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AN APPLICATION OF DIFFERENTIAL SUBORDINATIONS TO THE CLASS OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. For functions belonging to the generalization class $\mathcal{R}_p(k,\alpha,\lambda)$ of analytic functions in the open unit disc $\mathbb{U}=\{z: |z|<1\}$, which investigated in this paper, some applications of differential subordination are obtained which contain interesting property of Hurwitz-Lerch Zeta function.

1. Introduction and Definitions

Let $\mathcal{A}_p(k)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{m=p+k}^{\infty} a_m z^m \qquad (p,k \in \mathbb{N}=\{1,2,...\}) \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U}=\{z: |z|<1\}$.

Also, let $\mu_a(k)$ denote the class of analytic functions in $\mathbb{U}$ of the form

$$q(z) = a + \sum_{m=k}^{\infty} a_m z^m \qquad (z \in \mathbb{U}), \tag{1.2}$$

for some $a \in \mathbb{C}$ ($\mathbb{C}$ is the complex plane).

A function $f(z)$ in the class $\mathcal{A}_p(k)$ is said to be in the class $\mathcal{R}_p(k,\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{\alpha}{p} \qquad (z \in \mathbb{U}), \tag{1.3}$$

for some $\alpha\,(0 \le \alpha < p)$. The class $\mathcal{R}_p(1,0)=\mathcal{R}_p$, was introduced by Saitoh and Nunokawa [16] , Saitoh *et al.* [17] and Fukui *et al.* [7]. Also, the classes

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$\mathcal{R}_1(1,\alpha) = \mathcal{C}(\alpha)$, and $\mathcal{R}_1(1,0) = \mathcal{C}(0)$, were studied by Èzrohi [5] and MacGregor [13], respectively.

Further; a function $f(z)$ in the class $\mathcal{A}_p(k)$ is said to be in the class $\mathcal{P}_p(k,\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \qquad (z \in \mathbb{U}) , \tag{1.4}$$

for some $\alpha\,(0 \le \alpha < p)$. The classes $\mathcal{P}_p(1,\alpha) = \mathcal{P}(p,\alpha)$, and $\mathcal{P}_p(1,0) = \mathcal{P}_p$, were previously studied by Cho [3] and Umezawa [20], respectively. Also $\mathcal{P}_1(1,\alpha) = \mathcal{B}(\alpha)$, was introduced by Chen ([1], [2]) and Goel [8]. Moreover $\mathcal{P}_1(1,0) = \mathcal{B}(0)$, was studied by Goel [9] and Yamaguchi [21].

Now we define:

Definition 1.1. Suppose that $f(z) \in \mathcal{A}_p(k)$. Then the function $f(z)$ is said to be a member of the class $\mathcal{R}_p(k,\alpha,\lambda)$ if it satisfies

$$\operatorname{Re}\left\{(1-\lambda)\frac{f(z)}{z^p} + \lambda\,\frac{f'(z)}{p\,z^{p-1}}\right\} > \frac{\alpha}{p} \qquad (z \in \mathbb{U}) , \tag{1.5}$$

for some $\alpha\,(0 \le \alpha < p)$ and $\lambda \ge 0$.

We note that, $\mathcal{R}_p(k,\alpha,0) = R_p(k,\alpha)$, and $\mathcal{R}_p(k,\alpha,1) = \mathcal{P}_p(k,\alpha)$.
We shall also need the following definitions:

Definition 1.2. Let $f(z)$ and $F(z)$ be analytic functions. The function $f(z)$ is said to be ***subordinate*** to $F(z)$, written $f(z) \prec F(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| \le 1$, and such that $f(z) = F(w(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ ***if and only if*** $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Definition 1.3. Let $\Psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let $h(z)$ be univalent in $\mathbb{U}$. If $q(z) \in \mu_a(k)$ satisfies the differential subordination:

$$\Psi(\,q(z),\, z\,q'(z), z^2\,q''(z)\,;\, z) \prec\ h(z) \quad (z \in \mathbb{U}) , \tag{1.6}$$

then $q(z)$ will be called $(a,k)-solution$. The univalent function $s(z)$ is called $(a,k)-domainant$, if $q(z) \prec s(z)$ for all $q(z)$ satisfying (1.6) , $(a,k)-domainant\ \bar{s}(z) \prec s(z)$ for all $(a,k)-domainant\ s(z)$ of (1.6) is said to be $the\,best\ (a,k)-domainant$ of (1.6).

In this paper, an interesting property of functions in the class $\mathcal{R}_p(k,\alpha,\lambda)$ using the technique of differential subordination is obtained. An application of Hurwitz-Lerch Zeta function is also discussed.

2. Differential Subordination with $R_p(k,\alpha,\lambda)$

We require the following theorem due to Hallenbeck and Ruscheweyh [10] (see also [14])

Theorem 2.1. *Let* $h(z)$ *be convex in* $\mathbb{U}$ *, with* $h(0)=a,\ \gamma\neq 0$ *and* $\mathrm{Re}(\gamma)\geq 0$. *If* $q(z)\in\ \mu_a(k)$ *and*

$$q(z)+\frac{z\,q^{'}(z)}{\gamma}\ \prec\ h(z), \tag{2.1}$$

then

$$q(z)\prec S(z)\prec h(z),$$

where

$$S(z)=\frac{\gamma}{k\,z^{\frac{\gamma}{k}}}\int_0^z h(t)\,t^{\frac{\gamma}{k}-1}\,dt\ . \tag{2.2}$$

The function $S(z)$ *is convex and is* $the\ best\ (a,k)-domainint.$

Now, we prove the following lemma:

Theorem 2.2. *Let the function* $f(z)$ *defined by (1.1) be in the class* $\mathcal{R}_p(k,\alpha,\lambda)$ $(\lambda\geq 0)$, *then*

$$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\}>\left(\frac{2\alpha}{p}-1\right)+\frac{2(p-\alpha)}{p}\int_0^1\frac{dt}{1+t^{\frac{\lambda k}{p}}}\qquad(z\ \in\mathbb{U}), \tag{2.3}$$

for some $\alpha\,(\,0\leq\alpha<p)$.

The constant $\left(\frac{2\alpha}{p}-1\right)+\frac{2(p-\alpha)}{p}\int_0^1\frac{dt}{1+t^{\frac{\lambda k}{p}}}$ *is the best estimate.*

Proof. Defining the function $q(z)=\dfrac{f(z)}{z^p}$, we have $q(z)\in\ \mu_1(k)$.

If we take $\gamma=\dfrac{p}{\lambda}$, and the convex function $h(z)$ defined by

$$h(z)=\frac{1+\left(\frac{2\alpha}{p}-1\right)z}{1+z},\quad 0\leq\alpha<p\ , \tag{2.4}$$

then, we have

$$q(z)+\frac{z\,q^{'}(z)}{\gamma}\ =(1-\lambda)\frac{f(z)}{z^p}+\lambda\,\frac{f^{'}(z)}{p\,z^{p-1}}. \tag{2.5}$$

Since $f(z) \in R_p(k,\alpha,\lambda),$ we see that

$$q(z)+\frac{z\,q^{'}(z)}{\gamma}\prec h(z), \tag{2.6}$$

where $h(z)$ is defined by (2.4) with $h(0)=1.$

Applying Theorem 2.1, we obtain that $\frac{f(z)}{z^p}\prec S(z)$, where the convex function $S(z)$ defined by

$$S(z)=\frac{p}{\lambda k\; z^{\frac{p}{\lambda k}}}\int_0^z \frac{1+\left(\frac{2\alpha}{p}-1\right)t}{1+t}\; t^{\left(\frac{p}{\lambda k}-1\right)}\,dt. \tag{2.7}$$

Noting that $\mathrm{Re}\,\{h(z)\}>0$ and $S(z)\prec h(z),$ we have $\mathrm{Re}\,\{S(z)\}>0.$

This implies that

$$\begin{aligned}\inf_{z\in\mathbb{U}}\,\mathrm{Re}\,\{S(z)\}&=S(1)=\left(\frac{2\alpha}{p}-1\right)+\frac{2p}{\lambda k}\left(1-\frac{\alpha}{p}\right)\int_0^1\frac{u^{\left(\frac{p}{\lambda k}-1\right)}}{1+u}\,du\\&=\left(\frac{2\alpha}{p}-1\right)+\frac{2(p-\alpha)}{p}\int_0^1\frac{dt}{1+t^{\frac{\lambda k}{p}}}\qquad(z\in\mathbb{U}).\end{aligned} \tag{2.8}$$

Hence, the constant $\left(\frac{2\alpha}{p}-1\right)+\frac{2(p-\alpha)}{p}\int_0^1\frac{dt}{1+t^{\frac{\lambda k}{p}}}$ can't be replace by any larger one.

This completes the proof of Theorem 2.2.

Remark 2.1. Putting $p=k=\lambda=1,$ in Theorem 3.2, we have the results due to Saitoh [15].

3. An Application for Hurwitz-lerch Zeta Function

We can show that, for $0\le\alpha<p$, and for $z\in\mathbb{U}$, the function

$$f(z)=z^p\left\{\frac{2\alpha-p}{p}+\frac{2(p-\alpha)}{\lambda}\,\Phi(z,1,\frac{p}{\lambda})\right\} \tag{3.1}$$

is a member of the class $\mathcal{R}_p(1,\alpha,\lambda)$ $(\lambda>0)$, where $\Phi(z,s,b)$ is the *Hurwitz- Lerch Zeta function*, defined by (cf., eg., [18, p.121 *et seq.*]),

$$\Phi(z,s,b)=\sum_{k=0}^{\infty}\frac{z^k}{(k+b)^s}\,, \tag{3.2}$$

$(b \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \mathbb{Z}_0^- = \{0, -1, -2, \dots\},\ s \in \mathbb{C}$ when , $z \in \mathbb{U},\ \mathrm{Re}(s) > 1$ when $|z| = 1$). Recently, several properties of $\Phi(z, s, b)$ have been studied by Choi and Srivastava [4], Ferreira and López [6], Lin, Srivastava and Wang [11], Luo and Srivastava [12], and others.

Theorem 3.1. *Let* $\Phi(z, s, b)$ *is the Hurwitz-Lerch Zeta function, defined by* (3.2)*, then*

$$\mathrm{Re}\left\{\Phi\left(z, 1, m\right)\right\} > \Phi\left(-1, 1, m\right), \quad (|z| < 1;\ m > 0), \tag{3.3}$$

the constant $\Phi\left(-1, 1, m\right)$ *can't be replace by any larger one.*

Proof. Using Theorem 2.2, for the function $f(z)$ given by (3.1), we have

$$\mathrm{Re}\left\{\Phi\left(z, 1, \frac{p}{\lambda}\right)\right\} > \frac{\lambda}{p}\int_0^1 \frac{dt}{1 + t^{\frac{\lambda}{p}}}, \quad (|z| < 1;\ p \geq 1), \tag{3.4}$$

which is equivalent to

$$\mathrm{Re}\left\{\Phi\left(z, 1, m\right)\right\} > \Phi\left(-1, 1, m\right), \quad (|z| < 1;\ m > 0), \tag{3.4}$$

the constant $\Phi\left(-1, 1, m\right)$ can't be replace by any larger one.

Moreover, letting $\lambda = 2p$ in (3.4), we readily obtain the following *Hurwitz-Lerch Zeta* property:

Corollary 3.1. *Let* $\Phi(z, s, b)$ *is the Hurwitz- Lerch Zeta function, defined by (3.2), then*

$$\mathrm{Re}\left\{\Phi\left(z, 1, \frac{1}{2}\right)\right\} > \frac{\pi}{2} \quad \left(|z| < 1\right), \tag{3.5}$$

the constant $\frac{\pi}{2}$ *can't be replace by any larger one.*

Remark 3.1. More recently, Srivastava and Attiya [18] have shown some interesting results of an integral operator with the Hurwitz-Lerch Zeta functions.

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