TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 1, pp. 269-286, February 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

EQUIVALENCE OF NON-NEGATIVE RANDOM TRANSLATES OF AN IID RANDOM SEQUENCE

Aoi Honda, Yoshiaki Okazaki and Hiroshi Sato

Abstract. Let $\mathbf{X} = \{X_k\}$ be an IID random sequence and $\mathbf{Y} = \{Y_k\}$ be an independent random sequence also independent of $\mathbf{X}$. Denote by $\mu_{\mathbf{X}}$ and $\mu_{\mathbf{X}+\mathbf{Y}}$ the probability measures on the sequence space induced by $\mathbf{X}$ and $\mathbf{X}+\mathbf{Y} = \{X_k+Y_k\}$, respectively. The problem is to characterize $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ in terms of $\mu_{\mathbf{Y}}$ in the case where $\mathbf{X}$ is non-negative. Sato and Tamashiro[6] first discussed this problem assuming the existence of $f_{\mathbf{X}}(x) = \frac{d\mu_{X_1}}{dx}(x)$. They gave several necessary or sufficient conditions for $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ under some additional assumptions on $f_{\mathbf{X}}$ or on $\mathbf{Y}$.

The authors precisely improve these results. First they rationalize the assumption of the existence of $f_{\mathbf{X}}$. Then they prove that the condition (C.6) is necessary for wider classes of $f_{\mathbf{X}}$ with local regularities. They also prove if the p-integral $I_p^0(\mathbf{X}) < \infty$ and $\mathbf{Y} \in \ell_p^+$ a.s., then (C.6) is necessary and sufficient. Furthermore, in the typical case where $\mathbf{X}$ is exponentially distributed, they prove an explicit necessary and sufficient condition for $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$.

1. Introduction

For σ-finite measures μ and ν on a measurable space, $\mu \ll \nu$ means that μ is absolutely continuous with respect to ν, $\mu \perp \nu$ that they are singular, and $\mu \sim \nu$ that they are *equivalent* (mutually absolutely continuous). In the sequel, for a probability measure ν on $\mathbb{R}$ and some $-\infty \leq \theta < \infty$, we say "$\nu \sim m$ on $[\theta, \infty)$" if ν is supported by the half line $[\theta, \infty)$ and $\nu \sim m$ there, where m is the Lebesgue measure. If $\theta = -\infty$, then $[-\infty, \infty)$ should be read as $(-\infty, \infty)$.

Throughout this paper $\mathbf{X} = \{X_k\}$ denotes an independent identically distributed (IID) random sequence and $\mathbf{Y} = \{Y_k\}$ an independent random sequence, which

Accepted March 13, 2007.
Communicated by Hui-Hsiung Kuo.
2000 *Mathematics Subject Classification*: Primary 60G30, 28C20; Secondary 60B11, 46G12.
Key words and phrases: Non-negative random translates, Admissible, Kakutani's dichotomy theorem, Kitada-Sato criterion, p-integral, Exponential distribution.

is also independent of $\mathbf{X}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\mu_{\mathbf{X}}$ and $\mu_{\mathbf{X}+\mathbf{Y}}$ the probability measures on the sequence space induced by $\mathbf{X}$ and $\mathbf{X}+\mathbf{Y} = \{X_k + Y_k\}$, respectively. Furthermore, we always assume

$$\text{(C.0)} \qquad \mu_{X_k+Y_k} \sim \mu_{X_k}, k \geq 1,$$

where $\mu_{X_k+Y_k}$ and μ_{X_k} are the marginal distributions of $X_k + Y_k$ and X_k, respectively (see also (C.3)). $\mathbf{Y}$ is said to be *admissible* (*for* $\mathbf{X}$) if $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$.

Let $1 \leq p < \infty$ and $-\infty \leq \theta < \infty$. Let $f(x)$ be a probability density function on $\mathbb{R}$ which vanishes on $(-\infty, \theta)$ and $f(x) > 0$ a.e.(m) on (θ, ∞). Then we say $I_p^\theta(f) < \infty$ if $f(x)^{1/p}$ is absolutely continuous on $[\theta, \infty)$ and the p-integral defined by

$$I_p^\theta(f) := p^p \int_\theta^\infty \left| \frac{d}{dx} \left(f(x)^{\frac{1}{p}} \right) \right|^p dx < \infty.$$

In the case where $\theta = -\infty$, $I_p^{-\infty}(f)$ is simply denoted by $I_p(f)$. In particular $I_2(f)$ coincides with the Shepp's integral (Shepp[8]). For an IID random sequence $\mathbf{X} = \{X_k\}$, $I_p^\theta(\mathbf{X})$ is defined by $I_p^\theta(\mathbf{X}) := I_p^\theta(f_{\mathbf{X}})$, where $f_{\mathbf{X}}(x)$ is the probability density function of μ_{X_1} if exists.

For sequences of non-negative numbers $a_k \geq 0, 0 \leq p_k < 1,\ k \geq 1$, a Bernoulli sequence $\{\varepsilon(a_k, p_k)\}$ is an independent random sequence such that $\varepsilon(a_k, p_k)$ takes two values a_k and 0 with probability p_k and $1 - p_k$, respectively.

Kakutani's dichotomy theorem implies either $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ or $\mu_{\mathbf{X}+\mathbf{Y}} \perp \mu_{\mathbf{X}}$, and he also proved

$$\text{(C.1)} \qquad \sum_{k=1}^\infty \left(1 - \mathbb{E}\left[\sqrt{\frac{d\mu_{X_k+Y_k}}{d\mu_{X_k}}(X_k)} \right] \right) < \infty,$$

is necessary and sufficient for the admissibility of $\mathbf{Y}$ (Kakutani[2]).

On the other hand, defining $Z_k(x) := \dfrac{d\mu_{X_k+Y_k}}{d\mu_{X_k}}(x) - 1, k \geq 1$, Kitada and Sato[3] proved

$$\text{(C.2)} \qquad \textit{the almost sure convergence of } \sum_{k=1}^\infty Z_k(\mathbf{X}_k)$$

is necessary and sufficient for the admissibility of $\mathbf{Y}$.

The problem is to characterize the admissibility of $\mathbf{Y}$ only in terms of the distribution of $\mathbf{Y}$. In other words, the problem is to characterize the uniform integrability of the positive martingale

$$M_n = \prod_{k=1}^n \frac{d\mu_{X_k+Y_k}}{d\mu_{X_k}}(X_k), n \geq 1,$$

in terms of the distribution of $\mathbf{Y}$. We shall study the problem by utilizing the criteria (C.1) and (C.2).

The case where $\mathbf{Y}$ is a deterministic sequence $\boldsymbol{a} = \{a_k\}$ was first discussed systematically by [8]. He proved that $\mu_{\mathbf{X}+\boldsymbol{a}} \sim \mu_{\mathbf{X}}$ implies $\boldsymbol{a} \in \ell_2$, and that $\mu_{\mathbf{X}+\boldsymbol{a}} \sim \mu_{\mathbf{X}}$ for every $\boldsymbol{a} \in \ell_2$ if and only if $I_2(\mathbf{X}) < \infty$.

Define $\boldsymbol{a\varepsilon} := \{a_k\varepsilon_k\}$ where $\{\varepsilon_k\}$ is a Rademacher sequence and $\{a_k\}$ is a deterministic sequence. Then it was proved that the admissibility of $\boldsymbol{a\varepsilon}$ implies $\{a_k\} \in \ell_4$, and that $\mu_{\mathbf{X}+\boldsymbol{a\varepsilon}} \sim \mu_{\mathbf{X}}$ for every $\{a_k\} \in \ell_4$ if and only if

$$J_2(\mathbf{X}) := \int_{-\infty}^{\infty} \frac{f''_{\mathbf{X}}(x)^2}{f_{\mathbf{X}}(x)} dx < \infty$$

(Okazaki[4], Okazaki and Sato[5], Sato and Watari[7]). Furthermore, if $\mathbf{Y}$ is symmetric and $J_2(\mathbf{X}) < \infty$, then $\mathbf{Y} \in \ell_4$ a.s. implies the admissibility of $\mathbf{Y}$ ([7]).

Sato and Tamashiro[6] discussed the problem under the assumption of the existence of the density $f_{\mathbf{X}}(x) = \frac{d\mu_{X_1}}{dx}(x)$.

In Section 2, we shall prove that a Bernoulli sequence $\mathbf{Y} = \{\varepsilon(a_k, 1/2)\}$ is admissible for every $\{a_k\} \in \ell_2^+$ if and only if there exists $\theta \geq -\infty$ such that $\mu_{X_1} \sim m$ on $[\theta, \infty)$, $I_2^\theta(\mathbf{X}) < \infty$ and $f_{\mathbf{X}}(+\theta) := \lim_{x \searrow \theta} f_{\mathbf{X}}(x) = 0$ (Theorem 2.2). This shows that the assumption of the existence of $f_{\mathbf{X}}(x) = \frac{d\mu_{X_1}}{dx}(x)$ on $[\theta, \infty)$ in [6] is reasonable.

In Sections 3, 4 and 5, we assume $\theta = 0$, that is, $X_1 \geq 0$ a.s. and there exists $f_{\mathbf{X}}(x) = \frac{d\mu_{X_1}}{dx}(x)$ for $x \geq 0$. In this case, if $\mathbf{Y}$ is admissible for $\mathbf{X}$, then $\mathbf{Y}$ is necessarily non-negative, that is, $Y_k \geq 0$ a.s., $k \geq 1$, and no deterministic sequence is admissible unless trivial. On the other hand, if $\theta = 0$, the condition (C.0) is equivalent to

(C.3) $$\mathbb{P}(Y_k < \varepsilon) > 0 \text{ for every } \varepsilon > 0, k \geq 1.$$

In Section 3, we shall study the necessary condition for the admissibility of $\mathbf{Y}$. It is known that if $\mathbf{X}$ and $\mathbf{Y}$ are non-negative and $\mathbf{Y}$ is admissible, then we have

(C.4) $$\sum_{k=1}^{\infty} \mathbb{E}[Y_k : Y_k \leq \alpha]^2 < \infty,$$

(C.5) $$\sum_{k=1}^{\infty} \mathbb{P}(Y_k > \alpha)^2 < \infty,$$

for some (and any) $\alpha > 0$ (Hino[1], see also [3], [6]). [6] strengthened the necessary condition (C.5) to

(C.6) $$\sum_{k=1}^{\infty} \int_0^{\infty} \mathbb{P}(Y_k > x)^2 f_{\mathbf{X}}(x) dx < \infty,$$

in the case where $\mathbf{Y}$ is a Bernoulli sequence ([6], Theorem 3.1), or where $f_{\mathbf{X}}(+0) > 0$, $f_{\mathbf{X}}$ is absolutely continuous in an interval $[0, \delta]$ and $\text{ess.sup}_{0 \le x \le \delta} |f'_{\mathbf{X}}(x)| < \infty$ ([6], Theorem 3.3(B)). We shall prove (C.6) under new assumptions of the local increase (Theorem 3.1) or the integrability $\int_0^\delta x^{-2} f_{\mathbf{X}}(x)dx < \infty$ (Theorem 3.2) on $f_{\mathbf{X}}$, which include the case $f_{\mathbf{X}}(+0) = 0$. These results exhaust most cases of $f_{\mathbf{X}}$ and it is not known any examples of $f_{\mathbf{X}}$ where $\mathbf{Y}$ is admissible but (C.6) does not hold. We conjecture that (C.6) is a necessary condition for the admissibility of $\mathbf{Y}$ in general.

Furthermore, we shall strengthen (C.4) to

$$\text{(C.7)} \qquad \sum_{k=1}^{\infty} \int_0^\infty \mathbb{E}[Y_k \ : \ Y_k \le x\ ]^2 f_{\mathbf{X}}(x)dx < \infty.$$

(C.7) is necessary for admissibility of $\mathbf{Y}$ if $\mathbb{E}[|X_1|^2] < \infty$ (Theorem 3.4). However there exist examples of $\mathbf{X}$, with $\mathbb{E}[|X_1|^2] < \infty$ and admissible $\mathbf{Y}$ which do not satisfy (C.7) (Example 3.5). On the other hand, in general, (C.6) and (C.7) are not sufficient for the admissibility of $\mathbf{Y}$ (Example 5.4).

In Section 4, we shall study $\mathbf{X}$ with $I_p^0(\mathbf{X}) < \infty, 1 \le p \le 2$. We shall prove that if $I_p^0(\mathbf{X}) < \infty$ and $\mathbf{Y} \in \ell_p^+$ a.s., then $\mathbf{Y}$ is admissible if and only if (C.6) holds (Theorem 4.1).

In Section 5, we shall study the case where $\mathbf{X}$ is exponentially distributed, that is, $f_{\mathbf{X}}(x) = \lambda e^{-\lambda x} \mathbf{I}_{[0,\infty)}(x)$ for some $\lambda > 0$ as the most typical and simplest case. [6] gave a necessary and sufficient condition of this case for the admissibility of $\mathbf{Y}$ under the additional assumption

$$\text{(C.8)} \qquad \sum_{k=1}^{\infty} \mathbb{P}(Y_k > \alpha) < \infty,$$

for some $\alpha > 0$ ([6], Theorem 4.1). We shall give a necessary and sufficient condition for the admissibility of $\mathbf{Y}$ without any additional assumptions on $\mathbf{Y}$ (Theorem 5.1).

2. Non-negative Random Translates

To begin with, we shall prove the following lemma.

Lemma 2.1. *For a probability ν on $(\mathbb{R}, \mathcal{B})$, define $\nu_s(A) = \nu(A - s),\ A \in \mathcal{B},\ s \in \mathbb{R}_+$. Then $\nu_s \ll \nu$ for every $s \in \mathbb{R}_+$ if and only if $\nu \sim m$ on $[\theta, \infty)$ for some $-\infty \le \theta < \infty$.*

Proof. Assume $\nu_s \ll \nu$ for every $s \in \mathbb{R}_+$. Then for every open interval (a, b) such that $\nu((a, b)) = 0$, we have $\nu((a, b) - s) = \mu_s((a, b)) = 0$ for every $s \ge 0$.

Consequently there exists $-\infty \leq \theta < \infty$ such that ν is supported by the half line $[\theta, \infty)$.

Next we show $\nu \ll m$. Assume $m(A) = 0$ for $A \in \mathcal{B}$. Then we have

$$0 = m(A+1) = \int_{\mathbb{R}} d\nu(x) \int_0^1 \mathbf{I}_{A+1}(s+x)ds = \int_0^1 \nu\,(A+1-s)\,ds.$$

Hence there exists an $s \in [0,1)$ such that $\nu\,(A+1-s) = 0$. Since $\nu_{(1-s)} \ll \nu$, we have $\nu(A) = \nu_{(1-s)}(A+1-s) = 0$.

Finally we show $m \ll \nu$ on $[\theta, \infty)$. For every Borel set $A \subset [\theta, \infty)$ such that $\nu(A) = 0$, we have

$$0 = \int_0^\infty \nu_s(A)ds = \int_0^\infty ds \int_{[\theta,\infty)} \mathbf{I}_{A-s}(x)d\nu(x) = \int_{[\theta,\infty)} m((A-x)\cap\mathbb{R}_+)d\nu(x),$$

so that $F(x) := m((A-x)\cap\mathbb{R}_+) = 0$ a.s.$(d\nu)$. Then by the minimality of the support $[\theta, \infty)$, we can find a sequence $\theta_n \downarrow \theta$ such that $F(\theta_n) = 0$ and have

$$m(A) = \lim_n m((A-\theta_n)\cap\mathbb{R}_+) = 0.$$

The converse statement of the lemma is evident. ■

Theorem 2.2. *Let* $\mathbf{X} = \{X_k\}$ *be an IID random sequence of real (not necessarily non-negative) random variables and* $\mathbf{Y} = \left\{\varepsilon\,(a_k, 1/2)\right\}$ *be a Bernoulli sequence. Then we have*

(A) *The admissibility of* $\mathbf{Y}$ *implies* $\{a_k\} \in \ell_2^+$.

(B) $\mathbf{Y}$ *is admissible for every* $\{a_k\} \in \ell_2^+$ *if and only if there exits* $\theta \geq -\infty$ *such that* $\mu_{X_1} \sim m$ *on* $[\theta,\infty)$, $I_2^\theta(\mathbf{X}) < \infty$ *and* $f_{\mathbf{X}}(+\theta) = 0$.

Proof.

(A) is due to [6, Theorem 3.1], [1, Theorem 1.8].

(B) Since $a_k \geq 0$ is arbitrary, Lemma 1 is applicable to $\nu = \mu_{X_1}$ and we have $\mu_{X_1} \sim m$ on $[\theta, \infty)$ for some $\theta \geq -\infty$.

On the other hand, Kakutani's criterion (C.1) implies that $\mathbf{Y}$ is admissible for every $\{a_k\} \in \ell_2^+$ if and only if

$$\sum_{k=1}^\infty \int_{-\infty}^\infty \left|\sqrt{f_{\mathbf{X}}(x-a_k)} - \sqrt{f_{\mathbf{X}}(x)}\right|^2 dx < \infty$$

for every $\{a_k\} \in \ell_2^+$. Consequently we have $\sqrt{f_{\mathbf{X}}(x)}$ is absolutely continuous on $\mathbb{R}$ and $I_2^\theta(\mathbf{X}) < \infty$ by applying the arguments similar to [8]. ■

The condition $f_{\mathbf{X}}(+\theta) = 0$ is crucial since $I_2^{\theta}(\mathbf{X}) < \infty$ implies the absolute continuity of $\sqrt{f_{\mathbf{X}}(x)}$ on the whole real line $\mathbb{R}$. In fact, if $\mathbf{X}$ is exponentially distributed, where $f_{\mathbf{X}}(+0) > 0$ and $I_2^0(\mathbf{X}) < \infty$, then the Bernoulli sequence $\{\varepsilon(a_k, 1/2)\}$ is admissible if and only if $\{a_k\} \in \ell_1^+$ (Example 5.5).

3. Necessary Conditions for Admissibility

We shall discuss the necessity of (C.6) and show that (C.6) is necessary for the admissibility of $\mathbf{Y}$ under the various assumptions on $f_{\mathbf{X}}$.

By Kolmogorov's three series theorem, (C.2) is equivalent to the following two conditions.

$$\sum_{k=1}^{\infty} \mathbb{E}\big[|Z_k(\mathbf{X}_k)| : |Z_k(\mathbf{X}_k)| \geq \alpha\big] < \infty, \tag{3.1}$$

$$\sum_{k=1}^{\infty} \mathbb{E}\big[Z_k(\mathbf{X}_k)^2 : |Z_k(\mathbf{X}_k)| < \alpha\big] < \infty, \tag{3.2}$$

for some (and any) $\alpha > 0$.

In the following two theorems, we shall prove the necessity of (C.6) under the assumption of local regularities on $f_{\mathbf{X}}$. These results include the case $f_{\mathbf{X}}(+0) = 0$.

Theorem 3.1. *Assume that there exists some $\delta > 0$ such that $f_{\mathbf{X}}(x)$ is non-decreasing on the interval $[0, \delta]$. Then the admissibility of $\mathbf{Y}$ implies* (C.6).

Proof. Since $f_{\mathbf{X}}(x)$ is non-decreasing in $[0, \delta]$ and $f_{\mathbf{X}}(x) = 0$ for $x < 0$, we have for any $y > 0$,

$$0 \leq 1 - \frac{f_{\mathbf{X}}(x-y)}{f_{\mathbf{X}}(x)} \leq 1,$$

and (3.2) implies

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_0^{\delta} \mathbb{P}(Y_k > x)^2 f_{\mathbf{X}}(x)dx \\
&= \sum_{k=1}^{\infty} \int_0^{\delta} \mathbb{E}\left[1 - \frac{f_{\mathbf{X}}(x - Y_k)}{f_{\mathbf{X}}(x)} :\ Y_k > x\right]^2 f_{\mathbf{X}}(x)dx \\
&\leq \sum_{k=1}^{\infty} \int_0^{\delta} \mathbb{E}\left[1 - \frac{f_{\mathbf{X}}(x - Y_k)}{f_{\mathbf{X}}(x)}\right]^2 f_{\mathbf{X}}(x)dx \\
&= \sum_{k=1}^{\infty} \int_0^{\delta} \left(1 - \frac{\mathbb{E}[f_{\mathbf{X}}(x - Y_k)]}{f_{\mathbf{X}}(x)}\right)^2 f_{\mathbf{X}}(x)dx \\
&\leq \sum_{k=1}^{\infty} \mathbb{E}\left[Z_k(X_k)^2 :\ Z_k(X_k) \leq 1\right] < \infty.
\end{aligned}$$

■

Theorem 3.2. *Assume that* $\int_0^\delta x^{-2} f_{\mathbf{X}}(x)dx < \infty$ *for some* $\delta > 0$. *Then the admissibility of* $\mathbf{Y}$ *implies* (C.6).

Proof. By Chebyshev's inequality, we have $\mathbb{P}(x < Y_k \le \delta) \le \mathbb{E}[Y_k; Y_k \le \delta]/x$, and

$$\begin{aligned}&\sum_{k=1}^{\infty} \int_0^\delta \mathbb{P}(Y_k > x)^2 f_{\mathbf{X}}(x)dx \\ &= \sum_{k=1}^{\infty} \int_0^\delta \mathbb{P}(x < Y_k \le \delta)^2 f_{\mathbf{X}}(x)dx + \sum_{k=1}^{\infty} \int_0^\delta \mathbb{P}(x < Y_k, Y_k > \delta)^2 f_{\mathbf{X}}(x)dx \\ &=: A + B.\end{aligned}$$

Then we have by using (C.4) and (C.5)

$$A \le \sum_{k=1}^{\infty} \int_0^\delta x^{-2}\mathbb{E}[Y_k; Y_k \le \delta]^2 f_{\mathbf{X}}(x)dx = \sum_{k=1}^{\infty} \mathbb{E}[Y_k; Y_k \le \delta]^2 \int_0^\delta x^{-2} f_{\mathbf{X}}(x)dx < \infty,$$

$$B \le \sum_{k=1}^{\infty} \int_0^\delta \mathbb{P}(Y_k > \delta)^2 f_{\mathbf{X}}(x)dx = \sum_{k=1}^{\infty} \mathbb{P}(Y_k > \delta)^2 \int_0^\delta f_{\mathbf{X}}(x)dx < \infty. \qquad \blacksquare$$

The following theorem reformulates [6, Theorem 3.3(B)].

Theorem 3.3. *Assume that* $f_{\mathbf{X}}(+0) > 0$ *and there exist* $\delta > 0$ *and* $K > 0$ *satisfying*

$$|f_{\mathbf{X}}(y) - f_{\mathbf{X}}(x)| \le K|y - x| \quad \textit{for } x, y \in [0, \delta].$$

Then the admissibility of $\mathbf{Y}$ *implies* (C.6).

Proof. Taking δ sufficiently small, we may assume $K\delta < f_{\mathbf{X}}(+0)/2$. Then for $x \in [0, \delta]$ we have $0 < f_{\mathbf{X}}(+0)/2 < f_{\mathbf{X}}(x) < 3f_{\mathbf{X}}(+0)/2$ and

$$\begin{aligned}|Z_k(x)| &= \left| \frac{\mathbb{E}[f_{\mathbf{X}}(x - Y_k)]}{f_{\mathbf{X}}(x)} - 1 \right| \\ &= \left| \frac{\mathbb{E}[f_{\mathbf{X}}(x - Y_k) - f_{\mathbf{X}}(x) : Y_k < x]}{f_{\mathbf{X}}(x)} - \mathbb{P}(Y_k \ge x) \right| \\ &\le \frac{K\delta}{f_{\mathbf{X}}(x)} + 1 \le 2.\end{aligned}$$

Consequently by (3.2) and (C.4) we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_0^{\delta}\mathbb{P}(Y_k > x)^2 f_{\mathbf{X}}(x)dx \\
&= \sum_{k=1}^{\infty}\int_0^{\delta}\Big[\Big(\frac{\mathbb{E}[f_{\mathbf{X}}(x-Y_k)]}{f_{\mathbf{X}}(x)}-1\Big)f_{\mathbf{X}}(x)-\mathbb{E}[f_{\mathbf{X}}(x-Y_k)-f_{\mathbf{X}}(x) : Y_k\le x]\Big]^2\frac{dx}{f_{\mathbf{X}}(x)} \\
&\le 2\sum_{k=1}^{\infty}\int_0^{\delta} Z_k(x)^2 f_{\mathbf{X}}(x)dx + 4\frac{K^2\delta}{f_{\mathbf{X}}(+0)}\sum_{k=1}^{\infty}\mathbb{E}[Y_k : Y_k\le\delta]^2 \\
&\le 2\sum_{k=1}^{\infty}\mathbb{E}[Z_k(X_k)^2 : |Z_k(X_k)|\le 2] + 2K\sum_{k=1}^{\infty}\mathbb{E}[Y_k : Y_k\le\delta]^2 < \infty. \qquad \blacksquare
\end{aligned}$$

On the other hand, we have strengthen (C.4) to (C.7) as follows.

Theorem 3.4. *Assume* $\mathbb{E}[|X_1|^2] < \infty$ *and* $\mathbf{Y}$ *is admissible. Then we have* (C.7).

Proof. We have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_0^{\infty}\mathbb{E}[Y_k : Y_k\le x]^2 f_{\mathbf{X}}(x)dx \\
\le\ &\sum_{k=1}^{\infty}\int_0^{\delta}\mathbb{E}[Y_k : Y_k\le x]^2 f_{\mathbf{X}}(x)dx + 2\sum_{k=1}^{\infty}\int_{\delta}^{\infty}\mathbb{E}[Y_k : Y_k\le\delta]^2 f_{\mathbf{X}}(x)dx \\
&+2\sum_{k=1}^{\infty}\int_{\delta}^{\infty}\mathbb{E}[Y_k : \delta < Y_k\le x]^2 f_{\mathbf{X}}(x)dx =: A+2B+2C.
\end{aligned}$$

By (C.4) and (C.5), we have

$$\begin{aligned}
A &= \sum_{k=1}^{\infty}\mathbb{E}[Y_k : Y_k\le\delta]^2\int_0^{\delta} f_{\mathbf{X}}(x)dx < \infty, \\
B &\le \sum_{k=1}^{\infty}\mathbb{E}[Y_k : Y_k\le\delta]^2\int_{\delta}^{\infty} f_{\mathbf{X}}(x)dx < \infty, \\
C &= \sum_{k=1}^{\infty}\int_{\delta}^{\infty}\mathbb{E}[Y_k : \delta<Y_k\le x]^2 f_{\mathbf{X}}(x)dx \le \sum_{k=1}^{\infty}\int_{\delta}^{\infty}x^2\mathbb{P}(Y_k>\delta)^2 f_{\mathbf{X}}(x)dx<\infty. \qquad \blacksquare
\end{aligned}$$

The integrability $\mathbb{E}[X_1^2] < \infty$ is crucial in the above theorem. For instance, we have the following example.

Example 3.5. Let $f_{\mathbf{X}}(x) = 2/\{\pi(1+x^2)\}$ and $\mathbf{Y} = \big\{\varepsilon(k^3, 1/k^2)\big\}, k\ge 1$ be a Bernoulli sequence. Then $\mathbf{Y}$ is admissible but (C.7) does not hold.

Proof. Let $f_{\mathbf{X}}(x) = 2/\{\pi(1+x^2)\}$ and $\mathbf{Y} = \{a_k, p_k\}$, where $a_k = k^3$ and $p_k = 1/k^2$. By estimating Kakutani's criterion (C.1), we have

$$\begin{aligned}\infty &> \sum_{k=1}^{\infty}\int_0^{\infty}\left|\sqrt{\mathbb{E}[f_{\mathbf{X}}(x-Y_k)]}-\sqrt{f_{\mathbf{X}}(x)}\right|^2 dx\\ &= \sum_{k=1}^{\infty}\left(\sqrt{1-p_k}-1\right)^2\int_0^{a_k} f_{\mathbf{X}}(x)dx\\ &\quad + \sum_{k=1}^{\infty}\int_{a_k}^{\infty}\left|\sqrt{f_{\mathbf{X}}(x)+p_k(f_{\mathbf{X}}(x-a_k)-f_{\mathbf{X}}(x))}-\sqrt{f_{\mathbf{X}}(x)}\right|^2 dx = I_1+I_2.\end{aligned}$$

Since $1 \le \left(\sqrt{1-p_k}+1\right)^2 \le 4$, we have

$$I_1 = \sum_{k=1}^{\infty}\frac{(p_k)^2}{\left(\sqrt{1-p_k}+1\right)^2}\int_0^{a_k} f_{\mathbf{X}}(x)dx \le \sum_{k=1}^{\infty}\int_0^{\infty}\mathbb{P}(Y_k>y)^2 f_{\mathbf{X}}(y)dy \le 4I_1,$$

which shows $I_1 < \infty$ if and only if (C.6) holds. We have

$$\begin{aligned}I_2 &= \sum_{k=1}^{\infty}(p_k)^2\int_{a_k}^{\infty}\frac{(f_{\mathbf{X}}(x-a_k)-f_{\mathbf{X}}(x))^2}{\left(\sqrt{f_{\mathbf{X}}(x)+p_k(f_{\mathbf{X}}(x-a_k)-f_{\mathbf{X}}(x))}+\sqrt{f_{\mathbf{X}}(x)}\right)^2}dx\\ &\le \sum_{k=1}^{\infty}(p_k)^2\int_{a_k}^{\infty}\frac{a_k^2(2x-a_k)^2 f_{\mathbf{X}}(x)^2 f_{\mathbf{X}}(x-a_k)^2}{p_k f_{\mathbf{X}}(x-a_k)}dx\\ &= \sum_{k=1}^{\infty}(a_k)^2 p_k\int_{a_k}^{\infty}(2x-a_k)^2 f_{\mathbf{X}}(x)^2 f_{\mathbf{X}}(x-a_k)dx\\ &\le \left(\frac{2}{\pi}\right)^2\sum_{k=1}^{\infty}p_k\int_{a_k}^{\infty}\frac{a_k^2}{1+a_k^2}\frac{4x^2}{1+x^2}f_{\mathbf{X}}(x-a_k)dx\\ &\le \frac{16}{\pi^2}\sum_{k=1}^{\infty}p_k\int_0^{\infty}f_{\mathbf{X}}(x)dx = \frac{16}{\pi^2}\sum_{k=1}^{\infty}p_k.\end{aligned}$$

Consequently, if $\sum_{k=1}^{\infty}p_k < \infty$ then $I_1 \le \sum_{k=1}^{\infty}p_k^2 < \infty$ and $\mathbf{Y}$ is admissible.

On the other hand, for $a_k \ge 1$,

$$\begin{aligned}&\sum_{k=1}^{\infty}\int_0^{\infty}\mathbb{E}[Y_k \ :\ Y_k \le x]^2 f_{\mathbf{X}}(x)dx = \frac{2}{\pi}\sum_{k=1}^{\infty}(a_k p_k)^2\int_{a_k}^{\infty}\frac{1}{(1+x^2)}dx\\ &\ge \frac{1}{\pi}\sum_{k=1}^{\infty}(a_k p_k)^2\int_{a_k}^{\infty}\frac{1}{x^2}dx = \frac{1}{\pi}\sum_{k=1}^{\infty}a_k(p_k)^2,\end{aligned}$$

which implies that (C.7) is equivalent to $\sum_{k=1}^{\infty} a_k(p_k)^2 < \infty$ if $a_k \geq 1$. But $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ and $\sum_{k=1}^{\infty} a_k(p_k)^2 = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$, so that (C.7) does not hold. ■

Conversely, even if both (C.6) and (C.7) hold, $\mathbf{Y}$ is not necessarily admissible in general (Example 5.4).

4. The p-integral

In this section, we shall prove that (C.6) is necessary and sufficient condition for the admissibility of $\mathbf{Y}$ if $I_p^0(\mathbf{X}) < \infty$ and $\mathbf{Y} \in \ell_p^+$ a.s. for some $1 \leq p \leq 2$. In the case $p = 2$, [6] proved the sufficiency of (C.6), and the necessity of (C.6) under the condition $\text{ess.sup}_{0 \leq x \leq \delta} |f'_{\mathbf{X}}(x)| < \infty$.

Theorem 4.1. *Assume* $I_p^0(\mathbf{X}) < \infty$ *and* $\mathbf{Y} = \{Y_k\} \in \ell_p^+$ *a.s. for some* $1 \leq p \leq 2$. *Then* $\mathbf{Y}$ *is admissible if and only if* (C.6) *holds.*

Proof. Assume $\mathbf{Y} \in \ell_p^+$ a.s. Then Kolmogorov's three series theorem implies $\sum_{k=1}^{\infty} \mathbb{P}(Y_k > \delta) < \infty$ and $\sum_{k=1}^{\infty} \mathbb{E}[|Y_k|^p : Y_k \leq \delta] < \infty$. We have

$$\begin{aligned}
&\frac{1}{4}\sum_{k=1}^{\infty} \int_0^{\delta} \mathbb{P}(Y_k > x)^2 f_{\mathbf{X}}(x) dx \leq \sum_{k=1}^{\infty} \int_0^{\delta} \left| \frac{f_{\mathbf{X}}(x)\mathbb{P}(Y_k > x)}{\sqrt{f_{\mathbf{X}}(x)} + \sqrt{f_{\mathbf{X}}(x)\mathbb{P}(Y_k \leq x)}} \right|^2 dx \\
&\leq \sum_{k=1}^{\infty} \int_0^{\delta} \left| \sqrt{f_{\mathbf{X}}(x)} - \sqrt{f_{\mathbf{X}}(x)\mathbb{P}(Y_k \leq x)} \right|^2 dx \\
&\leq \sum_{k=1}^{\infty} \int_0^{\delta} 2\Big[\left(\sqrt{f_{\mathbf{X}}(x)} - \sqrt{\mathbb{E}[f_{\mathbf{X}}(x - Y_k)]} \right)^2 \\
&\quad + \left(\sqrt{\mathbb{E}[f_{\mathbf{X}}(x - Y_k)]} - \sqrt{f_{\mathbf{X}}(x)\mathbb{P}(Y_k \leq x)} \right)^2 \Big] dx.
\end{aligned}$$

The first term is finite by Kakutani's criterion (C.1).

On the other hand by inequality $\left| a^{\frac{1}{r}} - b^{\frac{1}{r}} \right|^r \geq \left| a^{\frac{1}{s}} - b^{\frac{1}{s}} \right|^s$, $a, b \geq 0,\ 0 < r \leq s$, we have for $q > 1$ such that $1/p + 1/q = 1$,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_0^{\delta} \left| \sqrt{\mathbb{E}[f_{\mathbf{X}}(x - Y_k) : Y_k \leq x]} - \sqrt{f_{\mathbf{X}}(x)\mathbb{P}(Y_k \leq x)} \right|^2 dx \\
&\leq \sum_{k=1}^{\infty} \int_0^{\delta} \left| \mathbb{E}[f_{\mathbf{X}}(x - Y_k) : Y_k \leq x]^{\frac{1}{p}} - (f_{\mathbf{X}}(x)\mathbb{P}(Y_k \leq x))^{\frac{1}{p}} \right|^p dx
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \int_0^{\delta} \left| \int_0^1 \frac{\mathbb{E}[f'_{\mathbf{X}}(x - sY_k)Y_k : Y_k \leq x]}{p\mathbb{E}[f_{\mathbf{X}}(x - sY_k) : Y_k \leq x]^{\frac{1}{q}}} ds \right|^p dx$$

$$\leq \sum_{k=1}^{\infty} \int_0^{\delta} dx \frac{1}{p^p} \int_0^1 \frac{1}{\mathbb{E}[f_{\mathbf{X}}(x - sY_k) : Y_k \leq x]^{\frac{p}{q}}}$$

$$\times \mathbb{E}\left[\frac{|f'_{\mathbf{X}}(x - sY_k)|Y_k}{f_{\mathbf{X}}(x - sY_k)^{\frac{1}{q}}} f_{\mathbf{X}}(x - sY_k)^{\frac{1}{q}} : Y_k \leq x \right]^p ds$$

$$\leq \frac{1}{p^p} \sum_{k=1}^{\infty} \int_0^{\delta} dx \int_0^1 \mathbb{E}\left[\frac{|f'_{\mathbf{X}}(x - sY_k)|^p |Y_k|^p}{f_{\mathbf{X}}(x - sY_k)^{\frac{p}{q}}} : Y_k \leq x \right] ds$$

$$\leq \frac{1}{p^p} \int_0^{\delta} \frac{|f'_{\mathbf{X}}(x)|^p}{f_{\mathbf{X}}(x)^{\frac{p}{q}}} dx \sum_{k=1}^{\infty} \mathbb{E}[|Y_k|^p : Y_k \leq \delta] < \infty.$$

Next we prove the converse. Since $\mathbf{Y} \in \ell_p^+$ a.s., by Kolmogorov's three series theorem, $\sum_{k=1}^{\infty} \mathbb{P}(Y_k > 1) < \infty$ and $\sum_{k=1}^{\infty} \mathbb{E}[|Y_k|^p : Y_k \leq 1] < \infty$, so that we have $\beta := \inf_k \mathbb{P}(Y_k \leq 1) > 0$ (see also (C.3)). In order to prove the theorem, we shall show Kakutani's criterion (C.1). Decompose

$$\left| \sqrt{\mathbb{E}[f_{\mathbf{X}}(x - Y_k)]} - \sqrt{f_{\mathbf{X}}(x)} \right|^2$$

$$\leq \left| \sqrt{\mathbb{E}[f_{\mathbf{X}}(x - Y_k) : Y_k > 1]} - \sqrt{f_{\mathbf{X}}(x)\mathbb{P}(Y_k > 1)} \right|^2$$

$$+2 \left| \sqrt{\mathbb{E}[f_{\mathbf{X}}(x - Y_k) : Y_k \leq x, Y_k \leq 1]} - \sqrt{f_{\mathbf{X}}(x)\mathbb{P}(Y_k \leq x, Y_k \leq 1)} \right|^2$$

$$+2 \left| \sqrt{f_{\mathbf{X}}(x)\mathbb{P}(Y_k \leq x, Y_k \leq 1)]} - \sqrt{f_{\mathbf{X}}(x)\mathbb{P}(Y_k \leq 1)} \right|^2$$

$$=: \mathbf{U}_k(x) + 2\mathbf{V}_k(x) + 2\mathbf{W}_k(x).$$

Then we have $\sum_{k=1}^{\infty} \int_0^{\infty} \mathbf{U}_k(x) dx \leq 2 \sum_{k=1}^{\infty} \mathbb{P}(Y_k > 1) < \infty$. For $q > 1$ defined by $1/p + 1/q = 1$ we have

$$\sum_{k=1}^{\infty} \int_0^{\infty} \mathbf{V}_k(x) dx \leq \frac{1}{p^p} \int_0^{\infty} \frac{|f'_{\mathbf{X}}(x)|^p}{f_{\mathbf{X}}(x)^{\frac{p}{q}}} dx \sum_{k=1}^{\infty} \mathbb{E}[|Y_k|^p : Y_k \leq 1] < \infty,$$

by the same way as the last part of the necessity, and finally

$$\sum_{k=1}^{\infty} \int_0^{\infty} \mathbf{W}_k(x) dx$$

$$= \sum_{k=1}^{\infty} \int_0^{\infty} \left| \frac{\mathbb{P}(Y_k \leq x, Y_k \leq 1) - \mathbb{P}(Y_k \leq 1)}{\sqrt{\mathbb{P}(Y_k \leq x, Y_k \leq 1)} + \sqrt{\mathbb{P}(Y_k \leq 1)}} \right|^2 f_{\mathbf{X}}(x) dx$$

$$\leq \frac{1}{\beta} \sum_{k=1}^{\infty} \int_0^{\infty} \mathbb{P}(x < Y_k)^2 f_{\mathbf{X}}(x) dx < \infty.$$

■

5. Exponential Distribution

In the case where $\mathbf{X}$ is exponentially distributed, we prove a necessary and sufficient condition for the admissibility of $\mathbf{Y}$ without any additional assumptions on $\mathbf{Y}$.

Theorem 5.1. *Let $\mathbf{X}$ be exponentially distributed and $\mathbf{Y}$ be non-negative. Then $\mathbf{Y}$ is admissible if and only if*

$$\sum_{k=1}^{\infty} \mathbb{P}(\gamma_k \leq Y_k) + \sum_{k=1}^{\infty} e^{-\lambda\gamma_k} + \sum_{k=1}^{\infty} \int_0^{\gamma_k} e^{\lambda y}\mathbb{P}(y < Y_k < \gamma_k)^2 dy < \infty,$$

where $\gamma_k := \sup\left\{x \geq 0 \;\middle|\mathbb{E}\left[e^{\lambda Y_k} : Y_k \leq x\right] < 2\right\}$.

Fact 5.2.

(i) If the distribution of Y_k is continuous, then

$$Z_k(x) = \mathbb{E}[e^{\lambda Y_k} : Y_k \leq x] - 1$$

is also continuous in x, and $\gamma_k < \infty$ implies $Z_k(\gamma_k) = 1$.

(ii) By definition we have $\mathbb{E}[e^{\lambda Y_k} : Y_k < \gamma_k] \leq 2$, and in particular, if $\gamma_k = \infty$ then we have $\mathbb{E}[e^{\lambda Y_k}] \leq 2$.

(iii) If $\gamma_k < \infty$, then we have $2 \leq \mathbb{E}[e^{\lambda Y_k} : Y_k \leq \gamma_k] < \infty$.

Proof of Theorem 5.1. We shall first prove the case where $\lambda = 1$. We use (C.2) for the admissibility of $\mathbf{Y}$. We show that (3.1) is equivalent to

$$\sum_{k=1}^{\infty} \mathbb{P}(\gamma_k \leq Y_k) + \sum_{k=1}^{\infty} e^{-\gamma_k} < \infty \tag{5.1}$$

and that under (3.1), (3.2) is equivalent to

$$\sum_{k=1}^{\infty} \int_0^{\gamma_k} e^{y}\mathbb{P}(y < Y_k < \gamma_k)^2 dy < \infty. \tag{5.2}$$

In order to prove that (3.1) implies (5.1), we have only to consider k with $\gamma_k < \infty$. By Fubini's theorem and Fact 5.2 (iii), we have

$$\begin{aligned} &\mathbb{E}[Z_k(X_k) : Z_k(X_k) \geq 1] = \int_{\gamma_k}^{\infty} e^{-x}\left(\mathbb{E}[e^{Y_k} : Y_k \leq x] - 1\right) dx \\ &= e^{-\gamma_k}\mathbb{E}[e^{Y_k} : Y_k \leq \gamma_k] + \mathbb{P}(Y_k > \gamma_k) - e^{-\gamma_k} \\ &\geq \mathbb{P}(Y_k > \gamma_k) + e^{-\gamma_k} \end{aligned} \tag{5.3}$$

which implies (5.1). On the other hand, since $Y_k \geq 0$ a.s. we have

$$\mathbb{E}[Z_k(X_k) : Z_k(X_k) \geq 1] = e^{-\gamma_k}\mathbb{E}[e^{Y_k} : Y_k < \gamma_k] + \mathbb{P}(Y_k \geq \gamma_k) - e^{-\gamma_k}$$
$$\geq e^{-\gamma_k}\mathbb{P}(Y_k < \gamma_k) + \mathbb{P}(\gamma_k \leq Y_k) - e^{-\gamma_k} = (1 - e^{-\gamma_k})\mathbb{P}(\gamma_k \leq Y_k).$$

Since (5.1) implies $\gamma_k \to \infty$ as $k \to \infty$, (3.1) implies (5.1).

Conversely, we have by Fact 5.2 (ii),

$$\mathbb{E}[Z_k(X_k) : Z_k(X_k) \geq 1] = e^{-\gamma_k}\mathbb{E}[e^{Y_k} : Y_k < \gamma_k] + \mathbb{P}(Y_k \geq \gamma_k) - e^{-\gamma_k}$$
$$\leq 2e^{-\gamma_k} + \mathbb{P}(Y_k \geq \gamma_k) - e^{-\gamma_k} \;=\; e^{-\gamma_k} + \mathbb{P}(Y_k \geq \gamma_k),$$

so that (5.1) implies (3.1). Therefore (3.1) is equivalent to (5.1).

Next, assume (3.1) and denote by $\{Y_k'\}$ an independent copy of $\{Y_k\}$. Then by Fubini's theorem, we have

$$\begin{aligned}
&\mathbb{E}[Z_k(X_k)^2 : Z_k(X_k) < 1] \\
&= \int_0^{\gamma_k} e^{-x}\left(\mathbb{E}[e^{Y_k+Y_k'} : Y_k, Y_k' \leq x] - 2\mathbb{E}[e^{Y_k} : Y_k \leq x] + 1\right) dx \\
&= \mathbb{E}\left[e^{Y_k+Y_k'}\int_{Y_k\vee Y_k'}^{\gamma_k} e^{-x}dx : Y_k, Y_k' \leq \gamma_k\right] \\
&\quad -2\mathbb{E}[e^{Y_k}\int_{Y_k}^{\gamma_k} e^{-x}dx : Y_k \leq \gamma_k] + \int_0^{\gamma_k} e^{-x}dx \\
&= \mathbb{E}[e^{Y_k\wedge Y_k'} - e^{-\gamma_k+Y_k+Y_k'} : Y_k, Y_k' \leq \gamma_k] \\
&\quad -2\mathbb{P}(Y_k \leq \gamma_k) + 2e^{-\gamma_k}\mathbb{E}[e^{Y_k} : Y_k \leq \gamma_k] + 1 - e^{-\gamma_k} \\
&= \mathbb{E}[e^{Y_k\wedge Y_k'} : Y_k, Y_k' \leq \gamma_k] - e^{-\gamma_k}\mathbb{E}[e^{Y_k} : Y_k \leq \gamma_k]^2 + \mathbb{P}(Y_k > \gamma_k) \\
&\quad -\mathbb{P}(Y_k \leq \gamma_k) + 2e^{-\gamma_k}\mathbb{E}[e^{Y_k} : Y_k \leq \gamma_k] - e^{-\gamma_k} \\
&= \mathbb{E}[e^{Y_k\wedge Y_k'} : Y_k, Y_k' < \gamma_k] - \mathbb{P}(Y_k < \gamma_k) - e^{-\gamma_k}\mathbb{E}[e^{Y_k} : Y_k < \gamma_k]^2 \\
&\quad +\mathbb{P}(Y_k \geq \gamma_k) + 2e^{-\gamma_k}\mathbb{E}[e^{Y_k} : Y_k < \gamma_k] - e^{-\gamma_k},
\end{aligned}$$

where $a \vee b := \max\{a, b\}$. By Fact 5.2 (ii) and by (5.1), the last four terms in the final expression are summable. Therefore, under (3.1), (3.2) is equivalent to the convergence of the series:

$$\sum_{k=1}^{\infty}\left\{\mathbb{E}[e^{Y_k\wedge Y_k'} : Y_k, Y_k' < \gamma_k] - \mathbb{P}(Y_k < \gamma_k)\right\}.$$

Since $\mathbb{P}(Y_k < \gamma_k) = \mathbb{P}(Y_k < \gamma_k, Y_k' < \gamma_k) + \mathbb{P}(Y_k < \gamma_k, Y_k' \geq \gamma_k)$, we have

$$\sum_{k=1}^{\infty} \left\{ \mathbb{E}[e^{Y_k \wedge Y_k'} : Y_k, Y_k' < \gamma_k] - \mathbb{P}(Y_k < \gamma_k) \right\}$$
$$= \sum_{k=1}^{\infty} \mathbb{E}[e^{Y_k \wedge Y_k'} - 1 : Y_k, Y_k' < \gamma_k] - \sum_{k=1}^{\infty} \mathbb{P}(Y_k < \gamma_k)\mathbb{P}(Y_k' \geq \gamma_k),$$

where the second sum in the right expression is finite by (5.1). Thus under (3.1), (3.2) is equivalent to

$$\int_0^{\gamma_k} e^y \mathbb{P}(y < Y_k < \gamma_k)^2 dy = \mathbb{E}\left[\int_0^{Y_k \wedge Y_k'} e^y dy : Y_k, Y_k' < \gamma_k \right]$$
$$= \sum_{k=1}^{\infty} \mathbb{E}[e^{Y_k \wedge Y_k'} - 1 : Y_k, Y_k' < \gamma_k] < \infty.$$

Therefore, (3.2) is equivalent to (5.2) under (3.1).

Combining (5.1) and (5.2), we obtain a necessary and sufficient condition for $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ as

$$\sum_{k=1}^{\infty} \mathbb{P}(\gamma_k \leq Y_k) + \sum_{k=1}^{\infty} e^{-\gamma_k} + \sum_{k=1}^{\infty} \int_0^{\gamma_k} e^y \mathbb{P}(y < Y_k < \gamma_k)^2 dy < \infty. \tag{5.4}$$

Finally We shall prove the case where $\lambda \neq 1$. In this case we have $\gamma_k = \sup\left\{x \geq 0 \mid \mathbb{E}\left[e^{\lambda\gamma_k} : Y_k \leq x\right] < 2\right\}$. We have $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ if and only if $\mu_{\lambda\mathbf{X}+\lambda\mathbf{Y}} \sim \mu_{\lambda\mathbf{X}}$. Since $d\mu_{\lambda X_1}(x) = e^{-x}\mathbf{I}_{[0,\infty)}(x)dx$, replacing $\mathbf{Y}$ with $\lambda\mathbf{Y}$ in (5.4), we have the conclusion. ■

As a corollary of Theorem , we obtain a necessary and sufficient condition in the case where $\mathbf{X}$ is exponentially distributed and $\mathbf{Y}$ is a Bernoulli sequence.

Corollary 5.3. *Let $\mathbf{X}$ be exponentially distributed and $\mathbf{Y} = \{\varepsilon(a_k, p_k)\}$ be a Bernoulli sequence. Then $\mathbf{Y}$ is admissible if and only if*

$$\sum_{k=1}^{\infty} \frac{e^{\lambda a_k} - 1}{(e^{(\lambda \vee \sigma_k)a_k} - 1)(e^{\sigma_k a_k} - 1)} < \infty,$$

where $\sigma_k := (1/a_k)\log\{(1 + p_k)/p_k\}$.

Example 5.4. Let $f_{\mathbf{X}}(x) = e^{-x}\mathbf{I}_{[0,\infty)}(x)$ and $\mathbf{Y} = \{\varepsilon(a_k, p_k)\}$, where $a_k = \log(k + 2), p_k = 1/(k + 1), k \geq 1$. Then (C.6) and (C.7) hold but $\mathbf{Y}$ is not admissible.

Proof. We have

$$\sum_{k=1}^{\infty}\int_0^{\infty}\mathbb{P}(Y_k>x)^2 f_{\mathbf{X}}(x)dx = \sum_{k=1}^{\infty}\int_0^{a_k} p_k^2 e^{-x}dx = \sum_{k=1}^{\infty} p_k^2(1-e^{-a_k})$$
$$= \sum_{k=1}^{\infty}\frac{1}{k^2+3k+2} < \infty$$

and

$$\sum_{k=1}^{\infty}\int_0^{\infty}\mathbb{E}[Y_k;Y_k\le x]^2 f_{\mathbf{X}}(x)dx = \sum_{k=1}^{\infty}\int_{a_k}^{\infty}(a_kp_k)^2 e^{-x}dx = \sum_{k=1}^{\infty} a_k^2p_k^2e^{-a_k}$$
$$= \sum_{k=1}^{\infty}\frac{(\log(k+2))^2}{(k+1)^2(k+2)} < \infty.$$

We show $\mathbf{Y}$ is not admissible. Since $\sigma_k := \frac{1}{a_k}\log\frac{1+p_k}{p_k} = 1$, $\mathbf{Y}$ is admissible if and only if

$$\sum_{k=1}^{\infty}\frac{e^{a_k}-1}{(e^{a_k}-1)(e^{a_k}-1)} = \sum_{k=1}^{\infty}\frac{1}{e^{a_k}-1} < \infty.$$

But

$$\sum_{k=1}^{\infty}\frac{1}{e^{a_k}-1} = \sum_{k=1}^{\infty}\frac{1}{k+1} = \infty.$$ ■

Example 5.5. Let $f_{\mathbf{X}}(x) = e^{-x}\mathbf{I}_{[0,\infty)}(x)$ and $\mathbf{Y} = \big\{\varepsilon(a_k,1/2)\big\}, k\ge 1$ be a Bernoulli sequence. Then $\mathbf{Y}$ is admissible if and only if $\boldsymbol{a} = \{a_k\}\in\ell_1^+$.

Proof. We may assume $a_k \le \log 3$. Then $\sigma_k = \frac{\log 3}{a_k}$ and

$$\sum_{k=1}^{\infty}\frac{e^{a_k}-1}{(e^{(1\vee\sigma_k)a_k}-1)(e^{\sigma_k a_k}-1)} = \sum_{k=1}^{\infty}\frac{e^{a_k}-1}{4}.$$

$\sum_{k=1}^{\infty}\frac{e^{a_k}-1}{4} < \infty$ if and only if $\boldsymbol{a} = \{a_k\}\in\ell_1^+$. ■

In the case where the distributions of all Y_k's are continuous, Theorem 5.1 is simplified as follows.

Theorem 5.6. *Let* $\mathbf{X}$ *be exponentially distributed and the distributions of* Y_k*'s be continuous. Then we have* $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$ *if and only if*

$$\sum_{k=1}^{\infty}\mathbb{P}(\gamma_k\le Y_k) + \sum_{k=1}^{\infty}\int_0^{\gamma_k} e^{\lambda y}\mathbb{P}(y<Y_k<\gamma_k)^2dy < \infty.$$

Proof. Let the distributions of Y_k's be continuous. Then by Fact 5.2 (i), we have $\mathbb{E}\left[e^{\lambda Y_k} : Y_k \le \gamma_k\right] - 1 = 1$ and it follows that

$$\lambda \int_0^{\gamma_k} e^y \mathbb{P}(y < Y_k \le \gamma_k) dy = \mathbb{E}\left[e^{\lambda Y_k} - 1 : Y_k \le \gamma_k\right]$$
$$= \mathbb{E}\left[e^{\lambda Y_k} : Y_k \le \gamma_k\right] - 1 + \mathbb{P}(Y > \gamma_k) = 1 + \mathbb{P}(Y_k > \gamma_k),$$

which implies

$$\mathbb{E}[e^{\lambda Y_k} - 1 : Y_k \le \gamma_k] = \lambda \int_0^{\gamma_k} e^{\lambda y} \mathbb{P}(y < Y_k \le \gamma_k) dy = 1 + \mathbb{P}(Y_k > \gamma_k).$$

If $\gamma_k < \infty$, then by the Schwarz inequality

$$1 \le [1 + \mathbb{P}(Y_k > \gamma_k)]^2 = \lambda^2 \left| \int_0^{\gamma_k} e^{\lambda y} \mathbb{P}(y < Y_k \le \gamma_k)\ dy \right|^2$$
$$\le \lambda^2 \int_0^{\gamma_k} e^{\lambda u} du \int_0^{\gamma_k} e^{\lambda y} \mathbb{P}(y < Y_k \le \gamma_k)^2 dy$$
$$\le \lambda e^{\lambda \gamma_k} \int_0^{\gamma_k} e^{\lambda y} \mathbb{P}(y < Y_k \le \gamma_k)^2 dy,$$

which implies $e^{-\lambda \gamma_k} \le \lambda \int_0^{\gamma_k} e^{\lambda y} \mathbb{P}(y < Y_k \le \gamma_k)^2 dy$. Therefore we have

$$\sum_{k=1}^{\infty} e^{-\lambda \gamma_k} = \sum_{k:\gamma_k < \infty} e^{-\lambda \gamma_k} \le \sum_{k=1}^{\infty} \int_0^{\gamma_k} e^{\lambda y} \mathbb{P}(y < Y_k \le \gamma_k)^2 dy$$

and if the distributions of Y_k's are continuous, then $\mathbb{P}(y < Y_k < \gamma_k) = \mathbb{P}(y < Y_k \le \gamma_k)$, and Theorem 5.1 implies the assertion. ■

Example 5.7. Let $f_{\mathbf{X}}(x) = e^{-x}\mathbf{I}_{[0,\infty)}(x)$ and Y_k obey to the uniform distribution

$$d\mu_k(y) = \frac{1}{a_k}\mathbf{I}_{[0,a_k]}(y)dy,\ k \ge 1,$$

where $a_k > 0$. Then we have $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ if and only if $\sum_{k=1}^{\infty} a_k < \infty$.

Proof. Let $\kappa > 0$ be the unique positive solution of $e^t = 1 + 2t$.

For a_k with $a_k \ge \tau$, we have $\gamma_k = \log(1 + 2a_k)$ and

$$\mathbb{P}(Y_k \ge \gamma_k) = \frac{1}{a_k} \int_{\gamma_k}^{a_k} dy = 1 - \frac{\log(1 + 2a_k)}{a_k},$$

and, by Theorem 5.6, $\sum_{k=1}^{\infty} \mathbb{P}(Y_k \ge \gamma_k) < \infty$ implies $\lim_k a_k = \kappa > 0$.

On the other hand, we have

$$\int_0^{\gamma_k} e^y \mathbb{P}(y < Y_k < \gamma_k)^2 dy = \frac{2}{a_k^2}\left[\mathrm{e}^{\gamma_k} - \left(1 + \gamma_k + \frac{\gamma_k^2}{2}\right)\right]$$
$$\geq \frac{|\log(1+2a_k)|^3}{3a_k^2} \to \frac{|\log(1+2\kappa)|^3}{3\kappa^2} > 0,$$

which contradicts to $\sum_{k=1}^{\infty}\int_0^{\gamma_k} e^y \mathbb{P}(y < Y_k < \gamma_k)^2 dy < \infty$. Therefore, but for finite number of a_k's, we may assume $a_k < \kappa$.

For a_k such that $a_k < \kappa$, we have $\gamma_k = \infty$ and

$$\int_0^{\gamma_k} e^y \mathbb{P}(y < Y_k < \gamma_k)^2 dy = \frac{2}{a_k^2}\left[\mathrm{e}^{a_k} - \left(1 + a_k + \frac{a_k^2}{2}\right)\right] \geq \frac{a_k}{3},$$

and Theorem 5.6 implies $\sum_{k=1}^{\infty} a_k < \infty$.

Conversely, $\sum_{k=1}^{\infty} a_k < \infty$ implies $\lim_k a_k = 0$ so that, without loss of generality, we may assume that $a_k < \kappa$ and $\gamma_k = \infty$. Then we have $\int_0^{\gamma_k} e^y \mathbb{P}(y < Y_k < \gamma_k)^2 dy \leq a_k e^{\kappa}/3 < \infty$ and Theorem 5.6 proves the example. ■

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Aoi Honda
Kyushu Institute of Technology,
680-4, Kawazu,
Iizuka 820-8502,
Japan
and
Université de Paris I Panthéon-Sorbonne,
90 rue de Tolbiac,
75634 Paris,
France
E-mail: aoi@ces.kyutech.ac.jp

Yoshiaki Okazaki
Kyushu Institute of Technology,
680-4, Kawazu,
Iizuka 820-8502,
Japan
E-mail: okazaki@ces.kyutech.ac.jp

Hiroshi Sato
Kyushu University,
Faculty of Mathematics,
Fukuoka 812-8581,
Japan
E-mail: sato@math.kyushu-u.ac.jp