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ON A WEIGHTED AND EXPONENTIAL GENERALIZATION OF RADO'S INEQUALITY

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Abstract. In this paper, a weighted and exponential generalization of Rado's inequality is established. As applications, the result is used to obtain a refinement of weighted power means inequality.

1. Introduction

The following inequality is known in the literature as Rado's inequality (see [1, p. 94]):

$$n(A_n(a) - G_n(a)) \geq (n-1)(A_{n-1}(a) - G_{n-1}(a)), \tag{1}$$

where $a_i > 0$ $(i = 1, 2, \ldots, n)$, $A_n(a) = (a_1 + a_2 + \cdots + a_n)/n$, $G_n(a) = \sqrt[n]{a_1 a_2 \cdots a_n}$.

Rado's inequality is one of the most important inequalities for means, it is well-known that this classical inequality can be applied to studying numerous inequalities related to arithmetic means and geometric means. Over the last decade Rado's inequality has received considerable attention from many researchers and has motivated a large number of research papers giving their simple proofs, various generalizations, improvements and analogues (see [1-7] and references therein).

Recently, it comes to our attention that an interesting generalization of Rado's inequality, which was proved by Bullen [8] (see also [9]), as follows

$$\begin{aligned} &\left(\sum_{i=1}^{n} \lambda_i\right)\left(\left(M_n^{[s]}(a,\lambda)\right)^r - \left(M_n^{[t]}(a,\lambda)\right)^r \right) \\ &\geq \left(\sum_{i=1}^{n-1} \lambda_i\right)\left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^r - \left(M_{n-1}^{[t]}(a,\lambda)\right)^r \right). \end{aligned} \tag{2}$$

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where $a_i > 0,\ \lambda_i > 0\ (i = 1, 2, \ldots, n),\ rst \neq 0,\ t/r \leq 1$ and $s/r \geq 1.$

In this paper, we shall establish an analogue of Bullen's inequality, which also involves weights and exponents, from which a number of Rado-type inequalities can be obtained by assigning appropriate values to the parameters. Finally, we provide an application of obtained result to the improvement of weighted power means inequality.

In order to prove the main result in Section 2, we need the following lemmas.

Lemma 1. (Weighted power means inequality [1, p. 76]). *If* $a_i > 0,\ \lambda_i > 0\ (i = 1, 2, \ldots, n),\ s \geq t,\ st \neq 0.$ *Then*

$$\left(\left(\sum_{i=1}^{n}\lambda_i a_i^s\right)\Big/\left(\sum_{i=1}^{n}\lambda_i\right)\right)^{1/s} \geq \left(\left(\sum_{i=1}^{n}\lambda_i a_i^t\right)\Big/\left(\sum_{i=1}^{n}\lambda_i\right)\right)^{1/t}. \tag{3}$$

Lemma 2. *If* $p \geq 1,\ a \geq b \geq c > 0,$ *then*

$$a^p - b^p + c^p \geq (a - b + c)^p. \tag{4}$$

If $p \geq 1,\ a \geq b > 0,\ c \geq b > 0,$ *then*

$$a^p - b^p + c^p \leq (a - b + c)^p. \tag{5}$$

The Lemma 2 will be used for improving Rado's inequality as a main tool. We show here two different proofs of Lemma 2: (I) Proof by direct method, (II) Proof by Jensen-Steffensen's inequality. These proofs are interesting, especially, the Proof II, an artful proof which is motivated by Professor Bullen's ideas, seems to be more simple.

Proof of Lemma 2 (I). Direct calculation gives

$$\begin{aligned}\frac{a^p - b^p + c^p}{(a-b+c)^p} &= \left(\frac{a}{a-b+c}\right)^p - \left(\frac{b}{a-b+c}\right)^p + \left(\frac{c}{a-b+c}\right)^p \\ &= \left(\frac{a}{a-b+c}\right)^p - \left(\frac{a}{a-b+c} + \frac{c}{a-b+c} - 1\right)^p + \left(\frac{c}{a-b+c}\right)^p \\ &:= x^p + y^p - (x + y - 1)^p,\end{aligned}$$

where $x = c/(a - b + c),\ y = a/(a - b + c).$

Define a function $f : (0, 1] \longrightarrow \mathbb{R}$ *by* $f(x) = x^p + y^p - (x + y - 1)^p$ *(where* p *and* y *are considered as the parameters).*

Case (I). *When* $p \geq 1,\ a \geq b \geq c > 0.$ *It implies that* $0 < x \leq 1,\ y \geq 1.$

Differentiating with respect to x*, we find*

$$f'(x) = px^{p-1} - p(x+y-1)^{p-1} \leq 0 \quad \text{for} \quad p \geq 1,\ 0 < x \leq 1,\ y \geq 1.$$

This means that $f(x)$ *is decreasing on* $(0,1]$*, we hence conclude that* $f(x) \geq f(1) = 1$ *for all* $x \in (0,1]$*,* $y \in [1,+\infty)$ *and* $p \in [1,+\infty)$*.*

Consequently, we have

$$\frac{a^p - b^p + c^p}{(a-b+c)^p} \; = \; x^p + y^p - (x+y-1)^p \; \geq \; 1,$$

which leads to inequality (4).

Case (II). *When* $p \geq 1$*,* $a \geq b > 0$*,* $c \geq b > 0$*. It implies that* $0 < x \leq 1$*,* $0 < y \leq 1$*.*

It is easy to verify that

$$f'(x) = px^{p-1} - p(x+y-1)^{p-1} \geq 0 \quad \text{for} \quad p \geq 1,\ 0 < x \leq 1,\ 0 < y \leq 1.$$

We can now assert $f(x) \leq f(1) = 1$*, since* $f(x)$ *is increasing on* $(0,1]$*. Further, we conclude that*

$$\frac{a^p - b^p + c^p}{(a-b+c)^p} \; = \; x^p + y^p - (x+y-1)^p \; \leq \; 1,$$

which yields the inequality (5). This completes the proof of Lemma 2.

Before the beginning of the Proof II, we shall introduce the Jensen-Steffensen's inequality (see [10,p.57]), as follows:

Proposition 1. *If* $f : I \to \mathbb{R}$ *is a convex function,* $x = (x_1, x_2, \ldots, x_n)$ *is real monotone* n*-tuple such that* $x_i \in I$ $(i = 1, 2, \ldots, n)$*, and* $p = (p_1, p_2, \ldots, p_n)$ *is a real* n*-tuple such that*

$$0 \leq P_k \leq P_n \quad (1 \leq k \leq n), \quad P_n > 0, \quad where \quad P_k = \sum_{i=1}^{k} p_i \, .$$

Then, the following inequality holds true:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \; \leq \; \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{6}$$

Let us now prove the Lemma 2.

Proof of Lemma 2 (II). Define a function:

$$f : (0,+\infty) \longrightarrow \mathbb{R} \quad \text{by} \quad f(x) = x^p \quad (p \geq 1).$$

It is easy to verify that f is a convex function on $(0, +\infty)$.

Case (I). *When $a \geq b \geq c > 0$. It implies that the combination $x = (a, b, c)$ is in decreasing order and $a, b, c \in (0, +\infty)$.*

Consider a 3-tuple $p = (1, -1, 1)$, we find

$$P_1 = 1, \quad P_2 = 1 + (-1) = 0, \quad P_3 = 1 + (-1) + 1 = 1,$$

and

$$0 \leq P_k \leq P_3 \quad (1 \leq k \leq 3), \quad P_3 > 0 \ .$$

Using Jensen-Steffensen's inequality gives

$$[1 \cdot a + (-1) \cdot b + 1 \cdot c]^p \ \leq \ 1 \cdot a^p + (-1) \cdot b^p + 1 \cdot c^p \ .$$

Inequality (4) is proved.

Case (II). *When $a \geq b > 0$ and $c \geq b > 0$. It implies that $a + c - b \geq a \geq b > 0$, we hence conclude that the combination $x = (a + c - b, \ a, \ b)$ is in decreasing order and $a + c - b, \ a, \ b \in (0, +\infty)$.*

Consider the same 3-tuple as above: $p = (1, -1, 1)$, it has been shown that

$$P_1 = 1, \quad P_2 = 0, \quad P_3 = 1, \quad 0 \leq P_k \leq P_3 \quad (1 \leq k \leq 3), \quad P_3 > 0 \ .$$

By applying Jensen-Steffensen's inequality, we obtain

$$[1 \cdot (a + c - b) + (-1) \cdot a + 1 \cdot b]^p \ \leq \ 1 \cdot (a + c - b)^p + (-1) \cdot a^p + 1 \cdot b^p \ ,$$

this is

$$a^p - b^p + c^p \ \leq \ (a - b + c)^p.$$

Inequality (5) is proved.

2. Main Results

As in [1-2] the following means for positive numbers $a_1, a_2, \ldots, a_n$ and positive weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ are defined by

$$M_n^{[s]}(a, \lambda) \ = \ \left(\left(\sum_{i=1}^{n} \lambda_i a_i^s \right) \Big/ \left(\sum_{i=1}^{n} \lambda_i \right) \right)^{1/s}, \ s \neq 0, \qquad \text{weighted power means,}$$

$$A_n(a,\lambda) = \left(\sum_{i=1}^{n} \lambda_i a_i\right) \Big/ \left(\sum_{i=1}^{n} \lambda_i\right), \qquad \text{weighted arithmetic means,}$$

$$H_n(a,\lambda) = \left(\sum_{i=1}^{n} \lambda_i\right) \Big/ \left(\sum_{i=1}^{n} \lambda_i a_i^{-1}\right), \qquad \text{weighted harmonic means,}$$

$$G_n(a,\lambda) = \left(\prod_{i=1}^{n} a_i^{\lambda_i}\right)^{1/\sum_{i=1}^{n} \lambda_i}, \qquad \text{extended geometric means.}$$

Our main result is stated in the following theorem.

Theorem 1. *Let* $a_i > 0,\ \lambda_i > 0\ (i = 1, 2, \ldots, n),\ st \neq 0.$ *Then for* $t/s \leq 1$ *and* $r/s \geq 1,$ *we have the inequality*

$$\begin{aligned} &\left(\sum_{i=1}^{n} \lambda_i\right)^{r/s} \left(\left(M_n^{[s]}(a,\lambda)\right)^r - \left(M_n^{[t]}(a,\lambda)\right)^r \right) \\ \geq &\left(\sum_{i=1}^{n-1} \lambda_i\right)^{r/s} \left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^r - \left(M_{n-1}^{[t]}(a,\lambda)\right)^r \right). \end{aligned} \tag{7}$$

Inequality (7) *is reversed for* $t/s \geq 1$ *and* $r/s \geq 1.$

Proof.

Case (I). When $t/s \leq 1$ and $r/s \geq 1.$

By Lemma 1 and the assumption $t/s \leq 1$, we conclude

$$\begin{aligned} &\left(\sum_{i=1}^{n} \lambda_i\right) \left(M_n^{[s]}(a,\lambda)\right)^s > \left(\sum_{i=1}^{n-1} \lambda_i\right) \left(M_{n-1}^{[s]}(a,\lambda)\right)^s \\ \geq &\left(\sum_{i=1}^{n-1} \lambda_i\right) \left(M_{n-1}^{[t]}(a,\lambda)\right)^s > 0. \end{aligned}$$

Based on the above inequalities and $r/s \geq 1,$ it follows from Lemma 2 that

$$\begin{aligned} &\left(\sum_{i=1}^{n} \lambda_i\right)^{r/s} \left(M_n^{[s]}(a,\lambda)\right)^r \\ &- \left(\sum_{i=1}^{n-1} \lambda_i\right)^{r/s} \left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^r - \left(M_{n-1}^{[t]}(a,\lambda)\right)^r\right) \end{aligned} \tag{8}$$

$$= \left(\sum_{i=1}^{n}\lambda_i\right)^{r/s}\left(\left(M_n^{[s]}(a,\lambda)\right)^s\right)^{r/s} - \left(\sum_{i=1}^{n-1}\lambda_i\right)^{r/s}\left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^s\right)^{r/s}$$
$$+ \left(\sum_{i=1}^{n-1}\lambda_i\right)^{r/s}\left(\left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right)^{r/s}$$
$$\geq \left[\left(\sum_{i=1}^{n}\lambda_i\right)\left(M_n^{[s]}(a,\lambda)\right)^s - \left(\sum_{i=1}^{n-1}\lambda_i\right)\left(M_{n-1}^{[s]}(a,\lambda)\right)^s\right.$$
$$\left. + \left(\sum_{i=1}^{n-1}\lambda_i\right)\left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right]^{r/s}$$
$$= \left(\sum_{i=1}^{n}\lambda_i\right)^{r/s}\left[\left(\lambda_n a_n^s + \left(\sum_{i=1}^{n-1}\lambda_i\right)\left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right)\Big/\left(\sum_{i=1}^{n}\lambda_i\right)\right]^{r/s}.$$

Using Lemma 1 with $t/s \leq 1$ gives

$$\left(\lambda_n a_n^s + \left(\sum_{i=1}^{n-1}\lambda_i\right)\left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right)\Big/\left(\sum_{i=1}^{n}\lambda_i\right)$$
$$\geq \left[\left(\lambda_n\left(a_n^s\right)^{t/s} + \left(\sum_{i=1}^{n-1}\lambda_i\right)\left(\left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right)^{t/s}\right)\Big/\left(\sum_{i=1}^{n}\lambda_i\right)\right]^{s/t}$$
$$= \left(M_n^{[t]}(a,\lambda)\right)^s.$$

Combining inequality (8) and the above inequality yields

$$\left(\sum_{i=1}^{n}\lambda_i\right)^{r/s}\left(M_n^{[s]}(a,\lambda)\right)^r - \left(\sum_{i=1}^{n-1}\lambda_i\right)^{r/s}\left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^r - \left(M_{n-1}^{[t]}(a,\lambda)\right)^r\right)$$
$$\geq \left(\sum_{i=1}^{n}\lambda_i\right)^{r/s}\left(M_n^{[t]}(a,\lambda)\right)^r,$$

which leads to the desired inequality (7).

Case (II). When $t/s \geq 1$ and $r/s \geq 1$.
From Lemma 1 and the assumption $t/s \geq 1$, we find

$$\left(\sum_{i=1}^{n}\lambda_i\right)\left(M_n^{[s]}(a,\lambda)\right)^s > \left(\sum_{i=1}^{n-1}\lambda_i\right)\left(M_{n-1}^{[s]}(a,\lambda)\right)^s > 0,$$

and

$$\left(\sum_{i=1}^{n-1}\lambda_i\right)\left(M_{n-1}^{[t]}(a,\lambda)\right)^s \geq \left(\sum_{i=1}^{n-1}\lambda_i\right)\left(M_{n-1}^{[s]}(a,\lambda)\right)^s > 0.$$

Similarly to the proof in Case (I), it follows respectively from Lemma 2 and Lemma 1 that

$$\left(\sum_{i=1}^{n}\lambda_i\right)^{r/s}\left(M_n^{[s]}(a,\lambda)\right)^r - \left(\sum_{i=1}^{n-1}\lambda_i\right)^{r/s}\left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^r - \left(M_{n-1}^{[t]}(a,\lambda)\right)^r\right)$$

$$\leq \left(\sum_{i=1}^{n}\lambda_i\right)^{r/s}\left[\left(\lambda_n a_n^s + \left(\sum_{i=1}^{n-1}\lambda_i\right)\left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right)\Big/\left(\sum_{i=1}^{n}\lambda_i\right)\right]^{r/s},$$

and

$$\left(\lambda_n a_n^s + \left(\sum_{i=1}^{n-1}\lambda_i\right)\left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right)\Big/\left(\sum_{i=1}^{n}\lambda_i\right) \leq \left(M_n^{[t]}(a,\lambda)\right)^s.$$

Combining the above inequalities gives

$$\left(\sum_{i=1}^{n}\lambda_i\right)^{r/s}\left(M_n^{[s]}(a,\lambda)\right)^r - \left(\sum_{i=1}^{n-1}\lambda_i\right)^{r/s}\left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^r - \left(M_{n-1}^{[t]}(a,\lambda)\right)^r\right)$$
$$\leq \left(\sum_{i=1}^{n}\lambda_i\right)^{r/s}\left(M_n^{[t]}(a,\lambda)\right)^r,$$

which leads to the reverse inequality of (7). The proof of Theorem 1 is complete.

Remark 1. It is worth noticing that inequality (7) and Bullen's inequality (2) do not imply each other, because inequality (7) and Bullen's inequality hold under different assumption conditions. For example, in Theorem 1, if $t = 1$, $s = -2$, $r = -3$, then $t/s < 1$ and $r/s > 1$. Under the same assumptions, it implies that $t/r < 1$ and $s/r < 1$, which is not suitable for the conditions of Bullen's inequality, hence Theorem 1 here is unable to be deduced from Bullen's inequality.

We give here some consequences from Theorem 1.

Putting $r = s$ in Theorem 1, we obtain

Corollary 1. *Let* $a_i > 0,\ \lambda_i > 0\ (i = 1, 2, \ldots, n),\ st \neq 0.$ *Then for* $t/s \leq 1$

we have the inequality

$$\left(\sum_{i=1}^{n}\lambda_i\right)\left(\left(M_n^{[s]}(a,\lambda)\right)^s-\left(M_n^{[t]}(a,\lambda)\right)^s\right)$$
$$\geq\left(\sum_{i=1}^{n-1}\lambda_i\right)\left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^s-\left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right). \tag{9}$$

Inequality (9) *is reversed for* $t/s \geq 1$.

In Corollary 1, putting $s=1$ gives

Corollary 2. *Let* $a_i>0,\ \lambda_i>0\ (i=1,2,\ldots,n),\ t\neq 0$. *Then for* $t\leq 1$ *we have the inequality*

$$\left(\sum_{i=1}^{n}\lambda_i\right)\left(A_n(a,\lambda)-M_n^{[t]}(a,\lambda)\right)\geq\left(\sum_{i=1}^{n-1}\lambda_i\right)\left(A_{n-1}(a,\lambda)-M_{n-1}^{[t]}(a,\lambda)\right). \tag{10}$$

Inequality (10) *is reversed for* $t\geq 1$.

Putting $t\to 0$ and $t=-1$ respectively in (10), and using the known result $\lim_{t\to 0} M_n^{[t]}(a,\lambda)=G_n(a,\lambda)$ (see [1, p. 74]), we get

Corollary 3. *Let* $a_i>0,\ \lambda_i>0\ (i=1,2,\ldots,n)$. *Then*

$$\left(\sum_{i=1}^{n}\lambda_i\right)(A_n(a,\lambda)-G_n(a,\lambda))\geq\left(\sum_{i=1}^{n-1}\lambda_i\right)(A_{n-1}(a,\lambda)-G_{n-1}(a,\lambda)), \tag{11}$$

$$\left(\sum_{i=1}^{n}\lambda_i\right)(A_n(a,\lambda)-H_n(a,\lambda))\geq\left(\sum_{i=1}^{n-1}\lambda_i\right)(A_{n-1}(a,\lambda)-H_{n-1}(a,\lambda)). \tag{12}$$

Remark 2. Clearly, Rado's inequality follows from inequality (11) with $\lambda_1=\lambda_2=\ldots=\lambda_n=1$. Corollary 1, Corollary 2 and Corollary 3 show that the inequality in Theorem 1 is a very general result.

3. Application to the Refinement of Weighted Power Means Inequality

The weighted power means inequality in Lemma 1 can be rewritten in a simplified form as

$$M_n^{[s]}(a,\lambda)\geq M_n^{[t]}(a,\lambda), \tag{13}$$

where $a_i > 0,\ \lambda_i > 0\ (i = 1, 2, \ldots, n),\ s \geq t,\ st \neq 0$.

It is well known that inequality (13) has important applications in many areas of pure and applied mathematics. This classical inequality was formulated without proof by J. Bienaymé [11] in 1840, and was first proved by D. Besso [12] in 1879. A shorter proof was given by D. S. Mitrinović and P. M. Vasić in [1].

As application of the foregoing results, we show here a refinement of the weighted power means inequality.

Theorem 2. *Let* $a_i > 0,\ \lambda_i > 0\ (i = 1, 2, \ldots, n),\ s \geq t,\ st \neq 0$. *Then*

$$\begin{aligned} & M_n^{[s]}(a,\lambda) \\ (14)\quad & \geq \left[\left(M_n^{[t]}(a,\lambda)\right)^s + \left(\sum_{i=1}^{n-1}\lambda_i \Big/ \sum_{i=1}^{n}\lambda_i\right)\left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^s - \left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right)\right]^{1/s} \\ & \geq M_n^{[t]}(a,\lambda). \end{aligned}$$

Proof. Applying Corollary 1 gives

$$\begin{aligned} & \left(M_n^{[s]}(a,\lambda)\right)^s \\ & \geq \left(M_n^{[t]}(a,\lambda)\right)^s + \left(\sum_{i=1}^{n-1}\lambda_i \Big/ \sum_{i=1}^{n}\lambda_i\right)\left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^s - \left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right) \\ & \geq \left(M_n^{[t]}(a,\lambda)\right)^s \end{aligned}$$

for $s \geq t,\ s > 0$.

$$\begin{aligned} & \left(M_n^{[s]}(a,\lambda)\right)^s \\ & \leq \left(M_n^{[t]}(a,\lambda)\right)^s + \left(\sum_{i=1}^{n-1}\lambda_i \Big/ \sum_{i=1}^{n}\lambda_i\right)\left(\left(M_{n-1}^{[s]}(a,\lambda)\right)^s - \left(M_{n-1}^{[t]}(a,\lambda)\right)^s\right) \\ & \leq \left(M_n^{[t]}(a,\lambda)\right)^s \end{aligned}$$

for $s \geq t,\ s < 0$.

Inequality (14) follows immediately from the above inequalities. Theorem 2 is proved.

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