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WEAK AND WEAK* TOPOLOGIES AND BRODSKII-MILMAN'S THEOREM ON HYPERSPACES

Thakyin Hu and Jui-Chi Huang

Abstract. Let K be a weakly compact, convex subset of a Banach space X with normal structure. Brodskii and Milman proved that there exists a point $p \in K$ which is fixed under all isometries of K onto K. Suppose now $WCC(X)$ is the collection of all non-empty weakly compact convex subsets of X. We shall define a certain weak topology $\mathcal{T}_w$ on $WCC(X)$ and have the above-mentioned result extended to the hyperspace $(WCC(X), \mathcal{T}_w)$

1. INTRODUCTION

Banach Contraction Principle and Schauder-Tychonof Theorem were published in the early 1900's. These theorems have important applications to various branches of mathematics. Suppose K is a weakly compact, convex subset with normal structure of a Banach space, Brodskii and Millman [3] proved that there exists a point $p \in K$ which is fixed under all isometries of K onto K, and Browder and Kirk ([4], [11]) proved that every non-expansive mapping of K into K has a fixed point. It is the main purpose of this paper to extend Brodskii-Milman's theorem to the hyperspace $WCC(X)$, where X is a Banach space and $WCC(X)$ is the collection of all non-empty weakly compact convex subsets of X.

2. NOTATIONS AND PRELIMINARIES

Let X be a Banach space, X^* its topological dual and $BCC(X)$ be the collection of all non-empty bounded, closed convex subsets of X. For $A, B \in BCC(X)$, define $N(A;\varepsilon) = \{x \in X : d(x,a) = \|x - a\| < \varepsilon$ for some $a \in A\}$ and $h(A,B) = \inf\{\varepsilon > 0 : A \subset N(B;\varepsilon)$ and $B \subset N(A;\varepsilon)\}$, equivalently,

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$h(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$. Then h is known as the Hausdorff metric and $(BCC(X), h)$ is known as the hyperspace over X. If $\dim(X) < \infty$ and $A_n \in BCC(X)$ is a bounded sequence (i.e. there exists $M < \infty$ such that $h(A_n, \{0\}) \leq M$ for all $n = 1, 2, \ldots$), Blaschke [2] proved that $\{A_n\}$ has a subsequence $\{A_{n_k}\}$ such that $\{A_{n_k}\}$ converges to some $A \in BCC(X)$. DeBlasi and Myjak [2] introduced the concept of weak convergence of a sequence in $BCC(X)$ and they proved an infinite dimensional version of Blaschke's theorem. Other notions of weak convergence of bounded, closed, convex sets have been studied by other mathematicians ([1, 13]). Let $WCC(X)$ be the collection of all non-empty weakly compact convex subsets of X and $CC(X)$ be the collection of all non-empty compact, convex subsets of X. For general X, we have $CC(X) \subsetneqq WCC(X) \subsetneqq BCC(X)$. If $X$is reflexive, we have $WCC(X) = BCC(X)$. If $\dim(X) < \infty$, we have $CC(X) = WCC(X) = BCC(X)$. Weak topologies have been introduced on the hyperspaces $CC(X)$, $WCC(X)$ and extensions of certain fixed point theorems are obtained ([7]-[11]). Suppose now $W^*CC(X^*)$ is the collection of all non-empty weak* compact, convex subsets of X^*. Because of the interplay between X and X^*, the notion of weak topology on $WCC(X)$ leads us naturally to consider the concept of weak* topology on $W^*CC(X^*)$. And we shall prove in the sequel that Brodskii-Milman's theorem can be extended to the hyperspaces $WCC(X)$ and $W^*CC(X^*)$. To continue our discussion, we let $\mathbb{Z}$ denote the complex plane and $CC(\mathbb{Z})$ the collection of all non-empty compact, convex subsets of $\mathbb{Z}$. First, observe that for each $x^* \in X^*$, the weak continuity and linearity of x^* imply that for each $A \in WCC(X)$ (i.e., A is a weakly compact, convex subset of X), we have $x^*(A) \in CC(\mathbb{Z})$ (i.e., $x^*(A)$ is a compact, convex subset of the complex plane $\mathbb{Z}$). Thus each x^* maps the space $WCC(X)$ into $CC(\mathbb{Z})$. Similarly each $x \in X$ maps the space $W^*CC(X^*)$ into $CC(\mathbb{Z})$.

Lemma 1.

(a) *Suppose* $A, B \in WCC(X)$. *Then* $h(x^*(A), x^*(B)) \leq \|x^*\| h(A,B)$ *for each* $x^* \in X^*$.

(b) *Suppose* $A^*, B^* \in W^*CC(X^*)$. *Then* $h(x(A^*), x(B^*)) \leq \|x\| h(A^*, B^*)$ *for each* $x \in X$.

Proof. Let $h(A,B) < r$. Then $A \subset N(B;r)$ and $B \subset N(A;r)$. Hence for each $a \in A$, there exists $b \in B$ such that $\|a - b\| < r$ and consequently, $\|x^*(a) - x^*(b)\| \leq \|x^*\| \|a - b\| \leq \|x^*\| \cdot r$, which in turn implies that $x^*(A) \subset N(x^*(B); \|x^*\| r)$. Similarly, $x^*(B) \subset N(x^*(A); \|x^*\| r)$. Thus $h(x^*(A), x^*(B)) \leq \|x^*\| h(A,B)$, and the proof is complete.

Suppose now $A, B \in WCC(X)$ with $B \not\subset A$, then there exists $b \in B$ but $b \notin A$. It follows from Hahn-Banach theorem that there exists $x^* \in X^*$ and real numbers r_1, r_2 such that Re $x^*(a) < r_1 < r_2 <$ Re $x^*(b)$ for all $a \in A$. Thus $|x^*(b) - x^*(a)| \geq |\text{Re } x^*(b) - \text{Re } x^*(a)| > r_2 - r_1$ for all $a \in A$ and consequently $x^*(b) \notin x^*(A)$ which implies $h(x^*(B), x^*(A)) > 0$ (i.e. $x^*(B) \neq x^*(A)$). The above brief discussion yields the following Lemma 2.

Lemma 2.

(a) $A = B$ *if and only if* $x^*(A) = x^*(B)$ *for each* $x^* \in X^*$*, where* $A, B \in WCC(X)$.

(b) $A^* = B^*$ *if and only if* $x(A^*) = x(B^*)$ *for each* $x \in X$*, where* $A^*, B^* \in W^*CC(X^*)$.

Definitions. Recall that the weak topology τ_w on X is defined to be the weakest topology which makes each $x^* : (X, \tau_w) \to (\mathbb{Z}, |\cdot|)$ continuous. It follows from Lemma 1 that each $x^* : (WCC(X), h) \to (CC(\mathbb{Z}), h)$ is continuous. Thus we may define $\mathcal{T}_w$ to be the weakest topology on the hyperspace $WCC(X)$ such that each $x^* : (WCC(X), \mathcal{T}_w) \to (CC(\mathbb{Z}), h)$ is continuous. Similarly, $\mathcal{T}_w^*$ is defined to be the weakest topology which makes each $x : (W^*CC(X^*), \mathcal{T}_w^*) \to (CC(\mathbb{Z}), h)$ continuous. A typical weak neighborhood ($\mathcal{T}_w$−neighborhood) of $A \in WCC(X)$ is denoted by $\mathcal{W}(A; x_1^*, \ldots, x_n^*; \varepsilon) = \{B \in WCC(X) : h(x_i^*(B), x_i^*(A)) < \varepsilon$ for $i = 1, 2, \ldots, n\}$, and a weak* neighborhood ($\mathcal{T}_w^*$ −neighborhood) of $A^* \in W^*CC(X^*)$ is denoted by $\mathcal{W}^*(A^*; x_1, \ldots, x_n; \varepsilon) = \{B^* \in W^*CC(X^*) : h(x_i(B^*), x_i(A^*)) < \varepsilon$ for $i = 1, 2, \ldots, n\}$. Also for $A, B \in WCC(X)$ and $\alpha \in \mathbb{Z}$, it follows from the continuity of addition and scalar multiplication that $A + B$ and αA belong to $WCC(X)$. Thus a subset $\mathcal{K} \subset WCC(X)$ is defined to be convex if for each $A_1, A_2, \ldots, A_n \in \mathcal{K}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1]$ with $\sum_{i=1}^{n} \alpha_i = 1$, we have $\sum_{i=1}^{n} \alpha_i A_i \in \mathcal{K}$. $\mathcal{K}$ is said to have normal structure ([6], [14]) if for each convex $\mathcal{M} \subset \mathcal{K}$, and $\mathcal{M}$ is not a singleten, then $\mathcal{M}$ has a non-diametral point (i.e., there exists $A \in \mathcal{M}$ such that $\sup\{h(A, B) : B \in \mathcal{M}\} < \text{diam}\mathcal{M} = \sup\{h(A, B) : A, B \in \mathcal{M}\}$.

Let $\overline{X} = \{\overline{x} = \{x\} : x \in X\}$ (i.e. $\overline{X}$ is the hyperspace consisting of singletons). Then $(\overline{X}, h)$ may be identified with $(X, \|\cdot\|)$, and $(\overline{X}, \mathcal{T}_w)$ may be identified with (X, τ_w) naturally. Thus theorems on hyperspaces are extensions of their counterparts on original underlying spaces. We remind our readers that we use small letters to denote elements of the underlying Banach spaces X and X^*; capital letters to denote subsets of X and X^* as well as elements of the hyperspaces $WCC(X)$ and $W^*CC(X^*)$; script letters to denote subsets of hyperspaces. Thus $B[0, r] = \{x \in X | \|x\| \leq r\}$ and $B^*[0, r]$ are closed balls of X and X^*; $\mathcal{B}[0, r] = \{A \in WCC(X) : h(A, \{0\}) \leq r\}$ and $\mathcal{B}^*[0, r]$ are closed balls of $WCC(X)$ and $W^*CC(X^*)$, respectively.

We shall need the following Lemma 3 which has been noted in ([6], [7]) and is easily verifiable.

Lemma 3. *Let* $A, B, C, D \in WCC(X)$ *and* $\alpha \in \mathbb{Z}$*. Then*

(a) $h(\alpha A, \alpha B) = |\alpha| h(A, B)$*, and*

(b) $h(A + B, C + D) \leq h(A, C) + h(B, D)$.

3. Main Results

We shall use the Uniform Boundedness Principle and Hahn-Banach Theorem to establish some fundamental properties of the hyperspaces. We prove that weakly compact subsets of $WCC(X)$ are weakly closed and bounded and weak* compact subsets of $W^*CC(X^*)$ are weak* closed and bounded. Also, we will prove that closed balls $\mathcal{U}[A, \delta]$ and $\mathcal{U}[A^*, \delta]$ are weakly closed and weak*-closed respectively. These properties are essential tools to establish the main theorem of this paper, namely, extension of Brodskii-Milman's theorem to the hyperspaces.

Theorem 1.

(a) *A weakly compact subset* $\mathcal{K} \subset WCC(X)$ *is weakly closed and bounded.*

(b) *A weak***-compact subset* $\mathcal{K}^* \subset W^*CC(X^*)$ *is weak***-closed and bounded.*

Proof. We shall prove only part (b) since the proof of part (a) is essentially the same. Suppose $\mathcal{K}^*$ is weak*-compact. Then $\mathcal{K}^*$ is weak*-closed since the weak*-topology $\mathcal{T}_w^*$ is Hausdorff. Also for each $x \in X$, it follows from the definition that $x : (W^*CC(X^*), \mathcal{T}_w^*) \to (CC(\mathbb{Z}), h)$ is continuous and hence $x(\mathcal{K}^*) = \{x(A^*) : A^* \in \mathcal{K}^*\}$ is a compact subset of the metric space $(CC(\mathbb{Z}), h)$, which implies the existence of some $M_x < \infty$ such that $\sup\{h(x(A^*), x(\{0\})) : A^* \in \mathcal{K}^*\} \leq M_x < \infty$. Note that $h(x(A^*), x(\{0\})) = \sup\{\|x(a^*)\| : a^* \in A^*\}$. Thus if we set $K^* = \bigcup_{A^* \in \mathcal{K}^*} A^* = \bigcup_{A^* \in \mathcal{K}^*} \{a^* : a^* \in A^*\} \subseteq X^*$, we have $\sup\{h(x(A^*), x(\{0\})) : A^* \in \mathcal{K}^*\} = \sup_{A^* \in \mathcal{K}^*} [\sup\{\|x(a^*)\| : a^* \in A^*\}] = \sup\{\|x(a^*)\| : a^* \in K^*\} \leq M_x < \infty$. Consequently, $K^* \subset X^*$ is a collection of linear functionals that is pointwise bounded at each $x \in X$. It follows now from the uniform boundedness principle that K^* is a bounded subset of X^*, i.e., $\sup\{\|a^*\| : a^* \in K^*\} \leq N < \infty$ for some N. now for each $A^* \in \mathcal{K}^*$, we have $h(A^*, \{0\}) = \sup\{\|a^*\| : a^* \in A^*\} \leq N$, since $A^* \subset K^*$. Thus $\mathcal{K}^*$ is a bounded subset of $(W^*CC(X^*), h)$ and the proof is complete.

Theorem 2.

(a) *The closed ball* $\mathcal{U}[A, \delta]$ *of the hyperspace* $(WCC(X), h)$ *is weakly closed* $($*i.e.* $\mathcal{T}_w$*-closed*$)$*,*

(b) *the closed ball* $\mathcal{U}[A^*, \delta]$ *of the hyperspace* $(W^*CC(X^*), h)$ *is weak***-closed* $($*i.e.,* $\mathcal{T}_w^*$*-closed*$)$.

Proof. We shall prove part (b) only since the proof of part (a) is similar. Let $B^* \notin \mathcal{U}[A^*, \delta]$ with $h(A^*, B^*) = \delta + r$ where $r > 0$. Since $h(A^*, B^*) = \max\{\sup_{x^* \in A^*} d(x^*, B^*), \sup_{x^* \in B^*} d(x^*, A^*)\}$, we shall consider the cases $h(A^*, B^*) = \sup_{x^* \in A^*} d(x^*, B^*)$ and $h(A^*, B^*) = \sup_{x^* \in B^*} d(x^*, A^*)$ separately.

Case 1. Suppose $h(A^*, B^*) = \sup_{x^* \in A^*} d(x^*, B^*)$, then there exists $a_0^* \in A^*$ such that $d(a_0^*, B^*) \geq h(A^*, B^*) - \frac{r}{3}$. It follows that for each $b^* \in B^*$, $\|a_0^* - b^*\| \geq d(a_0^*, B^*) \geq h(A^*, B^*) - \frac{r}{3} = (\delta + r) - \frac{r}{3} = \delta + \frac{2r}{3} > \delta + \frac{r}{3}$ and hence $b^* \notin N[a_0^*; \delta + \frac{r}{3}]$ showing that $N[a_0^*; \delta + \frac{r}{3}] \cap B^* = \varnothing$, where both $N[a_0^*; \delta + \frac{r}{3}]$ and B^* are weak*-compact convex sets. It follows now from the Hahn-Banach theorem that there exists $x \in X$ and real numbers r_1, r_2 such that Re $x(b^*) < r_1 < r_2 <$ Re $x(x^*)$ for $b^* \in B^*$ and $x^* \in N[a_0^*; \delta + \frac{r}{3}]$. Let $\varepsilon = \frac{r_2 - r_1}{2}$. Suppose now $A_k^* \in \mathcal{U}[A^*, \delta]$, we have $h(A_k^*, A^*) \leq \delta$ which in turn implies the existence of some $a_k^* \in A_k^*$ with $\|a_k^* - a_0^*\| < \delta + \frac{r}{3}$ or $a_k^* \in N(a_0^*; \delta + \frac{r}{3}) \subset N[a_0^*; \delta + \frac{r}{3}]$. Thus $|x(a_k^*) - x(b^*)| \geq |\text{Re } x(a_k^*) - \text{Re } x(b^*)| > r_2 - r_1 > \varepsilon$ for all $b^* \in B^*$, i.e. $x(a_k^*) \notin N(x(B^*); \varepsilon)$ which in turn implies $x(A_k^*) \not\subset N(x(B^*); \varepsilon)$ and hence $h(x(A_k^*), x(B^*)) \geq \varepsilon$. Thus $A_k^* \notin \mathcal{W}(B^*; x; \varepsilon)$ proving that each $B^* \notin \mathcal{U}[A^*, \delta]$ has a weak*-neighborhood $\mathcal{W}(B^*; x; \varepsilon)$ disjoint from $\mathcal{U}[A^*, \delta]$. Thus the complement of $\mathcal{U}[A^*, \delta]$ is weak*-open and hence $\mathcal{U}[A^*, \delta]$ is weak*-closed.

Case 2. Suppose $h(A^*, B^*) = \sup_{x^* \in B^*} d(x^*, A^*)$. It follows that there exists $b_0^* \in B^*$ such that $d(b_0^*, A^*) \geq h(A^*, B^*) - \frac{r}{3}$. Let $D^* = \bigcup_{a^* \in A^*} N[a^*; \delta + \frac{r}{3}] = A^* + N[0; \delta + \frac{r}{3}]$ where both A^* and $N[0; \delta + \frac{r}{3}]$ are weak*-compact, convex and hence D^* is also weak*-compact, convex. Now for each $x^* \in D^*$,there exists $a^* \in A^*$ with $\|x^* - a^*\| \leq \delta + \frac{r}{3}$. Thus $\|a^* - b_0^*\| \leq \|a^* - x^*\| + \|x^* - b_0^*\|$ which in turn implies that $\|x^* - b_0^*\| \geq \|a^* - b_0^*\| - \|a^* - x^*\| \geq d(b_0^*, A^*) - \|a^* - x^*\| \geq h(A^*, B^*) - \frac{r}{3} - \|a^* - x^*\| \geq (\delta + r) - \frac{r}{3} - (\delta + \frac{r}{3}) = \frac{r}{3}$. Consequently, $d(b_0^*, D^*) \geq \frac{r}{3}$ and we may apply Hahn-Banach theorem to get some $x \in X$ and real numbers r_1, r_2 such that Re $x(x^*) < r_1 < r_2 <$ Re $x(b_0^*)$ for all $x^* \in D^*$, which implies $|x(b_0^*) - x(x^*)| \geq |\text{Re } x(b_0^*) - \text{Re } x(x^*)| > r_2 - r_1 > \frac{r_2 - r_1}{2} = \varepsilon$ for all $x^* \in D^*$. Next, if $A_k^* \in \mathcal{U}[A^*, \delta]$ implies $h(A_k^*, A^*) \leq \delta < \delta + \frac{r}{3}$ and hence $A_k^* \subset N(A^*; \delta + \frac{r}{3}) \subset D^*$. Consequently, $|x(b_0^*) - x(a_k^*)| \geq r_2 - r_1 > \varepsilon$ for each $a_k^* \in A_k^*$, which implies $x(B^*) \not\subset N(x(A_k^*), \varepsilon)$. Thus $h(x(A_k^*), x(B^*)) \geq \varepsilon$ showing that $A_k^* \notin \mathcal{W}(B^*; x; \varepsilon)$. Therefore, the complement of $\mathcal{U}(A^*, \delta]$ is weak*-open and hence $\mathcal{U}[A^*, \delta]$ is weak*-closed and the proof is complete.

Theorem 3.

(a) *Suppose $\mathcal{K}$ is a non-empty, weakly compact (i.e. $\mathcal{T}_w$-compact), convex subset of $WCC(X)$ and $\mathcal{K}$ has normal structure. Then $\mathcal{K}$ contains a point A_0 which is fixed under all isometries of $\mathcal{K}$ onto $\mathcal{K}$.*

(b) *Suppose $\mathcal{K}^*$ is a non-empty, weak*-compact (i.e. $\mathcal{T}_w^*$-compact), convex subset of $W^*CC(X^*)$ and $\mathcal{K}^*$ has normal structure. Then $\mathcal{K}^*$ contains a point A_0^* which is fixed under all isometries of $\mathcal{K}^*$ onto $\mathcal{K}^*$.*

Proof. We shall prove part (a) only. Let $\mathcal{F} = \{T : \mathcal{K} \to \mathcal{K} \mid T$ is a surjective isometry$\}$. Observe that $T \in \mathcal{F}$ implies $T^{-1} \in \mathcal{F}$ since $T : \mathcal{K} \to \mathcal{K}$ is $1-1$, onto. We may now use Zorn's Lemma to obtain a set $\mathcal{K}_0 \subset \mathcal{K}$ which is minimal with respect to being non-empty, weakly compact, convex and invariant under T (i.e., $T(\mathcal{K}_0) \subset \mathcal{K}_0$) for each $T \in \mathcal{F}$. If $\mathcal{K}_0$ consists of a single element, we are done. Otherwise $0 < \text{diam}(\mathcal{K}_0) = d$. Since $\mathcal{K}_0$ is weakly compact, it follows from Theorem 1(a) that diam$(\mathcal{K}_0) = d < \infty$. Since $\mathcal{K}$ has normal structure, it follows that $\mathcal{K}_0$ has a non-diametral point, i.e., there exists $A_0 \in \mathcal{K}_0$ such that $\sup\{h(A_0, A) : A \in \mathcal{K}_0\} = d_1 < d$. Let $\mathcal{K}_1 = \mathcal{K}_0 \cap (\bigcap_{A \in \mathcal{K}_0} \mathcal{U}[A, d_1])$. Since $A_0 \in \mathcal{K}_1$, therefore $\mathcal{K}_1 \neq \varnothing$. $\mathcal{K}_1$ is convex since all sets involved are convex. Also each $\mathcal{U}[A, d_1]$ is weakly closed by Theorem 2. Thus $\mathcal{K}_1$ is weakly closed and hence weakly compact since it is contained in the weakly compact set $\mathcal{K}_0$. Since $T(\mathcal{K}_0) \subset \mathcal{K}_0$ for each $T \in \mathcal{F}$, for any given $B \in \mathcal{K}_0$, we have $T^{-1}(B) \in \mathcal{K}_0$ and $T(T^{-1}(B)) = B$ showing that $T(\mathcal{K}_0) = \mathcal{K}_0$ for each $T \in \mathcal{F}$. Next, we claim that $T(\mathcal{K}_1) \subset \mathcal{K}_1$ for each $\mathcal{K}_1$. To prove our claim, we let $B \in \mathcal{K}_1$ and $T \in \mathcal{F}$ be given, then for any $A \in \mathcal{K}_0$, we have $T^{-1}(A) \in \mathcal{K}_0$ and $h(T(B), A) = h(T(B), T(T^{-1}(A)) = h(B, T^{-1}(A)) \leq d_1$. Consequently, $h(T(B), A) \leq d_1$ for any $A \in \mathcal{K}_0$ and hence $T(B) \in \mathcal{K}_0 \cap \{\bigcap_{A \in \mathcal{K}_0} \mathcal{U}[A, d_1]\} = \mathcal{K}_1$ and the claim is proved. Thus $\mathcal{K}_1$ is a non-empty weakly compact, convex subset of $\mathcal{K}_0$ which is invariant under each $T \in \mathcal{F}$. Moreover, $d_1 < d$ implies that $\mathcal{K}_1 \subsetneqq \mathcal{K}_0$. That is a contradiction to the minimality of $\mathcal{K}_0$ and the theorem is proved.

Suppose X is uniformly convex or $\dim(X) < \infty$. Then it is well-known that X has normal structure ([6], [14]). We shall prove that the hyperspace $CC(X)$ has normal structure if $\dim(X) < \infty$.

Theorem 4. *Suppose* $\dim(X) < \infty$. *Then $CC(X)$ has normal structure.*

Proof. It follows from Blaschke's theorem that every closed and bounded subset $\mathcal{K} \subset (CC(X), h)$ is compact. Also every set $\mathcal{K}$ and its closure has the same diameter. Thus it is sufficient to prove that if $\mathcal{K} \subset CC(X)$ is h-compact and convex with diam$(\mathcal{K}) = d > 0$, then $\mathcal{K}$ has a non-diametral point $A_0 \in \mathcal{K}$ (i.e. $\sup\{h(A_0, A) : A \in \mathcal{K}\} = d_1 < d$). Assume the contrary, then every point of $\mathcal{K}$ is a diametral point. Let $A_1 \in \mathcal{K}$, A_1 is diametral and $\mathcal{K}$ is compact implies the existence of some $A_2 \in \mathcal{K}$ such that $h(A_1, A_2) = d$. By convexity of $\mathcal{K}$, $(A_1 + A_2)/2 \in \mathcal{K}$ and $(A_1 + A_2)/2$ is diametral. Let $A_3 \in \mathcal{K}$ be such that $h((A_1 + A_2)/2, A_3) = d$. Since $d = h((A_1+A_2)/2, A_3) = h((A_1+A_2)/2, (A_3+A_3)/2) \leq \frac{1}{2}h(A_1, A_3)+\frac{1}{2}h(A_2, A_3) \leq \frac{1}{2}d+\frac{1}{2}d = d$. It follows that $h(A_1, A_3) = h(A_2, A_3) = d$. Inductively, if $A_1, A_2, \ldots, A_n \in \mathcal{K}$ has been chosen such that $h(A_i, A_j) = d$ where $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. Then $(A_1+A_2+\cdots+A_n)/n \in \mathcal{K}$ is diametral implies the existence of $A_{n+1} \in \mathcal{K}$ such that $h(A_i, A_{n+1}) = d$ for $i = 1, 2, \ldots, n$. Consequently $\{A_i\}$ is an infinite sequence of $\mathcal{K}$ such that $h(A_i, A_j) = d$ for $i \neq j$. Thus $\{A_i\}$ is an infinite sequence that has no convergent subsequence. That is a contradiction to the compactness of $\mathcal{K}$, and hence the theorem is proved.

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Thakyin Hu
Department of Mathematics,
Tamkang University,
Tamsui, Taipei,
Taiwan 25137, R.O.C.

Jui-Chi Huang
Center for General Education,
Technology and Science Institute of Northern Taiwan,
No. 2, Xue Yuan Road, Peito,
Taipei 112, Taiwan, R.O.C.
E-mail: juichi@tsint.edu.tw
juichi.h0207@msa.hinet.net