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EXPLICIT BOUNDS ON SOME NEW NONLINEAR RETARDED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper, some new explicit bounds on solutions to a class of new nonlinear retarded integral inequalities are established, which can be used as effective tools in the study of certain nonlinear retarded integral equations. Applications examples are also indicated.

1. Introduction

It is well known that integral inequalities play a fundamental role in the theory of differential and integral equations. Among various types of integral inequalities, the Gronwall-Bellman type is particularly useful in that they provide explicit bounds for the unknown functions(see e.g. [1-3]). A specific branch of this type integral inequalities is originated by Ou-Iang. In his study of boundedness of solutions to linear second order differential equations, Ou-Iang [19] established and used the following nonlinear integral inequality which is now known as Ou-Iang's inequality in the literature.

Theorem A. ([19]). *Let u and f be real-valued, nonnegative, and continuous functions defined on $R_+ = [0, +\infty)$ and let $c \geq 0$ be a real constant. Then the nonlinear integral inequality*

$$u^2(t) \leq c^2 + 2\int_0^t f(s)u(s)ds, \qquad t \in R_+$$

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implies

$$u(t) \leq c + \int_0^t f(s)ds, \qquad t \in R_+.$$

While Ou-Iang's inequality is having a neat form and is interesting in its own right as an integral inequality, its importance lies equality heavily on its many beautiful applications in differential and integral equations(see, e.g., [24]). Since this, over the years, many generalizations of Ou-Iang's inequality to various situations have been established ; see for examples [4-7, 10, 12-14, 17-18, 20-22, 25-27] and the references cited therein.

Recently, in [23], Pachpatte has established the following useful linear Volterra-Fredholm type integral inequalities with retardation:

Theorem B. ([23]). *Let* $u(t) \in C(I, R_+)$, $a(t,s), b(t,s), c(t,s) \in C(D, R_+)$, $a(t,s), b(t,s)$ *are nondecreasing in* t *for each* $s \in I, h(t) \in C^1(I, I)$ *be nondecreasing with* $h(t) \leq t$ *on* $I, k \geq 0$ *be a constant, where* $I = [\alpha, \beta], R_+ = [0, \infty), D = \{(t,s) \in I^2 : \alpha \leq s \leq t \leq \beta\}$ *and suppose that*

$$u(t) \leq k + \int_{h(\alpha)}^{h(t)} a(t,s) \left[f(s)u(s) + \int_{h(\alpha)}^{s} c(s,\sigma)u(\sigma)d\sigma \right] ds + \int_{h(\alpha)}^{h(\beta)} b(t,s)u(s)ds,$$

for $t \in I$. *If*

$$p(t) = \int_{h(\alpha)}^{h(\beta)} b(t,s)\exp(A(s))ds < 1,$$

for $t \in I$, *where*

$$A(t) = \int_{h(\alpha)}^{h(t)} a(t,\xi) \left[f(\xi) + \int_{h(\alpha)}^{\xi} c(\xi,\sigma)d\sigma \right] d\xi,$$

for $t \in I$, *then*

$$u(t) \leq \frac{k}{1 - p(t)} \exp(A(t))$$

for $t \in I$.

When $a(t,s)f(s) = a(s), c(t,s) = 0, b(t,s) = b(s)$, and $h(t) = t$ the result in Theorem B will deduce the conclusion appeared in [2]. The aim of the present paper is to give some explicit bounds to some new nonlinear retarded integral inequalities involving which on the one hand generalize Ou-Iang's inequality to Volterra-Fredholm form at the first time to literatures and on the other hand give a handy and effective tool for the study of quantitative properties of solutions of differential and integral equations. We illustrate the usefulness of these inequalities by applying them to study the boundedness, uniqueness, and continuous dependence of the solutions of certain nonlinear retarded integral equations.

2. Nonlinear Retarded Integral Inequalities

In what follows, R denotes the set of real numbers,$R_+ = [0, +\infty), R_1 = [1, +\infty), I = [t_0, T]$; $C^i(M, S)$ denotes the class of all i-times continuously differentiable functions defined on set M with range in the set $S(i = 1, 2, \cdots)$ and $C^0(M, S) = C(M, S)$.

Theorem 2.1. *Let $u(t), a(t), b(t)$ and $c(t) \in C(I, R_+), \alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \le t$ on $I; \varphi \in C^1(R_+, R_+)$ with φ' nondecreasing and $\varphi'(u) > 0$ for $u > 0$; $w \in C(R_+, R_+)$ be a nondecreasing function with $w(u) > 0$ for $u > 0$ and $G_1(v) = \int_{v_0}^{v} \frac{ds}{w(s)}, v \ge v_0 > 0, G_1(+\infty) = \int_{v_0}^{+\infty} \frac{ds}{w(s)} = +\infty$; function*

$$H_1(t) = G_1 \circ \varphi^{-1}(2t - k) - G_1 \circ \varphi^{-1}(t) \tag{2.1}$$

is increasing for $t \ge k$. If $u(t)$ satisfies

$$\begin{aligned} &\varphi(u(t)) \\ &\le k + \int_{\alpha(t_0)}^{\alpha(t)} a(s)\varphi'(u(s)) \left[b(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(u(\tau))d\tau \right] ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(T)} a(s)\varphi'(u(s)) \left[b(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(u(\tau))d\tau \right] ds \end{aligned} \tag{2.2}$$

for $t \in I$, where $k \ge 0$ is a constant, then

$$\begin{aligned} u(t) \le G_1^{-1} \Bigg\{ & G_1 \left[H_1^{-1} \left(\int_{\alpha(t_0)}^{\alpha(T)} a(s)[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau]ds \right) \right] \\ & + \int_{\alpha(t_0)}^{\alpha(t)} a(s)[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau]ds \Bigg\} \end{aligned} \tag{2.3}$$

for $t \in I$, where G_1^{-1} and H_1^{-1} are inverse functions of G_1 and H_1, respectively.

Proof. Let $k > 0$ and define a positive nondecreasing function $z(t), t \in I$ by

$$\varphi(z(t)) \tag{2.4}$$
$$= k + \int_{\alpha(t_0)}^{\alpha(t)} a(s)\varphi'(u(s)) \left[b(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(u(\tau))d\tau\right] ds$$
$$+ \int_{\alpha(t_0)}^{\alpha(T)} a(s)\varphi'(u(s)) \left[b(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(u(\tau))d\tau\right] ds,$$

then

$$u(t) \le z(t),\ \ t \in I$$
$$\varphi(z(t_0)) = k + \int_{\alpha(t_0)}^{\alpha(T)} a(s)\varphi'(u(s)) \left[b(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(u(\tau))d\tau\right] ds. \tag{2.5}$$

By differentiation, we derive from (2.4) that

$$\varphi'(z(t))\frac{dz(t)}{dt} = \varphi'(u(\alpha(t)))a(\alpha(t)) \left[b(\alpha(t))w(u(\alpha(t))) + \int_{\alpha(t_0)}^{\alpha(t)} c(\tau)w(u(\tau))d\tau\right] \alpha'(t)$$
$$\le \varphi'(z(\alpha(t)))a(\alpha(t)) \left[b(\alpha(t))w(z(\alpha(t))) + \int_{\alpha(t_0)}^{\alpha(t)} c(\tau)w(z(\tau))d\tau\right] \alpha'(t)$$
$$\le \varphi'(z(t))a(\alpha(t))w(z(t)) \left[b(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} c(\tau)d\tau\right] \alpha'(t)$$

or

$$\frac{dz(t)}{w(z(t))} \le a(\alpha(t)) \left[b(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} c(\tau)d\tau\right] \alpha'(t)dt. \tag{2.6}$$

since $z(t) > 0, \alpha(t) \le t$ for $t \in I$, φ' is nondecreasing with $\varphi'(u) > 0$ for $u > 0$ and (2.5) holds.

By the definition of G_1, setting $t = s$ in (2.6) , integrating it with respect to s from t_0 to t and making change of variable we get

$$G_1(z(t)) \le G_1(z(t_0)) + \int_{\alpha(t_0)}^{\alpha(t)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau\right] ds \tag{2.7}$$

for $t \in I$.

Observing that

$$2\varphi(z(t_0)) - k = k + 2\int_{\alpha(t_0)}^{\alpha(T)} a(s) \left[b(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(u(\tau))d\tau\right] ds$$
$$= \varphi(z(T)),$$

and then from (2.7) we have

$$G_1 \circ \varphi^{-1}[2\varphi(z(t_0)) - k] = G_1 \circ \varphi^{-1}[\varphi(z(T))]$$
$$\leq G_1 \circ \varphi^{-1}[\varphi(z(t_0))] + \int_{\alpha(t_0)}^{\alpha(T)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau \right] ds$$

or

$$G_1 \circ \varphi^{-1}[2\varphi(z(t_0)) - k] - G_1 \circ \varphi^{-1}[\varphi(z(t_0))]$$
$$\leq \int_{\alpha(t_0)}^{\alpha(T)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau \right] ds. \tag{2.8}$$

Since $H_1(t) = G_1 \circ \varphi^{-1}(2t - k) - G_1 \circ \varphi^{-1}(t)$ is increasing for $t \geq k$, $H_1(t)$ has inverse function $H_1^{-1}(t)$ and then from (2.8) we get

$$z(t_0) \leq H_1^{-1} \left(\int_{\alpha(t_0)}^{\alpha(T)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau \right] ds \right). \tag{2.9}$$

Substituting (2.9) into (2.7) and combing with (2.5) we obtain the desired inequality (2.3). If $k = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of k and subsequently let $\varepsilon \to 0$. ■

Theorem 2.1′. *Let* $u(t), \alpha(t), a(t), b(t), c(t), w(u), G_1(u)$ *and* k *be as in Theorem 2.1. If* $u(t)$ *satisfying (2.2) for* $t \in I$, *and*

$$\widetilde{H}_1(t) = G_1 \circ \varphi^{-1}(2t - k) - G_1 \circ \varphi^{-1}(t) - \int_{\alpha(t_0)}^{\alpha(T)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau \right] ds$$

is increasing and $\widetilde{H}_1(t) = 0$ *has a solution* c_1 *for* $t \geq k$, *then*

$$u(t) \leq G_1^{-1} \left\{ G_1(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau \right] ds \right\} \tag{2.10}$$

for $t \in I$.

Proof. By the same steps from (2.4)-(2.8) in the proofs of Theorem 2.1, we have

$$u(t) \leq z(t), \tag{2.11}$$

$$z(t) \leq G_1^{-1} \left\{ G_1(z(t_0)) + \int_{\alpha(t_0)}^{\alpha(t)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau \right] ds \right\} \tag{2.12}$$

and

$$(2.13)\quad G_1 \circ \varphi^{-1}(2t-k) - G_1 \circ \varphi^{-1}(t) \le \int_{\alpha(t_0)}^{\alpha(T)} a(s)\left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau\right] ds.$$

for $t \in I$.

From the assumption of Theorem 2.1′ and (2.13), we have

$$\widetilde{H}_1(z(t_0)) \le 0 = \widetilde{H}_1(c_1).$$

Since $\widetilde{H}_1$ is increasing, $\widetilde{H}_1$ has inverse function $\widetilde{H}_1^{-1}$ and hence from the last inequality we get

$$z(t_0) \le c_1.$$

Substituting the last inequality into (2.12) and combing with (2.11) we get the desired inequality (2.10). ■

When $\varphi = u^p (p \ge 1$ is a constant) in Theorem 2.1, we have following

Corollary 2.2. *Let $u(t), \alpha(t), w(t), a(t), b(t)$ and k be as in Theorem 2.1, $p \ge 1$ be a constant. If $u(t)$ satisfies*

$$(2.14)\quad \begin{aligned} u^p(t) \le k &+ \int_{\alpha(t_0)}^{\alpha(t)} a(s)u^{p-1}(s)\left[b(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(u(\tau))d\tau\right] ds \\ &+ \int_{\alpha(t_0)}^{\alpha(T)} a(s)u^{p-1}(s)\left[b(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(u(\tau))d\tau\right] ds \end{aligned}$$

for $t \in I$, and

$$(2.15)\quad H_{11}(t) = G_1((2t-k)^{\frac{1}{p}}) - G_1(t^{\frac{1}{p}})$$

is increasing for $t \ge k$,then

$$(2.16)\quad \begin{aligned} u(t) \le G_1^{-1}\Bigg\{ & G_1\left[H_{11}^{-1}\left(\int_{\alpha(t_0)}^{\alpha(T)} a(s)[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau]ds\right)\right] \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} a(s)[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau]ds\Bigg\} \end{aligned}$$

for $t \in I$.

Remark 2.1. When $p = 2$, (2.14) is a Volterra-Fredholm-Ou-Iang type retarded inequality.

Corollary 2.3. *Let $w(t), a(t), b(t)$ and $\alpha(t)$ be as in Theorem 2.1, p and $k \geq 0$ be constants.If $u(t) \in C(I, R_1)$ and satisfies*

$$
\begin{aligned}
&u^p(t) \leq k \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} a(s)u^p(s) \left[b(s)w(\log u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(\log u(\tau))d\tau \right] ds \qquad (2.17)\\
&+ \int_{\alpha(t_0)}^{\alpha(T)} a(s)u^p(s) \left[b(s)w(\log u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(\log u(\tau))d\tau \right] ds
\end{aligned}
$$

for $t \in I$, and

$$H_{12}(t) = G_1\left(\frac{1}{p}\log(2t-k)\right) - G_1\left(\frac{1}{p}\log t\right)$$

is increasing for $t \geq k$, then

$$
\begin{aligned}
u(t) \leq G_1^{-1} \Bigg\{ & G_1 \left[H_{12}^{-1} \left(\int_{\alpha(t_0)}^{\alpha(T)} a(s)[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau]ds \right) \right] \qquad (2.18)\\
&+ \int_{\alpha(t_0)}^{\alpha(t)} a(s)[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau]ds \Bigg\}
\end{aligned}
$$

for $t \in I$.

Proof. Taking $v(t) = \log u(t)$, then (2.17) reduces to

$$
\begin{aligned}
e^{pv(t)} &\leq k + \int_{\alpha(t_0)}^{\alpha(t)} a(s)e^{pv(s)} \left[b(s)w(v(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(v(\tau))d\tau \right] ds \\
&+ \int_{\alpha(t_0)}^{\alpha(T)} a(s)e^{pv(s)} \left[b(s)w(v(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(v(\tau))d\tau \right] ds
\end{aligned}
$$

for $t \in I$, which is a special case of inequality (2.2) when $\varphi(v) = \exp(pv)$ and $H_1(t) = H_{12}(t) = G_1\left(\frac{1}{p}\log(2t-k)\right) - G_1\left(\frac{1}{p}\log t\right)$. By Theorem 2.1, we get the desired inequality (2.18) directly. ■

Corollary 2.4. *Let $u(t), a(t), b(t), \alpha(t)$ and p, k be as in Theorem 2.1. If $u(t)$ satisfies*

$$
\begin{aligned}
u^p(t) &\leq k + \int_{\alpha(t_0)}^{\alpha(t)} a(s)u^{p-1}(s) \left[b(s)u(s) + \int_{\alpha(t_0)}^{s} c(\tau)u(\tau)d\tau \right] ds \qquad (2.19)\\
&+ \int_{\alpha(t_0)}^{\alpha(T)} a(s)u^{p-1}(s) \left[b(s)u(s) + \int_{\alpha(t_0)}^{s} c(\tau)u(\tau)d\tau \right] ds
\end{aligned}
$$

for $t \in I$, *and*

$$A_p(T) = \exp \int_{\alpha(t_0)}^{\alpha(T)} pa(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau) d\tau \right] ds < 2,$$

then

$$u(t) \leq \frac{k}{2 - A_p(T)} \exp \int_{\alpha(t_0)}^{\alpha(t)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau) d\tau \right] ds \tag{2.20}$$

for $t \in I$.

Proof. In Corollary 2.2, by letting $w(u) = u$ we obtain

$$G_1(v) = \int_{v_0}^{v} \frac{ds}{w(s)} = \int_{v_0}^{v} \frac{ds}{s} = \ln \frac{v}{v_0}, v \geq v_0 > 0,$$
$$H_{11}(t) = G_1((2t-k)^{\frac{1}{p}}) - G_1(t^{\frac{1}{p}}) = \frac{1}{p} \ln \frac{2t-k}{t}, t \geq k$$

and hence

$$G_1^{-1}(v) = v_0 \exp v,$$
$$H_{11}^{-1}(t) = \frac{k}{2 - \exp(pt)}.$$

From inequality (2.16), we obtain inequality (2.20). ■

Corollary 2.5. *Let* $u(t), \alpha(t), a(t), b(t), c(t), p$ *and* k *be as in Theorem 2.1,* $0 < q < 1$ *be a constant. If* $u(t)$ *satisfies*

$$\begin{aligned} u^p(t) \leq k &+ \int_{\alpha(t_0)}^{\alpha(t)} a(s) u^{p-1}(s) \left[b(s) u^q(s) + \int_{\alpha(t_0)}^{s} c(\tau) u^q(\tau) d\tau \right] ds \\ &+ \int_{\alpha(t_0)}^{\alpha(T)} a(s) u^{p-1} \left[b(s) u^q(s) + \int_{\alpha(t_0)}^{s} c(\tau) u^q(\tau) d\tau \right] ds \end{aligned} \tag{2.21}$$

for $t \in I$, *then*

$$u(t) \leq \left\{ (c_{11})^{1-q} + (1-q) \int_{\alpha(t_0)}^{\alpha(t)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau) d\tau \right] ds \right\}^{\frac{1}{1-q}} \tag{2.22}$$

for $t \in I$, *where* c_{11} *is the solution of equation*

$$\begin{aligned} \widetilde{H}_1(t) = & \frac{1}{1-q} \left[(2t-k)^{\frac{1-q}{p}} - t^{\frac{1-q}{p}} \right] \\ & - \int_{\alpha(t_0)}^{\alpha(T)} a(s) \left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau) d\tau \right] ds = 0 \end{aligned} \tag{2.23}$$

for $t \geq k$.

Proof. By Theorem 2.1$'$, we only need prove (2.23) has a solution c_{11} for $t \geq k$. In fact, (2.21) is a special case of (2.2) with $\varphi = u^p, w = u^q$, so in Theorem 2.1$'$ we have (2.23). Taking $r = \frac{1-q}{p}$ and by computation we have

$$\widetilde{H}_1'(t) = \frac{1}{p}\frac{(2^{\frac{1}{1-r}}t)^{1-r} - (2t-k)^{1-r}}{[t(2t-k)]^{1-r}} > 0$$

for $t \geq k$,

$$\widetilde{H}_1(k) = -\int_{\alpha(t_0)}^{\alpha(T)} a(s)\left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau\right] ds < 0$$

and

$$\lim_{t\to+\infty} \widetilde{H}_1(t) = \lim_{t\to+\infty}\left\{\frac{t^{1-r}}{1-q}\left[\left(2-\frac{k}{t}\right)^{1-r} - 1\right] - \int_{\alpha(t_0)}^{\alpha(T)} a(s)\left[b(s) + \int_{\alpha(t_0)}^{s} c(\tau)d\tau\right] ds\right\} = +\infty,$$

so $\widetilde{H}_1(t) = 0$ has a unique solution $c_{11} > k$. ■

Using Theorem 2.1, we can get some more generalized results as following.

Theorem 2.4. *Let $u(t), a(t), b(t), c(t), \alpha(t), w(t), G_1, G_1^{-1}, H_1, H_1^{-1}$ and k be as in Theorem 2.1, $d(t), f(t)$ and $g(t) \in C(R_+, R_+)$. If $u(t)$ satisfies*

(2.24)
$$\varphi(u(t)) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} a(s)\varphi'(u(s))\left[b(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w(u(\tau))d\tau\right] ds + \int_{\alpha(t_0)}^{\alpha(T)} d(s)\varphi'(u(s))\left[f(s)w(u(s)) + \int_{\alpha(t_0)}^{s} g(\tau)w(u(\tau))d\tau\right] ds$$

for $t \in I$,then

$$u(t) \leq G_1^{-1}\left\{G_1\left[H_1^{-1}\left(\int_{\alpha(t_0)}^{\alpha(T)} a^*(s)[b^*(s) + \int_{\alpha(t_0)}^{s} c^*(\tau)d\tau]ds\right)\right] + \int_{\alpha(t_0)}^{\alpha(t)} a^*(s)[b^*(s) + \int_{\alpha(t_0)}^{s} c^*(\tau)d\tau]ds\right\} \qquad (2.25)$$

for $t \in I$, where $a^*(t), b^*(t)$ and $c^*(t) \in C(R_+, R_+)$ such that both $a(t)$ and $d(t)$ are less than or equal to $a^*(t)$, $b(t)$ and $f(t)$ are less than or equal to $b^*(t)$ and $c(t)$ and $g(t)$ are less than or equal to $c^*(t)$, respectively.

Proof. Form (2.24) and the assumptions we have

$$\varphi(u(t)) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} a^*(s)\varphi'(u(s)) \left[b^*(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c^*(\tau)w(u(\tau))d\tau \right] ds + \int_{\alpha(t_0)}^{\alpha(T)} a^*(s)\varphi'(u(s)) \left[b^*(s)w(u(s)) + \int_{\alpha(t_0)}^{s} c^*(\tau)w(u(\tau))d\tau \right] ds$$

An application of Theorem 2.1 to the last inequality yields (2.25). ■

Remark 2.2. (i) In Theorem 2.4, we can choice function $a^*(t) = a(t) + d(t)$ or $\max\{a(t), d(t)\}$ as well as in functions $b^*(t)$ and $c^*(t)$; (ii) From Theorem 2.4, we can get some useful conclusions similar to Theorem 2.1′-Corollary 2.3, but for space-saving, we omit the details here.

Theorem 2.5. *Let* $u(t), a(t), b(t), c(t), d(t), f(t), g(t)\alpha(t), a^*(t), b^*(t), c^*(t)$ *and* k *be as in Theorem 2.4,* $w_i \in C(R_+, R_+)$ *be nondecreasing functions with* $w_i(u) > 0$ *for* $u > 0, i = 1, 2$. *If* $u(t)$ *satisfies*

$$\begin{aligned} &\varphi(u(t)) \\ &\leq k + \int_{\alpha(t_0)}^{\alpha(t)} a(s)\varphi'(u(s)) \left[b(s)w_1(u(s)) + \int_{\alpha(t_0)}^{s} c(\tau)w_1(u(\tau))d\tau \right] ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(T)} d(s)\varphi'(u(s)) \left[f(s)w_2(u(s)) + \int_{\alpha(t_0)}^{s} g(\tau)w_2(u(\tau))d\tau \right] ds \end{aligned} \tag{2.26}$$

for $t \in I$, *there is a function* $W \in C(R_+, R_+)$ *is nondecreasing such that both* w_1 *and* w_2 *are less than or equal to* W, *and*

$$\begin{aligned} G_2(v) &= \int_{v_0}^{v} \frac{ds}{W(s)}, v \geq v_0 > 0, G_2(+\infty) = \int_{v_0}^{+\infty} \frac{ds}{W(s)} = +\infty, \\ H_2(t) &= G_2 \circ \varphi^{-1}(2t - k) - G_2 \circ \varphi^{-1}(t) \end{aligned}$$

is increasing for $t \geq k$, *then*

$$\begin{aligned} u(t) \leq G_2^{-1} \Bigg\{ & G_2 \left[H_2^{-1} \left(\int_{\alpha(t_0)}^{\alpha(T)} a^*(s)[b^*(s) + \int_{\alpha(t_0)}^{s} c^*(\tau)d\tau]ds \right) \right] \\ & + \int_{\alpha(t_0)}^{\alpha(t)} a^*(s)[b^*(s) + \int_{\alpha(t_0)}^{s} c^*(\tau)d\tau]ds \Bigg\} \end{aligned} \tag{2.27}$$

for $t \in I$, *where* G_2^{-1} *and* H_2^{-1} *are inverse functions of* G_2 *and* H_2, *respectively.*

Proof. From (2.26) and the assumptions, we can obtain

$$\begin{aligned}
&\varphi(u(t)) \\
(2.28)\qquad &\leq k+\int_{\alpha(t_0)}^{\alpha(t)} a^*(s)\varphi'(u(s))\left[b^*(s)W(u(s))+\int_{\alpha(t_0)}^{s} c^*(\tau)W(u(\tau))d\tau\right]ds \\
&\quad+\int_{\alpha(t_0)}^{\alpha(T)} a^*(s)\varphi'(u(s))\left[b^*(s)W(u(s))+\int_{\alpha(t_0)}^{s} c^*(\tau)W(u(\tau))d\tau\right]ds
\end{aligned}$$

for $t \in I$.

Now an application of Theorem 2.1 to (2.28) yields the desired inequality (2.27). ■

By same argument as in the proofs of Theorem 2.1′, we have the following result immediately.

Theorem 2.5′. *Let* $u(t), a(t), b(t), c(t), d(t), f(t), g(t)\alpha(t), a^*(t), b^*(t), c^*(t)$, $w_i (i = 1, 2)$ *and* k *be as in Theorem 2.4. If* $u(t)$ *satisfies (2.26) for* $t \in I$*,and*

$$\widetilde{H}_2(t) = G_2\circ\varphi^{-1}(2t-k)-G_2\circ\varphi^{-1}(t)-\int_{\alpha(t_0)}^{\alpha(T)} a^*(s)\left[b^*(s)+\int_{\alpha(t_0)}^{s} c^*(\tau)d\tau\right]ds$$

is increasing and $\widetilde{H}_2(t) = 0$ *has a solution* $c_2 \geq k$*, then*

$$u(t) \leq G_2^{-1}\left\{G_2(c_2)+\int_{\alpha(t_0)}^{\alpha(t)} a^*(s)\left[b^*(s)+\int_{\alpha(t_0)}^{s} c^*(\tau)d\tau\right]ds\right\} \tag{2.29}$$

for $t \in I$.

Corollary 2.6. *Let* $u(t), a(t), b(t), c(t), d(t), \alpha(t), a^*(t), b^*(t), c^*(t)$ *and* k *be as in Theorem 2.4,* $p \geq 1, 0 < q < 1$ *be constants. If* $u(t)$ *satisfies*

$$\begin{aligned}
u^p(t) &\leq k+\int_{\alpha(t_0)}^{\alpha(t)} a(s)u^{p-1}(s)\left[b(s)u(s)+\int_{\alpha(t_0)}^{s} c(\tau)u(\tau)d\tau\right]ds \\
&\quad+\int_{\alpha(t_0)}^{\alpha(T)} d(s)u^{p-1}\left[f(s)u^q(s)+\int_{\alpha(t_0)}^{s} g(\tau)u^q(\tau)d\tau\right]ds
\end{aligned} \tag{2.30}$$

for $t \in I$*, and*

$$\exp\int_{\alpha(t_0)}^{\alpha(T)} a^*(s)\left[b^*(s)+\int_{\alpha(t_0)}^{s} c^*(\tau)d\tau\right]ds < 2^{\frac{1}{p}},$$

then

$$u(t) \le \left\{ (1 + c_{21}^{1-q}) \exp\left((1-q) \int_{\alpha(t_0)}^{\alpha(t)} \sigma_1^*(s) \left[f^*(s) + \int_{\alpha(t_0)}^{s} \sigma_2^*(\tau) d\tau \right] ds \right) - 1 \right\}^{\frac{1}{1-q}} \tag{2.31}$$

for $t \in I$, where c_{21} is the solution of

$$\begin{aligned} \widetilde{H}_2(t) = & \frac{1}{1-q} \ln \frac{1 + (2t-k)^{\frac{1-q}{p}}}{1 + t^{\frac{1-q}{p}}} \\ & - \int_{\alpha(t_0)}^{\alpha(T)} a^*(s) \left[b^*(s) + \int_{\alpha(t_0)}^{s} c^*(\tau) d\tau \right] ds = 0 \end{aligned} \tag{2.32}$$

for $t \ge k$.

Proof. In Theorem 2.5′, by letting $w_1(u) = u, w_2(u) = u^q$ and $W = w_1 + w_2$ we obtain

$$G_2(v) = \int_{v_0}^{v} \frac{ds}{w_1(s) + w_2(s)} = \int_{v_0}^{v} \frac{ds}{s + s^q} = \frac{1}{1-q} \ln \frac{1 + v^{1-q}}{1 + v_0^{1-q}}, \ v \ge v_0 > 0 \tag{2.33}$$

and hence

$$G_2^{-1}(v) = \left[(1 + v_0^{1-q}) \exp((1-q)v) - 1 \right]^{\frac{1}{1-q}}. \tag{2.34}$$

By computation, we have

$$\begin{aligned} \widetilde{H}_2(t) = & \frac{1}{1-q} \ln \frac{1 + (2t-k)^{\frac{1-q}{p}}}{1 + t^{\frac{1-q}{p}}} - \int_{\alpha(t_0)}^{\alpha(T)} a^*(s) \left[b^*(s) + \int_{\alpha(t_0)}^{s} c^*(\tau) d\tau \right] ds, \\ \widetilde{H}_2'(t) = & \frac{k + 2t^{1-\frac{1-q}{p}} - (2t-k)^{1-\frac{1-q}{p}}}{[2t - k + (2t-k)^{1-\frac{1-q}{p}}](t + t^{1-\frac{1-q}{p}})} > 0 \end{aligned} \tag{2.35}$$

for $t \ge k$,

$$\widetilde{H}_2(k) = - \int_{\alpha(t_0)}^{\alpha(T)} a^*(s) \left[b^*(s) + \int_{\alpha(t_0)}^{s} c^*(\tau) d\tau \right] ds < 0 \tag{2.36}$$

and

$$\begin{aligned}\lim_{t\to+\infty}\widetilde{H}_2(t) &= \lim_{t\to+\infty}\left\{\frac{1}{1-q}\ln\frac{1+(2t-k)^{\frac{1-q}{p}}}{1+t^{\frac{1-q}{p}}}\right.\\ &\left.-\int_{\alpha(t_0)}^{\alpha(T)}a^*(s)\left[b^*(s)+\int_{\alpha(t_0)}^{s}c^*(\tau)d\tau\right]ds\right\}\\ &= \ln 2^{\frac{1}{p}}-\int_{\alpha(t_0)}^{\alpha(T)}a^*(s)\left[b^*(s)+\int_{\alpha(t_0)}^{s}c^*(\tau)d\tau\right]ds>0.\end{aligned} \tag{2.37}$$

By (2.35)-(2.37) we obtain that (2.32) has a solution $c_{21}>k$. Now by (2.29), (2.33) and (2.34) we can get the desired (2.31). ■

3. Some Applications

In this section, we apply our results to study the boundedness, uniqueness, and continuous dependence of the solutions of certain nonlinear retarded integral equations. First we consider the nonlinear retarded integral equation of the form

$$\begin{aligned}x^p(t) &= l(t)+\int_{t_0}^{t}F\Big(s,x(s-h(s)),\int_{t_0}^{s}m(\tau,x(\tau-h(\tau)))d\tau\Big)ds\\ &+\int_{t_0}^{T}G\Big(s,x(s-h(s)),\int_{t_0}^{s}n(\tau,x(\tau-h(\tau)))d\tau\Big)ds\end{aligned} \tag{3.1}$$

for $t\in I$, where $x,l\in C(I,R), h\in C^1(I,I)$ be nonincreasing with $t-h(t)\ge 0, h(t_0)=0, t-h(t)\in C^1(I,I), h'(t)<1, F,G\in C(I\times R^2,R), m,n\in C(I\times R,R), p\ge 1$ is a constant.

Following Theorem gives the bound on the solution of the equation (3.1).

Theorem 3.1. *Assume that the functions l,m,n,F and G in (3.1) satisfies the conditions*

$$|l(t)|\le k \tag{3.2}$$

$$|F(s,x,y)|\le a(s)|x|^{p-1}[b(s)|x|^q+|y|] \tag{3.3}$$

$$|m(s,x)|\le c(s)|x|^q \tag{3.4}$$

$$|G(s,x,y)|\le d(s)|x|^{p-1}[f(s)|x|^q+|y|] \tag{3.5}$$

$$|n(s,x)|\le g(s)|x|^q \tag{3.6}$$

where $k, a(s), b(s), c(s), d(t), f(t)$ *and* $g(s)$ *are as in Theorem 2.4,* $p \geq 1$ *and* $0 < q < 1$ *are constants, and let* $L = \max_{t \in I}\{1/(1-h'(t))\}$. *If* $x(t)$ *is a solution of equation (3.1) on* I, *then*

$$|x(t)| \leq \Big\{(c_{11}^*)^{1-q} + (1-q)L\int_{\alpha(t_0)}^{\alpha(t)} \sigma_{ad}(s)\left[M_{bf}(s) + L\int_{\alpha(t_0)}^{s}\sigma_{cg}(\tau)d\tau\right]ds\Big\}^{\frac{1}{1-q}} \tag{3.7}$$

for $t \in I$, *where* $\sigma_{ad}(\xi) = a(\xi + h(s)) + d(\xi + h(s)), \sigma_{cg}(\eta) = c(\eta + h(\tau)) + g(\eta + h(\tau)), M_{bf}(s) = \max\{b(\xi + h(s)), f(\xi + h(s))\}$ *and* c_{11}^* *is the solution of*

$$H_1^*(t) = \tfrac{1}{1-q}\left[(2t-k)^{\frac{1-q}{p}} - t^{\frac{1-q}{p}}\right] - L\int_{\alpha(t_0)}^{\alpha(t)}\sigma_{ad}(s)\left[M_{bf}(s) + L\int_{\alpha(t_0)}^{s}\sigma_{cg}(\tau)d\tau\right]ds = 0.$$

Proof. Using the conditions (3.2)-(3.6) to (3.1) we have

$$\begin{aligned}
|x(t)|^p &\leq k + \int_{t_0}^{t} a(s)|x(s-h(s))|^{p-1}\left[b(s)|x(s-h(s))|^q + \int_{t_0}^{s} c(\tau)|x(\tau - h(\tau))|^q d\tau\right]ds \\
&\quad + \int_{t_0}^{T} d(s)|x(s-h(s))|^{p-1}\left[f(s)|x(s-h(s))|^q + \int_{t_0}^{s} g(\tau)|x(\tau-h(\tau))|^q d\tau\right]ds \\
&\leq k + \int_{t_0}^{t} a(s)|x(s-h(s))|^{p-1}\left[b(s)|x(s-h(s))|^q + \int_{t_0-h(t_0)}^{s-h(s)} Lc(\eta + h(\tau))|x(\eta)|^q d\eta\right]ds \\
&\quad + \int_{t_0}^{T} d(s)|x(s-h(s))|^{p-1}\left[f(s)|x(s-h(s))|^q + \int_{t_0-h(t_0)}^{s-h(s)} Lg(\eta + h(\tau))|x(\eta)|^q d\eta\right]ds \\
&\leq k + \int_{t_0-h(t_0)}^{t-h(t)} a(\xi + h(s))|x(\xi)|^{p-1}\left[b(\xi+h(s))(|x(\xi)|^q\right.
\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&+\int_{t_0-h(t_0)}^{\xi} Lc(\eta+h(\tau))|x(\eta)|^q d\eta\Bigg] d\xi \\
&+\int_{t_0-h(t_0)}^{T-h(T)} d(\xi+h(s))|x(\xi)|^{p-1}\left[f(\xi+h(s))|x(\xi)|^q\right. \\
&\left.+\int_{t_0-h(t_0)}^{\xi} Lg(\eta+h(\tau))|x(\eta)|^q d\eta\right] d\xi \\
&\leq k+\int_{t_0-h(t_0)}^{t-h(t)} \sigma_{ad}(\xi)|x(\xi)|^{p-1}\left[M_{bf}(s)|x(\xi)|^q\right. \\
&\left.+\int_{t_0-h(t_0)}^{\xi} L\sigma_{cg}(\eta)|x(\eta)|^p d\eta\right] d\xi \\
&+\int_{t_0-h(t_0)}^{T-h(T)} \sigma_{ad}(\xi)|x(\xi)|^{p-1}\left[M_{bf}(s)|x(\xi)|^q\right. \\
&\left.+\int_{t_0-h(t_0)}^{\xi} L\sigma_{cg}(\eta)|x(\eta)|^q d\eta\right] d\xi
\end{aligned} \tag{3.8}
$$

for $t \in I$.

Now a suitable applications with of Corollary 2.5 to $|x(t)|$ in (3.8) yields (3.7). ■

Secondly, we consider the uniqueness of the solutions of (3.1).

Theorem 3.2. *Assume that the functions* m, n, F *and* G *in equation* (3.1) *satisfy the conditions*

$$|F(s,x,y)-F(s,\overline{x},\overline{y})| \leq a(s)|x^p-\overline{x}^p|^{\frac{p-1}{p}}\left(b(s)|x^p-\overline{x}^p|^{\frac{1}{p}}+|y-\overline{y}|\right) \tag{3.9}$$

$$|m(s,x)-m(s,\overline{x})| \leq c(s)|x^p-\overline{x}^p|^{\frac{1}{p}} \tag{3.10}$$

$$|G(s,x,y)-G(s,\overline{x},\overline{y})| \leq d(s)|x^p-\overline{x}^p|^{\frac{p-1}{p}}\left(f(s)|x^p-\overline{x}^p|^{\frac{1}{p}}+|y-\overline{y}|\right) \tag{3.11}$$

$$|n(s,x)-n(s,\overline{x})| \leq g(s)|x^p-\overline{x}^p|^{\frac{1}{p}} \tag{3.12}$$

$$A_p^*(T)=\exp\int_{\alpha(t_0)}^{\alpha(T)} pL\sigma_{ad}(s)\left[M_{bf}(s)+\int_{\alpha(t_0)}^{s} L\sigma_{cg}(\tau)d\tau\right] ds<2, \tag{3.13}$$

where $a, b, c, d, f, g, \sigma_{ad}, \sigma_{ad}, M_{bf}, L$ *and* p *are defined as in Theorem 3.1, then* (3.1) *has at most one positive solution on* I.

Proof. Let $x(t)$ and $\overline{x}(t)$ be two solutions of (3.1) on I, using the conditions (3.9)-(3.12) to (3.1) we have

$$
\begin{aligned}
&|x^p(t) - \overline{x}^p(t)| \\
&\leq \int_{t_0}^{t} a(s)|x^p(s-h(s)) - \overline{x}^p(s-h(s))|^{\frac{p-1}{p}} \\
&\qquad \Big[b(s)|x^p(s-h(s)) - \overline{x}^p(s-h(s))|^{\frac{1}{p}} \\
&\quad + \int_{t_0}^{s} c(\tau)|x^p(\tau-h(\tau)) - \overline{x}^p(\tau-h(\tau))|^{\frac{1}{p}} d\tau\Big]\, ds \\
&\quad + \int_{t_0}^{T} d(s)|x^p(s-h(s)) - \overline{x}^p(s-h(s))|^{\frac{p-1}{p}} \\
&\qquad \Big[f(s)|x^p(s-h(s)) - \overline{x}^p(s-h(s))|^{\frac{1}{p}} \\
&\quad + \int_{t_0}^{s} g(\tau)|x^p(\tau-h(\tau)) - \overline{x}^p(\tau-h(\tau))|^{\frac{1}{p}} d\tau\Big]\, ds
\end{aligned} \tag{3.14}
$$

Now making a change of variables on the right side of (3.14) and taking the similar procedure as in the proof of Theorem 3.1 we have

$$
\begin{aligned}
&|x^p(t) - \overline{x}^p(t)| \\
&\leq \int_{t_0-h(t_0)}^{t-h(t)} \sigma_{ad}(\xi)|x^p(\xi) - \overline{x}^p(\xi)|^{\frac{p-1}{p}} \Big[M_{bf}(\xi)|x^p(\xi) - \overline{x}^p(\xi)|^{\frac{1}{p}} \\
&\quad + \int_{t_0-h(t_0)}^{\xi} L\sigma_{cg}(\eta)|x^p(\eta) - \overline{x}^p(\eta)|^{\frac{1}{p}} d\eta\Big] d\xi \\
&\quad + \int_{t_0-h(t_0)}^{T-h(T)} \sigma_{ad}(\xi)|x^p(\xi) - \overline{x}^p(\xi)|^{\frac{p-1}{p}} \Big[M_{bf}(\xi)|x^p(\xi) - \overline{x}^p(\xi)|^{\frac{1}{p}} \\
&\quad + \int_{t_0-h(t_0)}^{\xi} L\sigma_{cg}(\eta)|x^p(\eta) - \overline{x}^p(\eta)|^{\frac{1}{p}} d\eta\Big] d\xi
\end{aligned} \tag{3.15}
$$

A suitable application of Corollary 2.4 to the function $|x^p(t) - \overline{x}^p(t)|^{\frac{1}{p}}$ in (3.15) yields that

$$|x^p(t) - \overline{x}^p(t)|^{\frac{1}{p}} \leq 0$$

for $t \in I$. Hence $x = \overline{x}$ on I. ■

Remark 3.1. In special case $p = 1$, the assumptions (3.9)-(3.12) reduce to Lipschitz type conditions.

Next, we consider the following retarded nonlinear integral equation

$$x^p(t) = l(t) + \int_{t_0}^{t} F\Big(s, x(s-h(s))\Big)ds + \int_{t_0}^{T} G\Big(s, x(s-h(s))\Big)ds \tag{3.16}$$

for $t \in I$, where $x, l \in C(I,R), h \in C^1(I,I)$ be nonincreasing with $t - h(t) \geq 0, h(t_0) = 0, t - h(t) \in C^1(I,I), h'(t) < 1, F, G \in C(I \times R, R), p \geq 1$ is a constant.

The following theorem investigate the continuous dependence of the solutions of (3.16) on the functions F and G. For this we consider the following variation of (3.16):

$$x^p(t) = \bar{l}(t) + \int_{t_0}^{t} \overline{F}\Big(s, x(s-h(s))\Big)ds + \int_{t_0}^{T} \overline{G}\Big(s, x(s-h(s))\Big)ds \tag{$\overline{3.16}$}$$

for $t \in I$, where $\bar{l} \in C(I,R), \overline{F}$ and $\overline{G} \in C(I \times R, R)$.

Theorem 3.3. *Consider (3.16) and* ($\overline{3.16}$). *If*

(i)

$$|F(s,v_1) - F(s,v_2)| \leq a(s)|v_1^p - v_2^p|, \qquad |G(s,v_1) - G(s,v_2)| \leq d(s)|v_1^p - v_2^p|$$

and

$$|F(s,\overline{x}) - \overline{F}(s,\overline{x})| \leq \frac{\varepsilon}{4(T-t_0)}, \qquad |G(s,\overline{x}) - \overline{G}(s,\overline{x})| \leq \frac{\varepsilon}{4(T-t_0)};$$

(ii) $|l(t) - \bar{l}(t)| \leq \frac{\varepsilon}{2}$;

(iii) $\bar{A}_p^*(T) = \exp \int_{\alpha(t_0)}^{\alpha(T)} pL\sigma_{ad}(s)ds < 2$

for all $s, t \in I$ and $v_1, v_2, \overline{x} \in R$, where $\varepsilon > 0$ is an arbitrary constant, then

$$|x^p(t) - \overline{x}^p(t)| \leq \frac{\varepsilon^p}{(2 - \bar{A}_p^*(T))^p} \exp \int_{t_0}^{t-h(t)} p\sigma_{ad}(s)ds \tag{3.17}$$

for $t \in I$. Hence x^p dependents continuously on F and G. In particular, if x does not change sign, it depends continuously on F and G.

Proof. Let $x(t)$ and $\overline{x}(t)$ be solutions of (3.16) and ($\overline{3.16}$), respectively. Then $x(t)$ satisfies (3.16) and $\overline{x}(t)$ satisfies ($\overline{3.16}$). Hence

$$\begin{aligned}
|x^p(t)-\overline{x}^p(t)| &\le |l(t)-\overline{l}(t)| + \int_{t_0}^{t} |F(s,x(s-h(s))) - \overline{F}(s,\overline{x}(s-h(s)))|ds \\
&\quad + \int_{t_0}^{T} |G(s,x(s-h(s))) - \overline{G}(s,\overline{x}(s-h(s)))|ds \\
&\le \frac{\varepsilon}{2} + \int_{t_0}^{t} |F(s,x(s-h(s))) - F(s,\overline{x}(s-h(s)))|ds \\
&\quad + \int_{t_0}^{t} |F(s,\overline{x}(s-h(s))) - \overline{F}(s,\overline{x}(s-h(s)))|ds \\
&\quad + \int_{t_0}^{T} |G(s,x(s-h(s))) - G(s,\overline{x}(s-h(s)))|ds \\
&\quad + \int_{t_0}^{T} |G(s,\overline{x}(s-h(s))) - \overline{G}(s,\overline{x}(s-h(s)))|ds \\
&\le \varepsilon + \int_{t_0}^{t} a(s)|x^p(s-h(s)) - \overline{x}^p(s-h(s))|ds \\
&\quad + \int_{t_0}^{T} d(s)|x^p(s-h(s)) - \overline{x}^p(s-h(s))|ds
\end{aligned}$$

by assumptions (i)-(iii).

Now by making a change of variable on the right of the last inequality and taking the similar procedure as in the proofs of Theorem 3.1 we have

$$\begin{aligned}
|x^p(t)-\overline{x}^p(t)| &\le \varepsilon + \int_{t_0}^{t-h(t)} \sigma_{ad}(\xi)|x^p(\xi)-\overline{x}^p(\xi)|d\xi \\
&\quad + \int_{t_0}^{T-h(T)} \sigma_{ad}(\xi)|x^p(\xi)-\overline{x}^p(\xi)|d\xi
\end{aligned} \tag{3.18}$$

for $t \in I$. Now by applying Corollary 2.4 with the case $c(\tau)=0$ to the function $|x^p(t)-\overline{x}^p(t)|^{\frac{1}{p}}$ we have

$$|x^p(t)-\overline{x}^p(t)|^{\frac{1}{p}} \le \frac{\varepsilon}{2-\bar{A}_p^*(T)} \exp \int_{t_0}^{t-h(t)} \sigma_{ad}(s)ds$$

or

$$|x^p(t)-\overline{x}^p(t)| \le \frac{\varepsilon^p}{(2-\bar{A}_p^*(T))^p} \exp \int_{t_0}^{t-h(t)} p\sigma_{ad}(s)ds$$

for $t \in I$.

Evidently, if function $\exp \int_{t_0}^{t-h(t)} p\sigma_{ad}(s)ds$ is bounded on I, so

$$|x^p(t) - \overline{x}^p(t)| \leq \varepsilon^p K$$

for some $K > 0$ and $t \in I$. Hence x^p depends continuously on F and G. ■

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