha f(x)| \le C_m(1+|x|)^{-m}$ for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le r$.

For example, the Meyer wavelet set consists of r-regular wavelets. Moreover if we take a large $N \in \mathbb{N}$ sufficiently, the Daubechies wavelet becomes r-regular.

Lemarié-Rieusset gave a characterization and an unconditional basis of $L^p(w)$ with $w \in A_p$ by the Daubechies wavelets in the case of one-variable. His proof is due to the boundedness of Calderón-Zygmund operators on $L^p(w)$. Following the same method, Aimar, Bernardis and Mart´n-Reyes showed that the result given by Lemarié-Rieusset was valid for 1-regular wavelets. More precisely, they obtained the next theorem.

Theorem 2.23. [cf. [1, 14]] *Let $w \in A_p$ and $\{\psi^e\}_{e\in E}$ be a wavelet set constructed by a multiresolution analysis such that each ψ^e is 1-regular. Then there exist two constants $0 < c \le C < \infty$ depending only on n, p, $A_p(w)$ and $\{\psi^e\}_{e\in E}$ such that for every $f \in L^p(w)$,*

$$c\,\|f\|_{L^p(w)} \le \left\| \left(\sum_{e\in E} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi_{j,k}^e\rangle \chi_{j,k}\right|^2 \right)^{1/2} \right\|_{L^p(w)} \le C\,\|f\|_{L^p(w)}\,.$$

Additionally the wavelet basis $\{\psi_{j,k}^e : e \in E,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^n\}$ forms an unconditional basis in $L^p(w)$.

On the other hand, Lemarié-Rieusset gave the next result. The result shows that we need not only wavelets but also scaling functions which construct wavelets if we consider $L^p(w)$ with $w \in A_p^{\mathrm{loc}}$. Although he proved it in the case of one-variable, it is true in the case of several-variables with obvious modifications applying tensor products.

Theorem 2.24. [cf. [14, Proposition 2 (ii)]] *Let $w \in A_p^{\mathrm{loc}}$, $m \in \mathbb{Z}$, φ be the Daubechies scaling function and $\{\psi^e\}_{e\in E}$ be the Daubechies wavelet set associated with φ. Define*

$$\mathcal{M}_{p,w,m}(f) := \left(\sum_{k\in\mathbb{Z}^n} \left| \langle f, \varphi_{m,k}\rangle \, \|\varphi_{m,k}\|_{L^p(w)} \right|^p \right)^{1/p} + \left\| \left(\sum_{e\in E} \sum_{j=m}^{\infty} \sum_{k\in\mathbb{Z}^n} \left| \langle f, \psi^e_{j,k}\rangle \chi_{j,k} \right|^2 \right)^{1/2} \right\|_{L^p(w)}.$$

Then there exist two constants $0 < c \le C < \infty$ depending only on n, p, $A_p^{\mathrm{loc}}(w)$, m and φ such that $c\|f\|_{L^p(w)} \le \mathcal{M}_{p,w,m}(f) \le C\|f\|_{L^p(w)}$ for all $f \in L^p(w)$. Additionally the sequence $\{\varphi_{m,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} \,:\, e \in E,\, j \ge m,\, k \in \mathbb{Z}^n\}$ forms an unconditional basis in $L^p(w)$.

Applying the characterizations and the constructions of the unconditional bases above, we can construct the greedy bases for $L^p(w)$. Namely the next theorem follows (cf. [9, Section 6]).

Theorem 2.25.

1. *Let $w \in A_p$ and $\{\psi^e\}_{e\in E}$ be a wavelet set constructed by a multiresolution analysis such that each ψ^e is 1-regular. Define $\widetilde{\psi^e}_{j,k} := \psi^e_{j,k}/\|\psi^e_{j,k}\|_{L^p(w)}$ for $e \in E$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. Then the sequence $\left\{\widetilde{\psi^e}_{j,k} \,:\, e \in E,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^n\right\}$ forms a greedy basis for $L^p(w)$.*
2. *Let $w \in A_p^{\mathrm{loc}}$, φ be the Daubechies scaling function and $\{\psi^e\}_{e\in E}$ be the Daubechies wavelet set associated with φ. Define $\tilde{\varphi}_{m,k} := \varphi_{m,k}/\|\varphi_{m,k}\|_{L^p(w)}$ and $\widetilde{\psi^e}_{j,k} := \psi^e_{j,k}/\|\psi^e_{j,k}\|_{L^p(w)}$ for $e \in E$, $j \ge m$ and $k \in \mathbb{Z}^n$. Then the sequence $\{\tilde{\varphi}_{m,k}\}_{k\in\mathbb{Z}^n} \cup \{\widetilde{\psi^e}_{j,k} \,:\, e \in E,\, j \ge m,\, k \in \mathbb{Z}^n\}$ forms a greedy basis for $L^p(w)$.*

In Section 5, we will construct the greedy bases for $L^{p,s}(w)$ by means of wavelets and scaling functions following the similar method.

3. The Characterization of $L^{p,s}(w)$ with $w \in A_p$ by Wavelets

Following statements in [8, Chapter 6], we can obtain the next characterization of $L^{p,s}(w)$ with $w \in A_p$ by wavelets.

Theorem 3.1. *Let $w \in A_p$ and $\{\psi^e\}_{e\in E}$ be a wavelet set constructed from a multiresolution analysis such that each wavelet ψ^e is $(s+1)$-regular. Then there exist two constants $0 < c \le C < \infty$ depending only on n, p, $A_p(w)$, s and $\{\psi^e\}_{e\in E}$ such that for all $f \in L^{p,s}(w)$,*

$$c\|f\|_{L^{p,s}(w)} \le \left\| \left(\sum_{e\in E} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} (1+2^{2js}) \left|\langle f, \psi^e_{j,k}\rangle \chi_{j,k}\right|^2 \right)^{1/2} \right\|_{L^p(w)} \le C\|f\|_{L^{p,s}(w)}.$$

Remark that we need some improvements on [8] to obtain Theorem 3.1. We use Theorem 2.15, [18, Theorem 1.1] and Theorem 2.23 described already, in addition, Lemma 3.3 and Proposition 3.4 as follows. We shall introduce the class of functions $\mathcal{R}^r(\mathbb{R}^n)$ in order to state them.

Definition 3.2. Let $r \in \mathbb{Z}_+$. The set $\mathcal{R}^r(\mathbb{R}^n)$ consists of all $f \in C^{r+1}(\mathbb{R}^n)$ satisfying that there exist constants $\varepsilon, \gamma > 0$ and $C_\alpha > 0$ for each $\alpha \in {\mathbb{Z}_+}^n$ with $|\alpha| \le r+1$ such that

1. $\displaystyle\int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0$ for every $|\alpha| \le r+1$,
2. $|f(x)| \le C_{(0,\dots,0)}(1+|x|)^{-(2+r+\gamma)n}$,
3. $|D^\alpha f(x)| \le C_\alpha(1+|x|)^{-(1+\varepsilon)n}$ for every $1 \le |\alpha| \le r+1$.

For example, if ψ^e is an $(r+1)$-regular wavelet constructed from a multiresolution analysis for some $r \in \mathbb{Z}_+$, then $\psi^e \in \mathcal{R}^r(\mathbb{R}^n)$ (cf. [17]).

Lemma 3.3. *Theorem 1.1. Let $r \in \mathbb{Z}_+$, $\{\Phi^e\}_{e\in E}$, $\{\psi^e\}_{e\in E} \subset \mathcal{R}^r(\mathbb{R}^n)$ and $w \in A_p$. Define*

$$\mathcal{W}\left[r, \{\Phi^e\}_e\right](f) := \left(\sum_{e\in E} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|2^{jr}\langle f, \Phi^e_{j,k}\rangle \chi_{j,k}\right|^2 \right)^{1/2}.$$

If $\{\psi^e\}_{e\in E}$ is a wavelet set, then there exists a constant $C > 0$ depending only on n, p, $A_p(w)$, r, $\{\Phi^e\}_{e\in E}$ and $\{\psi^e\}_{e\in E}$ such that for all $f \in L^p(w)$,

$$\left\|\mathcal{W}\left[r, \{\Phi^e\}_e\right](f)\right\|_{L^p(w)} \le C\left\|\mathcal{W}\left[r, \{\psi^e\}_e\right](f)\right\|_{L^p(w)}.$$

Hernández and Weiss proved Lemma 3.3 for the non-weighted case using the non-weighted version of Proposition 2.17 ([8, Theorem 4.9 and Theorem 6.21 in Chapter 6]). Going through the same arguments as [8] with Proposition 2.17, we can get Lemma 3.3.

We also have the following proposition.

Proposition 3.4. *Let $w \in A_p$ and $\{\Phi^e\}_{e\in E} \subset \mathcal{R}^s(\mathbb{R}^n)$. Then there exists a constant $C > 0$ depending only on n, p, $A_p(w)$, s and $\{\Phi^e\}_{e\in E}$ such that for all $f \in L^{p,s}(w)$,*

$$\|\mathcal{W}\left[s, \{\Phi^e\}_e\right](f)\|_{L^p(w)} \le C\|f\|_{L^{p,s}(w)}.$$

The next proposition and Lemma 3.3 are important in order to show Proposition 3.4.

Proposition 3.5. [[4, Theorem 1.2], cf. [3, 11]] *Let $w \in A_p$, $\lambda > n$ and $\{\phi_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$. Define $\phi^{**}_{j,\lambda}(f)(x) := \sup_{y\in\mathbb{R}^n}\left\{|\phi_j * f(x-y)|(1+2^j|y|)^{-\lambda}\right\}$ and assume the following:*

1. *There exists a constant $a > 0$ independent of j such that $\operatorname{supp}\mathcal{F}[\phi_j] \subset \left\{2^{j-a} \le |\xi| \le 2^{j+a}\right\}$ for all $j \in \mathbb{Z}$.*
2. *For each $\alpha \in {\mathbb{Z}_+}^n$, there exists a constant $C_\alpha > 0$ such that $|D^\alpha\mathcal{F}[\phi_j](\xi)| \le C_\alpha 2^{-j|\alpha|}$ for all $\xi \in \mathbb{R}^n$ and $j \in \mathbb{Z}$.*

Then there exists a constant $C > 0$ depending only on n, p, $A_p(w)$, s, λ and $\{\phi_j\}_{j\in\mathbb{Z}}$ such that for every $f \in L^{p,s}(w)$,

$$\left\|\left\{\sum_{j=-\infty}^{\infty}\left(2^{js}\phi^{**}_{j,\lambda}(f)\right)^2\right\}^{1/2}\right\|_{L^p(w)} \le C\,\|f\|_{L^{p,s}(w)}.$$

Proof of Proposition 3.4. Let $\{\psi^e\}_{e\in E}$ be the Meyer wavelet set described in Example 2.3.1. By Lemma 3.3, there exists a constant $C_0 > 0$ depending only on n, p, $A_p(w)$, s, $\{\Phi^e\}_e$ and $\{\psi^e\}_e$ such that for every $f \in L^{p,s}(w)$,

$$\|\mathcal{W}\left[s, \{\Phi^e\}_e\right](f)\|_{L^p(w)} \le C_0\,\|\mathcal{W}\left[s, \{\psi^e\}_e\right](f)\|_{L^p(w)}.$$

Denote $\phi^e_j(y) := 2^{jn}\psi^e(-2^jy)$ for $j \in \mathbb{Z}$ and $e \in E$. Take $\lambda > n$ arbitrarily. Following the same calculations as [8, Proof of Theorem 4.2 in Chapter 6], we have

$$\sum_{k\in\mathbb{Z}^n}\left|\langle f, \psi^e_{j,k}\rangle\chi_{j,k}(x)\right|^2 \le \left(1+\sqrt{n}\right)^{2\lambda}\phi^{e**}_{j,\lambda}(f)(x)^2.$$

Namely we obtain that

$$\mathcal{W}\left[s, \{\psi^e\}_e\right](f) \le \left(1+\sqrt{n}\right)^{\lambda}\left[\sum_{e\in E}\left\{\sum_{j=-\infty}^{\infty}\left(2^{js}\phi^{e**}_{j,\lambda}(f)\right)^2\right\}\right]^{1/2}.$$

Now remark that $\{\phi_j^e\}_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$ satisfies the assumptions of Proposition 3.5 . Hence there exists a constant $C_1 > 0$ depending only on n, p, $A_p(w)$, s, λ and $\{\psi^e\}_{e\in E}$ such that

$$\|\mathcal{W}\left[s, \{\psi^e\}_e\right](f)\|_{L^p(w)} \leq \left(1+\sqrt{n}\right)^\lambda (2^n-1)C_1 \|f\|_{L^{p,s}(w)}.$$

Therefore we get $\|\mathcal{W}\left[s, \{\Phi^e\}_e\right](f)\|_{L^p(w)} \leq (1+\sqrt{n})^\lambda (2^n-1)C_0C_1 \|f\|_{L^{p,s}(w)}$.

Proof of Theorem 3.1. Theorem 2.23 and Proposition 3.4 prove the right-hand side inequality. We will prove the left-hand side inequality. By Theorems 2.15 and 2.23, we have only to estimate $\left\|D^\beta f\right\|_{L^p(w)}$ for all $\beta \in {\mathbb{Z}_+}^n$ with $|\beta| = s$ and $f \in L^{p,s}(w)$. By the duality, it follows that

$$\left\|D^\beta f\right\|_{L^p(w)} = \sup_g \left\{\left|\int_{\mathbb{R}^n} D^\beta f(x)g(x)\,dx\right| : \|g\|_{L^{p'}(v)} \leq 1\right\},$$

where $v := w^{-1/(p-1)}$. As a result of [18, Theorem 1.1] and the right-hand side inequality, it suffices to prove that

$$\left|\int_{\mathbb{R}^n} D^\beta f(x)g(x)\,dx\right| \leq C\,\|\mathcal{W}\left[s, \{\psi^e\}_e\right](f)\|_{L^p(w)}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ with $\|g\|_{L^{p'}(v)} \leq 1$, where $C > 0$ is a constant independent of β, f and g. Because the wavelet basis $\{\psi_{j,k}^e : e \in E,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$, we obtain that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} D^\beta f(x)g(x)\,dx\right| = \left|\int_{\mathbb{R}^n} f(x)D^\beta g(x)\,dx\right| \\
&= \left|\int_{\mathbb{R}^n} \left\{\sum_{e\in E}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^e\rangle \psi_{j,k}^e(x)\right\}\right. \\
&\qquad \left.\cdot \left\{\sum_{e\in E}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} \langle D^\beta g, \overline{\psi_{j,k}^e}\rangle \overline{\psi_{j,k}^e(x)}\right\}\,dx\right| \\
&= \left|\sum_{e\in E}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^e\rangle \langle D^\beta g, \overline{\psi_{j,k}^e}\rangle\right| \\
&\leq \sum_{e\in E}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi_{j,k}^e\rangle \langle g, \overline{2^{js}(D^\beta\psi^e)_{j,k}}\rangle\right| \cdot \int_{\mathbb{R}^n} \chi_{j,k}(x)^2\,dx \\
&= \int_{\mathbb{R}^n} \sum_{e\in E}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} \left|2^{js}\langle f, \psi_{j,k}\rangle\chi_{j,k}(x) \cdot \langle g, \overline{(D^\beta\psi^e)_{j,k}}\rangle\chi_{j,k}(x)\right|\,dx.
\end{aligned}$$

Therefore by the Cauchy-Schwartz inequality and Hölder's inequality, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} D^\beta f(x)g(x)\,dx\right| &\le \int_{\mathbb{R}^n} \mathcal{W}\left[s,\{\psi^e\}_e\right](f)(x)\cdot\mathcal{W}\left[0,\left\{\overline{D^\beta\psi^e}\right\}_e\right](g)(x)\,dx\\ &\le \left\|\mathcal{W}\left[s,\{\psi^e\}_e\right](f)\right\|_{L^p(w)}\left\|\mathcal{W}\left[0,\left\{\overline{D^\beta\psi^e}\right\}_e\right](g)\right\|_{L^{p'}(v)}.\end{aligned}$$

Now let $\{\Psi^e\}_{e\in E}$ be a wavelet set constructed from a multiresolution analysis such that each Ψ^e is 1-regular. In view of Lemma 3.3 and Theorem 2.23, we get

$$\left\|\mathcal{W}\left[0,\left\{\overline{D^\beta\psi^e}\right\}_e\right](g)\right\|_{L^{p'}(v)} \le C_0\left\|\mathcal{W}\left[0,\{\Psi^e\}_e\right](g)\right\|_{L^{p'}(v)} \le C_1\|g\|_{L^{p'}(v)} \le C_1,$$

where $C_0, C_1 > 0$ are constants depending only on n, p, $A_p(w)$, s, $\{\psi^e\}_{e\in E}$ and $\{\Psi^e\}_{e\in E}$.

4. The Characterization of $L^{p,s}(w)$ with $w \in A_p^{\mathrm{loc}}$ by Wavelets and Scaling Functions

In this section, we characterize $L^{p,s}(w)$ with $w \in A_p^{\mathrm{loc}}$ by wavelets and scaling functions with proper smoothness and compact support. We have the next main result.

Theorem 4.1. *Let $w \in A_p^{\mathrm{loc}}$, φ be the Daubechies scaling function and $\{\psi^e\}_{e\in E}$ be the Daubechies wavelet set associated with φ. Suppose $\varphi \in C^{s+1}(\mathbb{R}^n)$ and $\{\psi^e\}_{e\in E} \subset C^{s+1}(\mathbb{R}^n)$. Define*

$$\mathcal{V}\left[s,\{\psi^e\}_e\right](f) := \left(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|2^{js}\langle f,\psi^e_{j,k}\rangle\chi_{j,k}\right|^2\right)^{1/2}$$

$$\mathcal{N}^s_{p,w}(f) := \left(\sum_{k\in\mathbb{Z}^n}\left|\langle f,\varphi_{0,k}\rangle\|\varphi_{0,k}\|_{L^p(w)}\right|^p\right)^{1/p} + \left\|\mathcal{V}\left[s,\{\psi^e\}_e\right](f)\right\|_{L^p(w)}.$$

Then there exist two constants $c, C > 0$ depending only on n, p, $A_p^{\mathrm{loc}}(w)$, s and φ such that $c\|f\|_{L^{p,s}(w)} \le \mathcal{N}^s_{p,w}(f) \le C\|f\|_{L^{p,s}(w)}$ for all $f \in L^{p,s}(w)$.

We need the following proposition in order to prove the characterization above.

Proposition 4.2. *Let $w \in A_p^{\mathrm{loc}}$ and $\{\Psi^e\}_{e\in E}$ be a set of functions in $\mathcal{R}^s(\mathbb{R}^n)$ with compact support. Then there exists a constant $C > 0$ depending only on n, p, $A_p^{\mathrm{loc}}(w)$, s and $\{\Psi^e\}_{e\in E}$ such that $\|\mathcal{V}\left[s,\{\Psi^e\}_e\right](f)\|_{L^p(w)} \le C\|f\|_{L^{p,s}(w)}$ for all $f \in L^{p,s}(w)$.*

Proof of Proposition 4.2. Let φ be the Daubechies scaling function in $C^s(\mathbb{R}^n)$ with $\operatorname{supp}\varphi = [0, 2N-1]^n$ for some positive integer $N \geq 2$ and write $f_l(x) := f(x)\varphi(x-l)$ for each $l \in \mathbb{Z}^n$. Then we have $\sum_{l\in\mathbb{Z}^n} \varphi(x-l) = 1$ by [8, Proposition 3.14 in Chapter 5] or [22, Proposition 2.17]. Thus we get the decomposition that $f = \sum_{l\in\mathbb{Z}^n} f_l$. Let us choose $m \in \mathbb{N}$ so that $\operatorname{supp}\Psi \subset [-m, m]^n$. Then we see that $\operatorname{supp}\mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l) \subset \prod_{\nu=1}^{n}[l_\nu - m, l_\nu + 2N + m] =: D_l$ for each $l \in \mathbb{Z}^n$. On the other hand, for all $x \in \mathbb{R}^n$, there exists a unique $L = L(x) \in \mathbb{Z}^n$ such that $x \in Q_{0,L}$. Denoting $\mathcal{A}(L) := \{l \in \mathbb{Z}^n : L_\nu + 1 - 2N - m \leq l_\nu \leq L_\nu + m \text{ for all } 1 \leq \nu \leq n\}$, we get that

$$\begin{aligned}
\left|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f)(x)\right|^p &\leq \left|\sum_{l\in\mathbb{Z}^n} \mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l)(x)\right|^p = \left|\sum_{l\in\mathcal{A}(L)} \mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l)(x)\right|^p \\
&\leq \sum_{l\in\mathcal{A}(L)} \left|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l)(x)\right|^p \cdot \sharp\mathcal{A}(L)^{p/p'} \\
&\leq (2m+2N)^{n(p-1)} \sum_{l\in\mathbb{Z}^n} \left|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l)(x)\right|^p .
\end{aligned}$$

Therefore it follows that

$$\begin{aligned}
&\left\|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f)\right\|_{L^p(w)}^p \\
&\leq (2m+2N)^{n(p-1)} \int_{\mathbb{R}^n} \sum_{l\in\mathbb{Z}^n} \left|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l)(x)\right|^p w(x)\,dx \\
&= (2m+2N)^{n(p-1)} \sum_{k\in\mathbb{Z}^n} \int_{Q_{0,k}} \sum_{l\in\mathbb{Z}^n} \left|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l)(x)\right|^p w(x)\,dx \\
&= (2m+2N)^{n(p-1)} \sum_{k\in\mathbb{Z}^n} \int_{Q_{0,k}} \sum_{l\in\mathcal{A}(k)} \left|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l)(x)\right|^p w(x)\,dx \\
&\leq (2m+2N)^{n(p-1)} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathcal{A}(k)} \int_{D_l} \left|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l)(x)\right|^p w(x)\,dx.
\end{aligned}$$

If we invoke Lemma 2.7, we get $\{w_l\}_{l\in\mathbb{Z}^n} \subset A_p$ such that $A_p(w_l) \leq 3^{np} A_p^{\mathrm{loc},2^n(m+N)^n}(w)$ and $w_l = w$ on D_l for all $l \in \mathbb{Z}^n$. Thus we obtain that

$$\left\|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f)\right\|_{L^p(w)}^p \leq (2m+2N)^{n(p-1)} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathcal{A}(k)} \left\|\mathcal{V}\left[s, \{\Psi^e\}_e\right](f_l)\right\|_{L^p(w_l)}^p .$$

Denote $G_l := \prod_{\nu=1}^{n}[l_\nu, l_\nu + 2N - 1] = \operatorname{supp}\varphi(x-l)$. Then we see that $w_l = w$ on G_l for every $l \in \mathbb{Z}^n$. Additionally by Proposition 3.4, there exists a constant $C_0 > 0$ depending only on n, p, $A_p^{\mathrm{loc}}(w)$, s and $\{\Psi^e\}_e$ such that for each $k \in \mathbb{Z}^n$ and $l \in \mathcal{A}(k)$,

$$\begin{aligned}
\|\mathcal{V}[s,\{\Psi^e\}_e](f_l)\|_{L^p(w_l)}^p &\le \|\mathcal{W}[s,\{\Psi^e\}_e](f_l)\|_{L^p(w_l)}^p \\
&\le C_0\,\|f_l\|_{L^{p,s}(w_l)}^p \\
&\le C_1 \sum_{|\alpha|\le s}\int_{G_l}|D^\alpha f(x)|^p\,w_l(x)\,dx \\
&= C_1 \sum_{|\alpha|\le s}\int_{G_l}|D^\alpha f(x)|^p\,w(x)\,dx,
\end{aligned}$$

where $C_1 > 0$ is a constant depending only on n, p, s, C_0 and φ. Thus we obtain that

$$\begin{aligned}
&\|\mathcal{V}[s,\{\Psi^e\}_e](f)\|_{L^p(w)} \\
&\le \left\{(2m+2N)^{n(p-1)}\cdot C_1\cdot(2N-1)^n(2m+2N)^n\sum_{|\alpha|\le s}\|D^\alpha f\|_{L^p(w)}^p\right\}^{1/p} \\
&\le C_2\|f\|_{L^{p,s}(w)},
\end{aligned}$$

where $C_2 > 0$ is a constant depending only on n, p, $A_p^{\mathrm{loc}}(w)$, s, $\{\Psi^e\}_e$ and φ.

Proof of Theorem 4.1. First we show the right-hand side inequality of the characterization. The estimate of $\|\mathcal{V}[s,\{\psi^e\}_e](f)\|_{L^p(w)}$ is shown by Proposition 4.2. We estimate the first term of $\mathcal{N}_{p,w}^s(f)$. Let $N \ge 2$ be the positive integer such that $\operatorname{supp}\varphi = \operatorname{supp}\psi^e = [0, 2N-1]^n$ for every $e \in E$. Denote $G_k := \prod_{\nu=1}^{n}[k_\nu, k_\nu + 2N - 1] = \operatorname{supp}\varphi_{0,k}$ and $v := w^{-1/(p-1)}$. By Hölder's inequality, we obtain that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n}\left|\langle f,\varphi_{0,k}\rangle\|\varphi_{0,k}\|_{L^p(w)}\right|^p \\
&= \sum_{k\in\mathbb{Z}^n}|\langle f,\varphi_{0,k}\rangle|^p\cdot\|\varphi_{0,k}\|_{L^p(w)}^p \\
&\le \sum_{k\in\mathbb{Z}^n}\int_{G_k}|f(x)|^p\,w(x)\,dx\cdot\left(\int_{G_k}|\varphi_{0,k}(x)|^{p'}v(x)\,dx\right)^{p-1}\cdot\|\varphi\|_{L^\infty(\mathbb{R}^n)}^p w(G_k)
\end{aligned}$$

$$
\begin{aligned}
&\le \|\varphi\|_{L^\infty(\mathbb{R}^n)}^{2p} \sum_{k\in\mathbb{Z}^n} \int_{G_k} |f(x)|^p\, w(x)\, dx \cdot w(G_k) v(G_k)^{p-1} \\
&\le \|\varphi\|_{L^\infty(\mathbb{R}^n)}^{2p} (2N-1)^{np} A_p^{\mathrm{loc},(2N-1)^n}(w) \sum_{k\in\mathbb{Z}^n} \int_{G_k} |f(x)|^p\, w(x)\, dx \\
&\le \|\varphi\|_{L^\infty(\mathbb{R}^n)}^{2p} (2N-1)^{n(p+1)} A_p^{\mathrm{loc},(2N-1)^n}(w) \|f\|_{L^p(w)}^p.
\end{aligned}
$$

Next we prove the left-hand side inequality. By the duality, we see that

$$
\|D^\alpha f\|_{L^p(w)} = \sup_g \left\{ \left| \int_{\mathbb{R}^n} D^\alpha f(x) g(x)\, dx \right| : \|g\|_{L^{p'}(v)} \le 1 \right\},
$$

for all $f \in L^{p,s}(w)$ and $\alpha \in \mathbb{Z}_+{}^n$ with $|\alpha| \le s$. Thus, in view of Theorem 2.19 and the right-hand side inequality, it suffices to show that

$$
\left| \int_{\mathbb{R}^n} D^\alpha f(x) g(x)\, dx \right| \le C \mathcal{N}_{p,w}^s(f)
$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$ with $\|g\|_{L^{p'}(v)} \le 1$, where $C > 0$ is a constant independent of α, f and g. Because $\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi_{j,k}^e : e \in E,\ j \in \mathbb{Z}_+,\ k \in \mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$, we obtain that

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^n} D^\alpha f(x) g(x)\, dx \right| = \left| \int_{\mathbb{R}^n} f(x) D^\alpha g(x)\, dx \right| \\
&= \Bigg| \int_{\mathbb{R}^n} \left\{ \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{0,k}\rangle \varphi_{0,k}(x) + \sum_{e\in E} \sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^e\rangle \psi_{j,k}^e(x) \right\} \\
&\quad \times \left\{ \sum_{k\in\mathbb{Z}^n} \langle D^\alpha g, \varphi_{0,k}\rangle \varphi_{0,k}(x) + \sum_{e\in E} \sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^n} \langle D^\alpha g, \psi_{j,k}^e\rangle \psi_{j,k}^e(x) \right\} dx \Bigg| \\
&= \left| \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{0,k}\rangle \langle D^\alpha g, \varphi_{0,k}\rangle + \sum_{e\in E} \sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^e\rangle \langle D^\alpha g, \psi_{j,k}^e\rangle \right|.
\end{aligned}
$$

We estimate $\left| \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{0,k}\rangle \langle D^\alpha g, \varphi_{0,k}\rangle \right|$ first. By Hölder's inequality, we see that

$$
1 = \int_{\mathbb{R}^n} |\varphi_{0,k}(x)|^2\, dx \le \|\varphi_{0,k}\|_{L^p(w)} \|\varphi_{0,k}\|_{L^{p'}(v)}
$$

for every $k \in \mathbb{Z}^n$. We shall also remark that $|\langle D^\alpha g, \varphi_{0,k}\rangle| = |\langle g, (D^\alpha \varphi)_{0,k}\rangle|$. Using Hölder's inequality again, we are led to

$$\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{0,k}\rangle \langle D^\alpha g, \varphi_{0,k}\rangle\right| \\
&\le \sum_{k\in\mathbb{Z}^n} \left|\langle f, \varphi_{0,k}\rangle \|\varphi_{0,k}\|_{L^p(w)} \cdot \langle g, (D^\alpha\varphi)_{0,k}\rangle \|\varphi_{0,k}\|_{L^{p'}(v)}\right| \\
&\le \left(\sum_{k\in\mathbb{Z}^n} \left|\langle f, \varphi_{0,k}\rangle \|\varphi_{0,k}\|_{L^p(w)}\right|^p\right)^{1/p} \cdot \left(\sum_{l\in\mathbb{Z}^n} \left|\langle g, (D^\alpha\varphi)_{0,l}\rangle \|\varphi_{0,l}\|_{L^{p'}(v)}\right|^{p'}\right)^{1/p'}.
\end{aligned}$$

Note that $\operatorname{supp}\varphi_{0,l}$, $\operatorname{supp}(D^\alpha\varphi)_{0,l} \subset G_l$ for each $l \in \mathbb{Z}^n$. The same calculation as the proof of the right-hand side inequality works and we obtain

$$\sum_{l\in\mathbb{Z}^n} \left|\langle g, (D^\alpha\varphi)_{0,l}\rangle \|\varphi_{0,l}\|_{L^{p'}(v)}\right|^{p'} \quad \le \quad C_1^{p'} \|g\|_{L^{p'}(v)}^{p'},$$

where $C_1 := \max_{|\alpha|\le s} \|D^\alpha\varphi\|^2_{L^\infty(\mathbb{R}^n)} (2N-1)^{n(1+1/p')} A_{p'}^{\mathrm{loc},(2N-1)^n}(v)^{1/p'}$. Hence we get

$$\left|\sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{0,k}\rangle \langle D^\alpha g, \varphi_{0,k}\rangle\right| \le C_1 \left(\sum_{k\in\mathbb{Z}^n} \left|\langle f, \varphi_{0,k}\rangle \|\varphi_{0,k}\|_{L^p(w)}\right|^p\right)^{1/p} \|g\|_{L^{p'}(v)}.$$

Next we estimate $\left|\sum_{e\in E}\sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^n} \langle f, \psi^e_{j,k}\rangle \langle D^\alpha g, \psi^e_{j,k}\rangle\right|$. By straightforward calculations and Hölder's inequality, we obtain that

$$\begin{aligned}
&\left|\sum_{e\in E}\sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^n} \langle f, \psi^e_{j,k}\rangle \langle D^\alpha g, \psi^e_{j,k}\rangle\right| \\
&\le \sum_{e\in E}\sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi^e_{j,k}\rangle \langle g, 2^{j|\alpha|}(D^\alpha\psi^e)_{j,k}\rangle\right| \cdot \int_{\mathbb{R}^n} \chi_{j,k}(x)^2\, dx \\
&= \int_{\mathbb{R}^n} \sum_{e\in E}\sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^n} \left|2^{js}\langle f, \psi^e_{j,k}\rangle \chi_{j,k}(x) \cdot 2^{j(|\alpha|-s)} \langle g, (D^\alpha\psi^e)_{j,k}\rangle \chi_{j,k}(x)\right| dx \\
&= \int_{\mathbb{R}^n} \sum_{e\in E}\sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^n} \left|2^{js}\langle f, \psi^e_{j,k}\rangle \chi_{j,k}(x) \cdot \langle g, (D^\alpha\psi^e)_{j,k}\rangle \chi_{j,k}(x)\right| dx \\
&\le \int_{\mathbb{R}^n} \mathcal{V}\left[s, \{\psi^e\}_e\right](f)(x) \cdot \mathcal{V}\left[0, \{D^\alpha\psi^e\}_e\right](g)(x)\, dx \\
&\le \left\|\mathcal{V}\left[s, \{\psi^e\}_e\right](f)\right\|_{L^p(w)} \left\|\mathcal{V}\left[0, \{D^\alpha\psi^e\}_e\right](g)\right\|_{L^{p'}(v)}.
\end{aligned}$$

Now remark that $\operatorname{supp} \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}}) \subset \prod_{\nu=1}^{n} [l_\nu - 2N+1, l_\nu + 2] =: E_l$ for each $l \in \mathbb{Z}^n$. On the other hand, for all $x \in \mathbb{R}^n$, there exists a unique $L = L(x) \in \mathbb{Z}^n$ such that $x \in Q_{0,L}$. Denote $\mathcal{B}(L) := \{l \in \mathbb{Z}^n : L_\nu - 1 \le l_\nu \le L_\nu + 2N - 1$ for all $1 \le \nu \le n\}$. Then we get

$$\begin{aligned}
|\mathcal{V}[0, \{D^\alpha \psi^e\}_e](g)(x)|^{p'} &\le \left| \sum_{l \in \mathbb{Z}^n} \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}})(x) \right|^{p'} \\
&= \left| \sum_{l \in \mathcal{B}(L)} \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}})(x) \right|^{p'} \\
&\le \sum_{l \in \mathcal{B}(L)} \left| \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}})(x) \right|^{p'} \cdot \sharp \mathcal{B}(L)^{p'/p} \\
&\le (2N+1)^{n(p'-1)} \sum_{l \in \mathbb{Z}^n} \left| \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}})(x) \right|^{p'}.
\end{aligned}$$

Therefore it follows that

$$\begin{aligned}
&\|\mathcal{V}[0, \{D^\alpha \psi^e\}_e](g)\|_{L^{p'}(v)}^{p'} \\
&\le (2N+1)^{n(p'-1)} \int_{\mathbb{R}^n} \sum_{l \in \mathbb{Z}^n} \left| \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}})(x) \right|^{p'} v(x)\, dx \\
&= (2N+1)^{n(p'-1)} \sum_{k \in \mathbb{Z}^n} \int_{Q_{0,k}} \sum_{l \in \mathbb{Z}^n} \left| \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}})(x) \right|^{p'} v(x)\, dx \\
&= (2N+1)^{n(p'-1)} \sum_{k \in \mathbb{Z}^n} \int_{Q_{0,k}} \sum_{l \in \mathcal{B}(k)} \left| \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}})(x) \right|^{p'} v(x)\, dx \\
&\le (2N+1)^{n(p'-1)} \sum_{k \in \mathbb{Z}^n} \sum_{l \in \mathcal{B}(k)} \int_{E_l} \left| \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}})(x) \right|^{p'} v(x)\, dx.
\end{aligned}$$

By Lemma 2.7, we can construct $\{v_l\}_{l \in \mathbb{Z}^n} \subset A_{p'}$ such that $v_l = v$ on E_l and $A_{p'}(v_l) \le 3^{np'} A_{p'}^{\mathrm{loc},(2N+1)^n}(v)$ for every $l \in \mathbb{Z}^n$. In addition, let $\{\Psi^e\}_{e \in E}$ be a wavelet set constructed from a multiresolution analysis such that each Ψ^e is 1-regular. By virtue of Lemma 3.3 and Theorem 2.23, we have that

$$\begin{aligned}
\left\| \mathcal{V}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}}) \right\|_{L^{p'}(v_l)}^{p'} &\le \left\| \mathcal{W}[0, \{D^\alpha \psi^e\}_e](g \cdot \chi_{Q_{0,l}}) \right\|_{L^{p'}(v_l)}^{p'} \\
&\le C_2 \left\| \mathcal{W}[0, \{\Psi^e\}_e](g \cdot \chi_{Q_{0,l}}) \right\|_{L^{p'}(v_l)}^{p'} \\
&\le C_3 \left\| g \cdot \chi_{Q_{0,l}} \right\|_{L^{p'}(v_l)}^{p'} = C_3 \left\| g \cdot \chi_{Q_{0,l}} \right\|_{L^{p'}(v)}^{p'},
\end{aligned}$$

where $C_2, C_3 > 0$ are constants depending only on n, p, $A_p^{\mathrm{loc}}(w)$, s, $\{\psi^e\}_{e\in E}$ and $\{\Psi^e\}_{e\in E}$. Now write $C_4 := (2N+1)^{n/p} C_3^{1/p'}$. Then it follows that

$$\begin{aligned}&\|\mathcal{V}\left[0, \{D^\alpha \psi^e\}_e\right](g)\|_{L^{p'}(v)}^{p'} \\ \leq\ &(2N+1)^{n(p'-1)} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathcal{B}(k)} C_3 \left\|g\cdot\chi_{Q_{0,l}}\right\|_{L^{p'}(v)}^{p'} = C_4^{p'} \|g\|_{L^{p'}(v)}^{p'}.\end{aligned}$$

Namely we get

$$\left|\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} < f, \psi_{j,k}^e >< D^\alpha g, \psi_{j,k}^e >\right| \leq C_4 \left\|\mathcal{V}\left[s, \{\psi^e\}_e\right](f)\right\|_{L^p(w)} \|g\|_{L^{p'}(v)}.$$

Consequently we have

$$\left|\int_{\mathbb{R}^n} D^\alpha f(x) g(x)\, dx\right| \leq \max\{C_1, C_4\} \mathcal{N}_{p,w}^s(f) \|g\|_{L^{p'}(v)} \leq \max\{C_1, C_4\} \mathcal{N}_{p,w}^s(f). \blacksquare$$

5. The Greedy Bases of $L^{p,s}(w)$

As is mentioned in [9], in the case of weighted L^p spaces $L^p(w)$, if we have the characterization and the unconditional basis of $L^p(w)$ by wavelets and scaling functions, then we can establish the greedy basis given by them following the same statements as [5]. In this section, we show the similar arguments are applicable to $L^{p,s}(w)$.

Theorem 5.1.

1. *Let $w \in A_p$ and $\{\psi^e\}_{e\in E}$ be a wavelet set constructed from a multiresolution analysis such that each ψ^e is $(s+1)$-regular. Then the following (a) and (b) hold:*

   (a) *The wavelet basis $\{\psi_{j,k}^e : e \in E, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ forms an unconditional basis for $L^{p,s}(w)$.*

   (b) *Define $\widetilde{\psi^e}_{j,k} := \psi_{j,k}^e / \|\psi_{j,k}^e\|_{L^{p,s}(w)}$ for $e \in E$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. Then the sequence $\left\{\widetilde{\psi^e}_{j,k} : e \in E, j \in \mathbb{Z}, k \in \mathbb{Z}^n\right\}$ forms a greedy basis for $L^{p,s}(w)$.*

2. *Let $w \in A_p^{\mathrm{loc}}$, φ be the Daubechies scaling function and $\{\psi^e\}_{e\in E}$ be the Daubechies wavelet set associated with φ such that $\varphi \in C^{s+1}(\mathbb{R}^n)$ and $\{\psi^e\}_{e\in E} \subset C^{s+1}(\mathbb{R}^n)$. Then the following (a) and (b) hold:*

(a) *The sequence* $\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} \,:\, e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$ *forms an unconditional basis for* $L^{p,s}(w)$.

(b) *Define* $\tilde{\varphi}_{0,k} := \varphi_{0,k}/\|\varphi_{0,k}\|_{L^{p,s}(w)}$ *and* $\widetilde{\psi^e}_{j,k} := \psi^e_{j,k}/\|\psi^e_{j,k}\|_{L^{p,s}(w)}$ *for* $e \in E$, $j \in \mathbb{Z}_+$ *and* $k \in \mathbb{Z}^n$. *Then the sequence* $\{\tilde{\varphi}_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\widetilde{\psi^e}_{j,k} \,:\, e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$ *forms a greedy basis for* $L^{p,s}(w)$.

Proof of Theorem 5.1. We shall concentrate on *2*, *1* being proved similarly.

To begin with, we shall prove *(a)*. It suffices to check the following two conditions:

(I) There exists a constant $C>0$ independent of f, A and B such that $\|T_{A,B}f\|_{L^{p,s}(w)} \le C\,\|f\|_{L^{p,s}(w)}$ for all $f \in L^{p,s}(w)$ and all finite subsets $A \subset \mathbb{Z}^n$ and $B \subset E \times \mathbb{Z}_+ \times \mathbb{Z}^n$, where $T_{A,B}f := \sum_{k\in A} \langle f, \varphi_{0,k}\rangle\, \varphi_{0,k} + \sum_{(e,j,k)\in B} \langle f, \psi^e_{j,k}\rangle\, \psi^e_{j,k}$.

(II) The set $\operatorname{span}\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \operatorname{span}\{\psi^e_{j,k} \,:\, e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$ is dense in $L^{p,s}(w)$.

We show the condition (I) first. By the orthonormality and Theorem 4.1, we obtain

$$c\,\|T_{A,B}f\|_{L^{p,s}(w)} \le \mathcal{N}^s_{p,w}(T_{A,B}f) \le \mathcal{N}^s_{p,w}(f) \le C\|f\|_{L^{p,s}(w)},$$

where $0 < c \le C < \infty$ are the constants appearing in Theorem 4.1.

Next we check (II). We give an outline of the proof. It suffices to prove that for all $f \in L^{p,s}(w)$,

$$\lim_{A\nearrow\mathbb{Z}^n,\, B\nearrow E\times\mathbb{Z}_+\times\mathbb{Z}^n} \mathcal{N}^s_{p,w}(f - T_{A,B}f) = 0,$$

since $c\,\|f - T_{A,B}f\|_{L^{p,s}(w)} \le \mathcal{N}^s_{p,w}(f - T_{A,B}f)$ by Theorem 4.1. We split $\mathcal{N}^s_{p,w}(f - T_{A,B}f)$ by $\mathcal{N}^s_{p,w}(f - T_{A,B}f) = \mathcal{N}_1(f - T_{A,B}f) + \mathcal{N}_2(f - T_{A,B}f)$ with

$$\mathcal{N}_1(f) := \left(\sum_{k\in\mathbb{Z}^n} \left|\langle f, \varphi_{0,k}\rangle\, \|\varphi_{0,k}\|_{L^p(w)}\right|^p\right)^{1/p}$$
$$\text{and} \quad \mathcal{N}_2(f) := \|\mathcal{V}[s, \{\psi^e\}_e](f)\|_{L^p(w)}.$$

The orthonormality of the system $\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} \,:\, e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$ with regard to the L^2-inner product, the boundedness of $\mathcal{N}_\nu(f - T_{A,B}f)$ for each $\nu = 1, 2$ and Lebesgue's dominated convergence theorem give us the desired result.

Next we prove *(b)*. The proof we give here is essentially the same as [9, Section 5.3], and based on [5, the proof of Lemma 4.1]. We prepare the following two lemmas.

Lemma 5.2. *Let $w \in A_p^{\rm loc}$. Then w satisfies the dyadic reverse doubling condition, i.e., there exists a constant $d > 1$ independent of I and I' such that $d\, w(I') \le w(I)$ for all dyadic cubes $I,\ I'$ with $I' \subsetneqq I$.*

The proof of Lemma 5.2 is found in [7, p. 141] or [20, Proof of Corollary 1.1]. We also have the next estimate.

Lemma 5.3. *Let $w \in A_p^{\rm loc}$, φ be the Daubechies scaling function and $\{\psi^e\}_{e\in E}$ be the Daubechies wavelet set associated with φ such that $\varphi \in C^{s+1}(\mathbb{R}^n)$ and $\{\psi^e\}_{e\in E} \subset C^{s+1}(\mathbb{R}^n)$. Define*

$$\widetilde{\mathcal{V}}\,[s, \{\psi^e\}_e]\,(f) := \left(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \left| w(Q_{j,k})^{-1/p} \left\|\psi^e_{j,k}\right\|_{L^{p,s}(w)} \langle f, \psi^e_{j,k}\rangle \chi_{Q_{j,k}} \right|^2 \right)^{1/2}.$$

Then we have

$$c\left\|\widetilde{\mathcal{V}}\,[s, \{\psi^e\}_e]\,(f)\right\|_{L^p(w)} \le \|\mathcal{V}\,[s, \{\psi^e\}_e]\,(f)\|_{L^p(w)} \le C\left\|\widetilde{\mathcal{V}}\,[s, \{\psi^e\}_e]\,(f)\right\|_{L^p(w)}$$

for all $f \in L^{p,s}(w)$, where $0 < c \le C < \infty$ are the constants appearing in Theorem 4.1.

Proof of Lemma 5.3. For each $e \in E$, $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$, we have that

$$\mathcal{N}^s_{p,w}(\psi^e_{j,k}) = \left\|2^{js}\chi_{j,k}\right\|_{L^p(w)} = 2^{js+jn/2} w(Q_{j,k})^{1/p}.$$

On the other hand, by Theorem 4.1, we obtain

$$C^{-1}\mathcal{N}^s_{p,w}(\psi^e_{j,k}) \le \left\|\psi^e_{j,k}\right\|_{L^{p,s}(w)} \le c^{-1}\mathcal{N}^s_{p,w}(\psi^e_{j,k}).$$

Namely it follows that $C^{-1}2^{js} \le w(Q_{j,k})^{-1/p}\left\|\psi^e_{j,k}\right\|_{L^{p,s}(w)} 2^{-jn/2} \le c^{-1}2^{js}$. This estimate shows the desired result. ■

Now let us return to the proof of Theorem 5.1 *(b)*. In view of Theorem 5.1 *(a)* and Theorem 2.12, it is enough to prove that $\{\tilde\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\widetilde{\psi^e}_{j,k} \,:\, e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$ is democratic. We see that $\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} \,:\, e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$ forms an unconditional basis for $L^{p,s}(w)$ by *(a)*. Thus for all $f \in L^{p,s}(w)$ we can write

$$f = \sum_{k\in\mathbb{Z}^n} a_k(f)\tilde\varphi_{0,k} + \sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} b^e_{j,k}(f)\widetilde{\psi^e}_{j,k},$$

where $a_k(f) := \langle f, \varphi_{0,k}\rangle \|\varphi_{0,k}\|_{L^{p,s}(w)}$ and $b^e_{j,k}(f) := \langle f, \psi^e_{j,k}\rangle \|\psi^e_{j,k}\|_{L^{p,s}(w)}$. Now define

$$\widetilde{\mathcal{N}}^s_{p,w}(f) := \left(\sum_{k\in\mathbb{Z}^n} \left|\langle f, \varphi_{0,k}\rangle \|\varphi_{0,k}\|_{L^{p,s}(w)}\right|^p\right)^{1/p} + \left\|\widetilde{\mathcal{V}}\left[s, \{\psi^e\}_e\right](f)\right\|_{L^p(w)}.$$

By Theorem 4.1, we see that $c\|\varphi_{0,k}\|_{L^{p,s}(w)} \le \mathcal{N}^s_{p,w}(\varphi_{0,k}) = \|\varphi_{0,k}\|_{L^p(w)} \le \|\varphi_{0,k}\|_{L^{p,s}(w)}$. Thus Lemma 5.3 gives us $c'\|f\|_{L^{p,s}(w)} \le \widetilde{\mathcal{N}}^s_{p,w}(f) \le C'\|f\|_{L^{p,s}(w)}$, where $0 < c' \le C' < \infty$ are constants depending only on n, p, $A_p^{\mathrm{loc}}(w)$, s and φ. Then we have

$$\begin{aligned} c'\|f\|_{L^{p,s}(w)} &\le \left(\sum_{k\in\mathbb{Z}^n} |a_k(f)|^p\right)^{1/p} \\ &+ \left\|\left(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \left|w(Q_{j,k})^{-1/p} b^e_{j,k}(f)\chi_{Q_{j,k}}\right|^2\right)^{1/2}\right\|_{L^p(w)} \\ &\le C'\|f\|_{L^{p,s}(w)}. \end{aligned} \tag{3}$$

Let us denote $\tilde{\varphi}_Q := \tilde{\varphi}_{j,k}$ and $\widetilde{\psi^e}_Q := \widetilde{\psi^e}_{j,k}$ for a dyadic cube $Q = Q_{j,k}$. Now we take finite subsets $A \subset \{Q_{0,k}\}_{k\in\mathbb{Z}^n}$ and $\Lambda \subset \{(e, Q_{j,k}) : e \in E,\, j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n\}$, and write $g := \sum_{I\in A} \tilde{\varphi}_I + \sum_{(e,J)\in\Lambda} \widetilde{\psi^e}_J$. We also denote $B := \big\{Q_{j,k} : (e, Q_{j,k}) \in \Lambda \text{ for some } e \in E\big\}$. Note that $\sharp B \le \sharp\Lambda \le (2^n-1)\sharp B$. Using (3), we obtain

$$\begin{aligned} &c'\|g\|_{L^{p,s}(w)} \\ &\le (\sharp A)^{1/p} + \left\|\left(\sum_{(e,J)\in\Lambda} \left|w(J)^{-1/p}\chi_J\right|^2\right)^{1/2}\right\|_{L^p(w)} \\ &\le (\sharp A)^{1/p} + (2^n-1)^{1/2}\left\{\int_{\bigcup_{J'\in B} J'} \left(\sum_{J\in B} w(J)^{-2/p}\chi_J(x)\right)^{p/2} w(x)\,dx\right\}^{1/p}. \end{aligned} \tag{4}$$

For each $x \in \bigcup_{J\in B} J$, $J_1(x)$ denotes the minimal dyadic cube in B with regard to the inclusion relation that contains x. Then we get

$$\textstyle\sum_{J\in B} w(J)^{-2/p}\chi_J(x) \le \sum_{r=0}^{\infty} w(J_r)^{-2/p}, \tag{5}$$

where $J_0 := J_1(x)$, J_r is a dyadic cube satisfying $J_{r-1} \subset J_r$ and $2^n |J_{r-1}| = |J_r|$ for every $r \in \mathbb{N}$. By Lemma 5.2, there exists a constant $d > 1$ such that $w(J_r) \geq d\, w(J_{r-1}) \geq \ldots \geq d^r\, w(J_0) = d^r\, w(J_1(x))$ for all $r \in \mathbb{N}$. Thus we have

$$\sum_{r=0}^{\infty} w(J_r)^{-2/p} \leq \sum_{r=0}^{\infty} (d^r w(J_1(x)))^{-2/p} = C_0 w(J_1(x))^{-2/p}, \tag{6}$$

where $C_0 := (1 - d^{-2/p})^{-1}$. Following (5) and (6), we obtain

$$\begin{aligned}
&\int_{\bigcup_{J' \in B} J'} \left(\sum_{J \in B} w(J)^{-2/p} \chi_J(x) \right)^{p/2} w(x)\, dx \\
&\leq \int_{\bigcup_{J' \in B} J'} \left(C_0 w(J_1(x))^{-2/p} \right)^{p/2} w(x)\, dx \\
&= C_0^{p/2} \int_{\bigcup_{J' \in B} J'} w(J_1(x))^{-1} w(x)\, dx.
\end{aligned} \tag{7}$$

Now we set $\tilde{J} := \left\{ x \in \bigcup_{J' \in B} J' \;:\; J_1(x) = J \right\}$ for each $J \in B$. Then, since $\tilde{J} \subset J$ and $\bigcup_{J' \in B} J' = \bigcup_{J \in B} \tilde{J}$, it follows that

$$\begin{aligned}
\int_{\bigcup_{J' \in B} J'} w(J_1(x))^{-1} w(x)\, dx &= \int_{\bigcup_{J \in B} \tilde{J}} w(J_1(x))^{-1} w(x)\, dx \\
&\leq \sum_{J \in B} \int_{\tilde{J}} w(J_1(x))^{-1} w(x)\, dx \\
&= \sum_{J \in B} \int_{J} w(J)^{-1} w(x)\, dx \\
&= \sharp B.
\end{aligned} \tag{8}$$

Following (4)-(8), we have $c\|g\|_{L^{p,s}(w)} \leq (\sharp A)^{1/p} + C_0^{1/2} (2^n - 1)^{1/2} (\sharp B)^{1/p}$. Hence there exists a constant $C_1 > 0$ independent of g, A and Λ such that

$$\|g\|_{L^{p,s}(w)} \leq C_1 \left\{ \sharp A + \sharp \Lambda \right\}^{1/p}. \tag{9}$$

On the other hand, applying (3) to $f = g$ again, we have

$$
\begin{aligned}
&C'\|g\|_{L^{p,s}(w)} \\
&\geq (\sharp A)^{1/p} + \left\| \left(\sum_{(e,J)\in\Lambda} \left| w(J)^{-1/p}\chi_J \right|^2 \right)^{1/2} \right\|_{L^p(w)} \\
&\geq (\sharp A)^{1/p} + \left\{ \int_{\bigcup\limits_{J'\in B} J'} \left(\sum_{J\in B} w(J)^{-2/p}\chi_J(y) \right)^{p/2} w(x)\,dx \right\}^{1/p}.
\end{aligned}
\tag{10}
$$

For each $x \in \bigcup\limits_{J\in B} J$, we have

$$
\left(\sum_{J\in B} w(J)^{-2/p}\chi_J(x) \right)^{p/2} \geq w(J_1(x))^{-1}. \tag{11}
$$

Now going through the same argument as (5)-(6), we get

$$
\sum_{J\in B} w(J)^{-1}\chi_J(x) \leq C_0' w\left(J_1(x)\right)^{-1}, \tag{12}
$$

where $C_0' > 0$ is a constant depending only on p and d. Following (10)-(12), we obtain

$$
\begin{aligned}
C'\|g\|_{L^{p,s}(w)} &\geq (\sharp A)^{1/p} + \left(\int_{\bigcup\limits_{J'\in B} J'} {C_0'}^{-1} \sum_{J\in B} w(J)^{-1}\chi_J(x)w(x)\,dx \right)^{1/p} \\
&= (\sharp A)^{1/p} + \left({C_0'}^{-1} \sum_{J\in B} w(J)^{-1} \int_J w(x)\,dx \right)^{1/p} \\
&= (\sharp A)^{1/p} + {C_0'}^{-1/p} (\sharp B)^{1/p}.
\end{aligned}
$$

Namely there exists a constant $C_2 > 0$ independent of g, A and Λ such that

$$
C_2\|g\|_{L^{p,s}(w)} \geq \{\sharp A + \sharp\Lambda\}^{1/p}. \tag{13}
$$

Following (9) and (13), we get $C_2^{-1}\ \{\sharp A + \sharp\Lambda\}^{1/p} \leq \|g\|_{L^{p,s}(w)} \leq C_1\ \{\sharp A + \sharp\Lambda\}^{1/p}$. Consequently we have proved that the sequence $\{\tilde{\varphi}_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\widetilde{\psi^e}_{j,k} : e \in E,\ j \in \mathbb{Z}_+,\ k \in \mathbb{Z}^n\}$ is democratic. ∎

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