TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 1, pp. 253-268, February 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON ANALYTIC PROPERTIES AND CHARACTER ANALOGS OF HARDY SUMS

Yilmaz Simsek

Abstract. The aim of this paper is to define $Y(h,k)$ sum which is related to the Hardy's sums $s_5(h,k)$. On the semi-group G, matrix operation of this sum is defined. Substituting mediants of Farey fractions into the matrix operation, $Y(h,k)$ sum is generalized. By using contour integration, the reciprocity theorem of the $Y(h,k)$ sum is proved. Moreover, by using $L(1,\chi)$ function and Gauss sums, generalized character analogs of the Hardy sums are found.

1. Introduction, Definitions and Notations

In [9], Iwasawa gave fundamental properties of the Dirichlet L-function, Dirichlet character and generalized Bernoulli numbers. In [23], Srivastava and Choi gave fundamental properties of the zeta functions, L-functions and Bernoulli numbers. Zhang [22] studied on the asymptotic behavior of the mean value of Dedekind sums, and gave the relation between Dedekind sums and $L(1,\chi)$. In [6], Berndt and Goldberg studied analytic properties of Hardy sums. They proved the relations between Hardy sums and finite trigonometric sums. In [12, 14, 15, 17, 18], the author studied on Hardy sums. He gave relations between these sums, generalized Bernoulli numbers, Hurwitz zeta function and L-function.

We summarize our work as follows:

In Section 2, we give analytic properties of the Hardy's sums $s_5(h,k)$ and define new sum $Y(h,k)$ and matrix operation of this sums. We prove Reciprocity Theorem of the $Y(h,k)$.

In section 3, we construct infinite series representations for $Y(h,k)$. We also prove the relation between $Y(h,k)$, $L(1,\chi)$, $G(n,\chi)$(Gauss sum),and $B_{1,\chi}$.

The Dedekind sum $s(h,k)$, arising in the theory of the Dedekind-eta function, is defined by

Received February 18, 2002, accepted July 26, 2007.
Communicated by H. M. Srivastava.
2000 *Mathematics Subject Classification*: Primary 11F20; Secondary 11B68, 11S40, 11S80, 30D05.
Key words and phrases: $((x))$-function, Dedekind sum, Hardy sums, Farey fraction, Dirichlet character and L-function, Generalized Bernoulli numbers.

$$s(h,k)=\sum_{j \bmod k}((\frac{j}{k}))((\frac{hj}{k})),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & x \text{ is not an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

and h is an integer and k is a positive integer and $(h,k)=1$ cf. ([1, 16, 18, 19]). Asai [4] defined $D(h,k)$ sum which is related to the Hardy's sums $s(h,k)$. He also defined matrix operation of this sum. He gave some properties of this sum.

The most important property of Dedekind sums is the following reciprocity theorem.

If h and k are positive integers with $(h,k)=1$, then

$$s(h,k)+s(k,h)=-\frac{1}{4}+\frac{1}{12}(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}). \tag{1.1}$$

Proofs of (1.1) were given by Apostol [1] and the author [18] see also cf. ([6, 20]).

In 1905, Hardy [8], by using contour integration, proved some reciprocity theorems and started eleven more reciprocity theorems for some similar arithmetical sums. In recent years, five of Hardy's reciprocity theorems were studied by Berndt [5]. Goldberg [7] proved these reciprocity theorems from Berndt's transformation formula [5], which are related to the logarithms of the classical theta functions (cf. [21], Chapter 21). Berndt and Goldberg [6] studied on theta functions and Hardy sums. They proved reciprocity theorems of these sums. They also evaluated certain non-absolutely convergent-double series in terms of these sums. In recent years, proof of Hardy's reciprocity theorems which do not depend on Berndt's transformations formulae have been given by Apostol and Vu [3], Bernd and Goldberg [6], Sitaramachandraro [20] and by the author ([12, 14, 18]).

Hardy sums (Hardy-Berndt sums) are defined by (cf. [6, 7, 20, 3, 14, 17, 18, 15]):

Let h and k be integers with $(h,k)=1$.

$$\begin{aligned} S(h,k) &= \sum_{a \bmod k}(-1)^{a+1+[\frac{ah}{k}]}, \\ s_1(h,k) &= \sum_{a \bmod k}(-1)^{[\frac{ah}{k}]}((\frac{a}{k})), \\ s_2(h,k) &= \sum_{a \bmod k}(-1)^{a}((\frac{a}{k}))((\frac{ah}{k})), \\ s_3(h,k) &= \sum_{a \bmod k}(-1)^{a}((\frac{ah}{k})), \\ s_4(h,k) &= \sum_{a \bmod k}(-1)^{[\frac{ah}{k}]}, \\ s_5(h,k) &= \sum_{j=1}^{k}(-1)^{j+[\frac{hj}{k}]}((\frac{j}{k})). \end{aligned} \tag{1.2}$$

The reciprocity theorem of the $s_5(h,k)$ sum is given by the following theorem:

Theorem 1. *Let h and k be integers with $(h,k)=1$.*
If h and k are odd, then

$$s_5(h,k)+s_5(k,h)=\frac{1}{2}-\frac{1}{2hk}. \tag{1.3}$$

Different proofs of the above theorem were given by cf. ([6, 7, 20, 3, 14, 17, 18]).

Noted that by using (1.1), Sitaramachandraro [20] proved Theorem 1. He also expressed each of the Hardy sums in terms of Dedekind sums. He proved the following theorem:

Theorem 2. *Let h and k be coprime positive integers. If $h+k$ is even, then*

$$s_5(h,k)=-10s(h,k)+4s(2h,k)+4s(h,2k). \tag{1.4}$$

and if $h+k$ is odd, then

$$s_5(h,k)=0. \tag{1.5}$$

Farey fractions are defined as follows cf. (see for detail [1] and [13]):

The set of Farey fractions of order n, denoted by F_n, is the set of reduced fractions in the closed interval $[0,1]$ with denominators $\leq n$, listed in increasing order of magnitude.

If $h/k<H/K$ are adjacent Farey fractions, then it is known that $hK-kH=-1$. The mediant of adjacent Farey fractions $h/k<H/K$ is $(h+H)/(k+K)$. It satisfies the inequality $h/k<(h+H)/(k+K)<H/K$. The following inequality can be obtained by repeating the calculation of mediants n-times successively [13]:

$$\frac{h}{k}<\frac{h+H}{k+K}<\ldots<\frac{h+nH}{k+nK}<\frac{H}{K}. \tag{1.6}$$

We give arithmetic properties of the $s(h,k)$ and $s_5(h,k)$ as follows:

Dedekind sums have the following properties cf. ([1], [12]) :

Since $((-x))=-((x))$, we have

$$s(-h,k)=-s(h,k)$$

and

$$s(h,-k)=s(h,k).$$

If $hh^{'}\equiv 1(\bmod k)$, then

$$s(h^{'},k)=s(h,k).$$

If $h/k < H/K$ are adjacent Farey fractions, then $hK - kH = -1$. Thus, we have

$$hK \equiv -1(\, mod\, k), \text{ and } Hk \equiv 1(\, mod\, K), \tag{1.7}$$

and also that

$$\frac{H}{K} - \frac{h}{k} = \frac{1}{kK}.$$

By using the above relations, we have

$$s(h,k) = -s(K,k),$$

$$s(H,K) = s(k,K).$$

Observe that by using (1.6) and the above relations, generalized Dedekind sums are defined cf. [13].

By using arithmetic properties of the Dedekind sums and Theorem 2, we obtain similar relations for $s_5(h,k)$ as follows cf. [12]:

By using (1.7), we have

$$s_5(h,k) = -s_5(K,k),$$

and

$$s_5(H,K) = s_5(k,K).$$

By substituting the above relations into (1.3), we have

$$s_5(h,k) - s_5(H,K) = -\frac{1}{2} + \frac{1}{2kK}$$

cf. [12]. By using above relations, we obtained the following theorem:

Theorem 3. ([12]). *If $h/k < H/K$ are adjacent Farey fractions, then we have*

$$s_5(h+H, k+K) = \frac{s_5(h,k) + s_5(H,K)}{2} + \frac{k-K}{4kK(k+K)}. \tag{1.8}$$

For each pair (h,k) of relatively prime integers with positive k, or equivalently, for each reduced fraction h/k, we define new sum, $Y(h,k)$ as follows:

$$Y(h,k) = 4ks_5(h,k),$$

where $s_5(h,k)$ is the Hardy's sums.

Berndt and Goldberg[6] represented Hardy sums by the finite trigonometric sums. By using these representations, we define $Y(h,k)$ sum by the finite trigonometric sum as follows:

$$Y(h,k) = 2 \sum_{\substack{j=1 \\ j \neq \frac{k+1}{2}}}^{k} \tan\left(\frac{\pi h(2j-1)}{2k}\right) \cot\left(\frac{\pi(2j-1)}{2k}\right) \tag{1.9}$$

By using arithmetic properties of the $s(h,k)$ and $s_5(h,k)$ sums, we obtain arithmetic properties of the $Y(h,k)$ sums as follows:

$$Y(-h,k) = -Y(h,k), \tag{1.10}$$

if $hK \equiv -1(\, mod\, k)$, then

$$Y(K,k) = -Y(h,k), \tag{1.11}$$

if $Hk \equiv 1(\, mod\, K)$, then

$$Y(k,K) = Y(H,K). \tag{1.12}$$

2. Main Theorems on $Y(h,k)$

Here, we can use notations of the Asai[4] and Salie[11].

Let G denote the semi-group consisting of all two-by-two matrices with non-negative integers coefficients and determinant one. If $\begin{bmatrix} h & H \\ k & K \end{bmatrix}$ is an element of G, then h/k and H/K are both non-negative reduced fractions, the former of which may happen to be $\infty = 1/0$, then these, h/k, H/K are so-called adjacent Farey fractions, and vice-versa cf. [4].

The reciprocity theorem of the $Y(h,k)$ sums is given as follows:

Theorem 4. *If h and k are positive odd integers and $(h,k) = 1$, then we have*

$$hY(h,k) + kY(k,h) = 2hk - 2. \tag{2.1}$$

Proof. We shall give just a brief sketch as the details are similar to those in cf. ([6, 18]).

We define $F(z) = \cot\pi z \tan\pi hz \tan\pi kz$. Denote by $\mathfrak{R}$ the contour obtained from the rectangle of vertices $\pm iB$, $1/2 \pm iB$. The function $F(z)$ has pole $z = 0$ and $z = 1/2$ on this contour; therefore, if we want to integrate $F(z)$, we have to modify the contour by indentations at this points. We take as indentations identical small semicircles leaving $z = 0$ inside, $z = 1/2$ outside. Clearly, $F(z+1) = F(z)$, so that the integrals along the vertical sides (including the indentations) cancel each other. Also, $\lim_{B\to+\infty} \cot(x+iB) = -i$ and $\lim_{B\to+\infty} \tan(x+iB) = i$ uniformly for $0 \le x \le 1/2$, so that $\lim_{B\to+\infty} F(x+iB) = i$, and similarly $\lim_{B\to-\infty} F(x+iB) = -i$, both uniformly for $0 \le x \le 1/2$. Observe that $F(z) = F(x+iy)$ is holomorphic on $\mathfrak{R}$, t follows that $\int_{\mathfrak{R}} F(z)dz$ is independent of B, consequently, $\int_{\mathfrak{R}} F(z)dz = \lim_{B\to+\infty} \int_{\mathfrak{R}} F(z)dz = 2i$, and by the Residue Theorem, we have

$$\frac{1}{2\pi i}\int_{\mathfrak{R}} F(z)dz = \frac{1}{\pi} = \mathfrak{I}, \tag{2.2}$$

where $\mathfrak{I}$ stands for the sum of the residues of $F(z)$ at its singularities inside $\mathfrak{R}$.

Next, we calculate $\mathfrak{I}$ by the use of the Residue Theorem. The integrand has simple poles at $z_1 = (2n-1)/(2h), (n = 1, 2, ..., h-1)$ and $z_2 = (2m-1)/(2k), (m = 1, 2, ..., k-1)$. The poles at $z \neq 0$ are indeed all simple, because $(2n-1)/(2h) = (2m-1)/(2k)$ is ruled out by $(h, k) = 1$. $F(z)$ has a double pole at $z_3 = 1/2$.

We now pass to computation of the residues $(Res(F(z), z_k)(k = 1, 2, 3)$ denote the residue of the integrand of $\mathfrak{I}$). The residues at the z_1, z_2, and z_3 are

$$\operatorname{Re}s(F(z), z_1) = \frac{1}{h\pi}\tan(\frac{k\pi(2n-1)}{2h})\cot(\frac{\pi(2n-1)}{2h}),$$

$$\operatorname{Re}s(F(z), z_2) = \frac{1}{k\pi}\tan(\frac{h\pi(2m-1)}{2k})\cot(\frac{\pi(2m-1)}{2k}), \operatorname{Res}(F(z), z_3) = \frac{1}{hk\pi},$$

respectively. Now, by using Residue Theorem,

$$\mathfrak{I} = \frac{1}{k\pi}\sum_{\substack{m=1\\ m\neq\frac{k+1}{2}}}^{k}\tan(\frac{h\pi(2m-1)}{2k})\cot(\frac{\pi(2m-1)}{2k}) + \frac{1}{h\pi}\sum_{\substack{n=1\\ n\neq\frac{h+1}{2}}}^{h}\tan(\frac{k\pi(2n-1)}{2h})\cot(\frac{\pi(2n-1)}{2h}) + \frac{1}{hk\pi}. \tag{2.3}$$

Now, for $(h, k) = 1$, by using (1.9) in (2.3) we have

$$\mathfrak{I} = \frac{1}{2k\pi}Y(h, k) + \frac{1}{2h\pi}Y(k, h) + \frac{1}{2kh\pi}.$$

Combine the above with (2.2), we obtain (2.1). Thus we arrive at the desired result. ∎

Observe that by using the definition of $Y(h, k)$ and (1.3), we arrive at the another proof of the above theorem.

Lemma 1. *If $H/K < h/k$ are both non-negative adjacent Farey fractions, then we have*

$$Y(h + H, k + K) = Y(h, k) + Y(H, K) + k - K. \tag{2.4}$$

Proof. We multiply both sides of (1.8) by $4(k + K)$, we have

$$Y(h + H, k + K) = \frac{Y(h, k) + Y(H, K)}{2} \tag{2.5}$$

$$+2Ks_5(h,k)+2ks_5(H,K)+\frac{k-K}{kK}.$$

By using arithmetic properties of $s_5(h,k)$ in (1.3), we obtain

$$-2s_5(K,k)-2s_5(k,K)+1=\frac{1}{kK}. \tag{2.6}$$

By substituting (2.6) into (2.5) and after some elementary calculations, then we arrive at (2.4). ■

Theorem 5. *If* $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is an element of* G *and* h/k *is a non-negative reduced fraction, then we have*

$$Y(ah+bk,ch+dk)=hY(a,c)+kY(b,d)+dY(h,k)-cY(k,h)-dh+ck.$$

Proof. The proof of this theorem is similar to that of Asai [4]. Let $A=T_+=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or $T_-=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Since the two elements T_+ and T_- generate the semi-group G, the general formula is obtained by mathematical induction method. In fact we deduce the formula of the case T_{+A} or T_{-A} from one of the cases A, where it must be noticed that the element $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$ can be obtained by exchanging T_+, and T_- for each other in the word expression of A by then. The proof is completed. ■

Remark 1. Lemma 1 is special case of Theorem 5, namely, $h/k=1/1$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is replaced by $\begin{bmatrix} h & H \\ k & K \end{bmatrix}$.

We define a matrix operation of the $Y(h,k)$ sum. This matrix operation is similar to that of Asai [4].

For each element $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of G let us put

$$Y(A)=\begin{bmatrix} Y(a,c) & Y(b,d) \\ Y(c,a) & Y(d,b) \end{bmatrix},$$

and

$$C(A)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(Y(A)A^{-1}-I)$$

where I denotes the identity matrix and A^{-1} denotes inverse of a matrix A.

Let A and B be elements of G. We give $C(AB)$ operation by the following lemma.

Lemma 2. *If A and B are elements of G, then we have*

$$C(AB) = C(A) + AC(B)A^{-1}.$$

Proof. Here, the proof of this lemma is similar to that of Asai [4]. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} h & H \\ k & K \end{bmatrix}$ and $AB = \begin{bmatrix} u & v \\ x & y \end{bmatrix}$, from Theorem 5, we have

$$Y(u,x) = (d,-c)\begin{pmatrix} Y(h,k) \\ Y(k,h) \end{pmatrix} + (Y(a,c), Y(b,d))\begin{pmatrix} h \\ k \end{pmatrix} - (d,-c)\begin{pmatrix} h \\ k \end{pmatrix}.$$

Thus, we have

$$Y(x,u) = (-b,a)\begin{pmatrix} Y(h,k) \\ Y(k,h) \end{pmatrix} + (Y(c,a), Y(d,b))\begin{pmatrix} h \\ k \end{pmatrix} - (-b,a)\begin{pmatrix} h \\ k \end{pmatrix},$$

since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} k \\ h \end{pmatrix} = \begin{pmatrix} u \\ x \end{pmatrix}$. If we replace h and k with H and K, respectively in these right-hand sides, then u and x are replaced by v and y respectively. Hence, we get

$$Y(AB) = Y(A)B + (A^{-1})^t Y(B) - (A^{-1})^t B, \tag{2.7}$$

where A^t denotes transpose of a matrix A, since

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(A^{-1})^t = A\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \qquad \blacksquare$$

By using (2.7), we obtain the following relations:

Theorem 6. *If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an element of G, h/k and H/K are non-negative reduced adjacent fractions, then we have*

$$\begin{aligned} &Y(ah+bk, ch+dk) \\ &= hY(a,c) + kY(b,d) + dY(h,k) - cY(k,h) - dh + ck, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &Y(ch+dk, ah+bk) \\ &= hY(c,a) + kY(d,b) - bY(h,k) + aY(k,h) - ak + bh, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &Y(aH+bK, cH+dK) \\ &= HY(a,c) + KY(b,d) + dY(H,K) - cY(K,H) - dH + cK, \end{aligned} \tag{2.10}$$

$$\begin{aligned}&Y(cH+dK, aH+bK)\\ &= HY(a,c)+KY(d,b)-bY(H,K)+aY(K,H)-aK+bH.\end{aligned}\tag{2.11}$$

Proof. By applying Theorem 3 and arithmetic properties of $Y(h,k)$, after some elementary calculations, thus we arrive at (2.8)-(2.11). ■

Observe that by substituting (2.7) and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, $B = \begin{bmatrix} h & H \\ k & K \end{bmatrix} \in G$ into the definitions of $Y(A)$ and $Y(AB)$, we obtain the another proof of the above theorem.

Remark 2. If we replace h/k and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $1/1$ and $\begin{bmatrix} h & H \\ k & K \end{bmatrix}$, respectively, in (2.8), thus we obtain (2.4).

Theorem 7.
If $h/k < ((n-1)h+H)/((n-1)k+K)$ are both non-negative adjacent Farey fractions, then we have

$$\begin{aligned}Y(nh+H, nk+K) &= nY(h,k)+Y((n-1)h+H,(n-1)k+K)\\ &\quad +(2-n)k-K,\end{aligned}\tag{2.12}$$

or

$$\begin{aligned}Y(nh+H, nk+K) &= nY(h,k)+Y(H,K)-nK+\frac{n(3-n)k}{2},\\ Y(nk+K, nh+H) &= Y(k,h)+Y((n-1)k+K,(n-1)h+H)\\ &\quad +H+(n-2)h.\end{aligned}\tag{2.13}$$

Proof. For the proof, we use (2.7).
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} h & (n-1)h+H \\ k & (n-1)k+K \end{bmatrix}$ be elements of G. From the definitions of $Y(A)$ and (2.7), we have

$$\begin{aligned}&Y(a((n-1)h+H)+b((n-1)k+K), c((n-1)h+H)+d((n-1)k+K))\\ &= ((n-1)h+H)Y(a,c)+((n-1)k+K)Y(b,d)+dY((n-1)h\\ &+H,(n-1)k+K)-cY((n-1)k+K,(n-1)h+H)-d((n-1)h+H)\\ &+c((n-1)k+K),\end{aligned}\tag{2.14}$$

$$\begin{aligned}&Y(c((n-1)h+H+d((n-1)k+K), a((n-1)h+H)+b((n-1)k+K)\\ &= ((n-1)h+H)Y(c,a)+((n-1)k+K)Y(d,b)-bY((n-1)h\\ &+H,(n-1)k+K)+aY((n-1)k+K,(n-1)h+H)\\ &+b((n-1)h+H)-a((n-1)k+K.\end{aligned}\tag{2.15}$$

Thus, by replacing h/k and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $1/1$ and $\begin{bmatrix} h & (n-1)h+H \\ k & (n-1)k+K \end{bmatrix}$ respectively in (2.8) and (2.9) then we have (2.12) and (2.13). Thus we arrive at the desired result. ■

Remark 3. The above theorem also can be proved by mathematical induction. We also note that by replacing $((n-1)h+H)/((n-1)k+K) < H/K$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $1/1$ and $\begin{bmatrix} h & (n-1)h+H \\ k & (n-1)k+K \end{bmatrix}$ in (2.14) and (2.15) then we obtain (2.12) and (2.13).

Corollary 1. *If $(h+(n-1)H)/(k+(n-1)K) < H/K$ are both non-negative adjacent Farey fractions, then*

$$Y(h+nH,k+nK) = Y(H,K)+Y(h+(n-1)H,k+(n-1)K)+k+(n-2)K,$$

$$Y(k+nK,h+nH) = Y(K,H)+Y(k+(n-1)K,h+(n-1)H)-h+(2-n)H.$$

The proof of this Corollary is similar to that of Theorem 7.

3. Character Analogs of the Hardy Sums

In this section, we give relations generalized bernoulli numbers, Dirichlet L-functions and Gauss sums. We use the notations of cf. ([2, 9, 22, 23]). Here, we prove the following Theorems:

Theorem 8. *Let h and k be positive integers with $(h,k)=1$. Let χ be any Dirichlet character with conductor $2k$ and $\chi(-1)=-1$. If h and k are odd, then*

$$Y(h,k)=\frac{16k}{\pi i\phi(2k)} \sum_{\substack{j=1 \\ 2j-1\not\equiv 0 \,(\mathrm{mod}\, k)}}^{\infty} \frac{1}{2j-1} \sum_{\substack{\chi(-1)=-1 \\ \chi(\mathrm{mod}\, 2k)}} B_{1,\overline{\chi}} G(h(2j-1)-k,\chi),$$

where $\phi(x)$ denotes Euler function.

Theorem 9. *Let h and k be positive integers with $(h,k)=1$. Let χ be Dirichlet character with conductor $2k$ and $\chi(-1)=-1$. If $h+k$ is odd, then*

$$S(h,k)=\frac{8}{\pi i\phi(2k)} \sum_{j=1}^{\infty} \frac{1}{2j-1} \sum_{\substack{\chi(-1)=-1 \\ \chi(\mathrm{mod}\, 2k)}} B_{1,\overline{\chi}} G(h(2j-1)-k,\chi),$$

if h is even and k is odd, then

$$s_1(h,k)=\frac{4}{\pi i\phi(2k)}\sum_{\substack{j=1\\2j-1\not\equiv 0(\bmod k)}}^{\infty}\frac{1}{2j-1}\sum_{\substack{\chi(-1)=-1\\ \chi(\bmod 2k)}}B_{1,\overline{\chi}}G(h(2j-1),\chi),$$

if h is odd and k is even, then

$$s_2(h,k)=\frac{i}{\pi\phi(2k)}\sum_{j=1}^{\infty}\frac{1}{j}\sum_{\substack{\chi(-1)=-1\\ \chi(\, mod\, 2k)}}B_{1,\overline{\chi}}G(2hj-k,\chi),$$

if k is odd, then

$$s_3(h,k)=\frac{2}{\pi i\phi(2k)}\sum_{j=1}^{\infty}\frac{1}{j}\sum_{\substack{\chi(-1)=-1\\ \chi(\, mod\, 2k)}}B_{1,\overline{\chi}}G(2hj-k,\chi),$$

if h is odd, then

$$s_4(h,k)=\frac{8i}{\pi\phi(2k)}\sum_{j=1}^{\infty}\frac{1}{2j-1}\sum_{\substack{\chi(-1)=-1\\ \chi(\, mod\, 2k)}}B_{1,\overline{\chi}}G(h(2j-1),\chi),$$

and if h and k are odd, then

$$s_5(h,k)=\frac{4}{\pi i\phi(2k)}\sum_{\substack{j=1\\2j-1\not\equiv 0(\bmod k)}}^{\infty}\frac{1}{2j-1}\sum_{\substack{\chi(-1)=-1\\ \chi(\bmod 2k)}}B_{1,\overline{\chi}}G(h(2j-1)-k,\chi).$$

We need the following properties of Dirichlet L-function, Gauss sums, $G(n,\chi)$ and generalized Bernoulli number, $B_{1,\chi}$.

Let χ be a primitive Dirichlet character with conductor k. The Gauss sums define as follows:

$$G(z,\chi)=\sum_{m=1}^{k-1}\chi(h)e^{\frac{2\pi imz}{k}}.$$

Put $G(\chi)=G(1,\chi)$ cf. [9].

Theorem 10. ([2, Theorem 8.9]). *Let* $(n, k) = 1$.

$$G(n, \chi) = \overline{\chi}(n) G(\chi). \tag{3.1}$$

Dirichlet L-function is defined by (cf. [9], [23], [24])

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}, \qquad \mathrm{Re}(s) > 1.$$

Theorem 11. ([9, p. 6]). *If* $\chi \not\equiv 1$, *then*

$$L(1, \chi) \neq 0, \infty$$

and its value is explicitly given as follows

$$L(1, \chi) = -\frac{G(\chi)}{k} \sum_{a} \overline{\chi}(a) \log(1 - \zeta^a)$$

if $\chi(-1) = 1$,

$$L(1, \chi) = -\frac{G(\chi)}{k} \sum_{a} \overline{\chi}(a) \log|1 - \zeta^a|$$

if $\chi(-1) = -1$,

$$L(1, \chi) = \frac{\pi i G(\chi)}{k^2} \sum_{a} \overline{\chi}(a) a \tag{3.2}$$

where $\zeta = e^{\frac{2\pi i}{k}}$, *and the sum is taken over all integers* a *such that* $1 \leq a \leq k$, $(a, k) = 1$.

The generalized Bernoulli polynomials and numbers are defined by (cf. [9], [24])

$$B_{n,\chi}(x) = k^{n-1} \sum_{m=1}^{k} \chi(m) B_n(\frac{m - k + x}{k}), n \geq 0;$$

in particular $(x = 0)$

$$B_{n,\chi} = k^{n-1} \sum_{m=1}^{k} \chi(m) B_n(\frac{m - k}{k}), n \geq 0. \tag{3.3}$$

Lemma 3. ([9]). *If* $\chi(-1) = -1$, *then*

$$B_{1,\chi} = \frac{1}{k} \sum_{m=1}^{k} \chi(m) m. \tag{3.4}$$

Theorem 12. ([22]). *Let k be an integer ≥ 3, and let χ be any Dirichlet character with conductor k and $\chi(-1) = -1$. Then we have*

$$L(1, \chi) = \frac{\pi}{2k} \sum_{m=1}^{k} \chi(m) \cot(\frac{\pi m}{k}).$$

We note that

$$\sum_{m=1}^{k} \chi(m) \cot(\frac{\pi m}{k}) = 0$$

if $\chi(-1) = 1$. For $(m, k) = 1$, from Theorem 12, and the orthogonality of characters with conductor k, we have the following equation

$$\cot(\frac{\pi m}{k}) = \frac{2k}{\pi \phi(k)} \sum_{\substack{\chi(-1) = -1 \\ \chi(\bmod k)}} \overline{\chi}(m) L(1, \chi). \tag{3.5}$$

Theorem 13. ([6, Theorem 1]). *Let h and k be positive integers with $(h, k) = 1$. If $h + k$ is odd, then*

$$S(h, k) = \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{2j-1} \tan\left(\frac{\pi h(2j-1)}{2k}\right),$$

if h is even and k is odd, then

$$s_1(h, k) = \frac{-2}{\pi} \sum_{\substack{j = 1 \\ 2j-1 \not\equiv 0 (\bmod k)}}^{\infty} \frac{1}{2j-1} \cot\left(\frac{\pi h(2j-1)}{2k}\right),$$

if h is odd and k is even, then

$$s_2(h, k) = -\frac{1}{2\pi} \sum_{\substack{j = 1 \\ 2j \not\equiv 0 \, (\bmod k)}}^{\infty} \frac{1}{j} \tan\left(\frac{\pi h j}{k}\right),$$

if k is odd, then

$$s_3(h, k) = \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{1}{j} \tan\left(\frac{\pi h j}{k}\right),$$

if h is odd, then

$$s_4(h,k)=\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j-1}\cot\left(\frac{\pi h(2j-1)}{2k}\right),$$

and if h and k are odd, then

$$s_5(h,k)=\frac{2}{\pi}\sum_{\substack{j=1\\ 2j-1\not\equiv 0(\mathrm{mod}\,k)}}^{\infty}\frac{1}{2j-1}\tan\left(\frac{\pi h(2j-1)}{2k}\right). \tag{3.6}$$

By Using (3.1), (3.2), (3.4) and (3.5) Theorem 13 is represented by Dirichlet character, Gauss sums and $B_{1,\chi}$ which is given by Theorem 9.

Proof. [Proof of Theorem 8] By using (3.6) and definition of $Y(h,k)$, we have

$$Y(h,k)=\frac{8k}{\pi}\sum_{\substack{j=1\\ 2j-1\not\equiv 0(\mathrm{mod}\,k)}}^{\infty}\frac{1}{2j-1}\tan\left(\frac{\pi h(2j-1)}{2k}\right).$$

Substituting well-known equation

$$\tan\pi x=-\cot\pi(x-\frac{1}{2})$$

in the above, we obtain

$$Y(h,k)=\frac{-8k}{\pi}\sum_{\substack{j=1\\ 2j-1\not\equiv 0(\mathrm{mod}\,k)}}^{\infty}\frac{1}{2j-1}\cot\left(\frac{\pi(h(2j-1)-k)}{2k}\right). \tag{3.7}$$

Substituting (3.5) into (3.7), we have

$$\begin{aligned} Y(h,k)=\frac{-32k^2}{\pi^2\phi(2k)}&\sum_{\substack{j=1\\ 2j-1\not\equiv 0(\mathrm{mod}\,k)}}^{\infty}\\ \frac{1}{2j-1}&\sum_{\substack{\chi(-1)=-1\\ \chi\,(\mathrm{mod}\,2k)}}\overline{\chi}(h(2j-1)-k)L(1,\chi). \end{aligned} \tag{3.8}$$

By using (3.2) in (3.8), we have

$$Y(h,k) = \frac{8}{\pi i \phi(2k)} \sum_{\substack{j=1 \\ 2j-1 \not\equiv 0 (\operatorname{mod} k)}}^{\infty} \frac{1}{2j-1} \sum_{\substack{\chi(-1)=-1 \\ \chi(\,mod\, 2k)}} \overline{\chi}(h(2j-1)-k,\chi) G(\chi) \sum_{m=1}^{2k} m\overline{\chi}(m).$$

Using (3.1) and (3.4) in the above, thus we arrive at the desired result. ■

The proof of Theorem 9 follows precisely along the same lines as the proof of Theorem 8, and so we omit it.

ACKNOWLEDGMENT

I would like to thank the referee for his/her comments. The author is supported by the research fund of Akdeniz University.

REFERENCES

1. T. M. Apostol, *Modular functions and Dirichlet series in Number Theory*, Springer-Verlag, 1976.
2. T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, 1976.
3. T. M. Apostol and T. H. Vu, Elementary proofs of Berndt's reciprocity laws, *Pacific J. Math.*, **98** (1982), 17-23.
4. T. Asai, Some arithmetic on Dedekind sums, *J. Mat. Soc. Japan*, **38(1)** (1986), 163-172.
5. B. C. Berndt, Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan, *J. Reine Angew. Math.*, **303/304** (1978), 332-150.
6. B. C. Berndt, and L. A. Goldberg, Analytic Properties of arithmetic Sums arising in the theory of the classical theta-functions, *SIAM. J. Math. Anal.*, **15** (1984), 143-150.
7. L. A. Goldberg, *Transformation of Theta-functions and analogues of Dedekind sums*, Thesis, University of Illinois Urbana, 1981.
8. G. H. Hardy, On certain series of discontinues functions, connected with the modular functions, *Quart. J. Math.*, **36** (1905), 93-123, (= Collected papers, Vol. IV, pp. 362-392. Clarendon Press Oxford, 1969).
9. K. Iwasawa, *Lectures on p-adic L-functions*, Princeton Univ. press, 1972.

10. M. R. Pettet and R. Sitaramachandra Rao, Three-term relations for Hardy sums, *J. Number Theory*, **25** (1989), 328-339.
11. H. Salie, Zum wertevorrat der Dedekindschen summen, *Math. Zeitschr*, **72** (1959), 61-75.
12. Y. Simsek, *Dedekind ve Hardy toplamlari nin genellestrilmesi*(= *Generalized Dedekind and Hardy sums*), Thesis, Cukurova University, Adana, 1993.
13. Y. Simsek, A note on Dedekind sums, *Bull. Cal. Math. Soc.*, **85** (1993), 567-572.
14. Y. Simsek, Theorems on three-term relations for Hardy sums, *Turkish J. Math.*, **22** (1998), 153-162.
15. Y. Simsek, Relation between theta-function Hardy sums Eisenstein and Lambert series in the transformation formula of $\log \eta_{g,h}(z)$, *J. Number Theory*, **99(2)** (2003), 338-360.
16. Y. Simsek, Generalized Dedekind sums associated with the Abel sum and the Eisenstein and Lambert series, *Advan. Stud. Contemp. Math.*, **9** (2004), 125-137.
17. Y. Simsek, On generalized Hardy sums, $S_5(h,k)$, *Ukrainian Math. J.*, **56(10)** (2004), 1434-1440.
18. Y. Simsek, Remarks on reciprocity laws of the Dedekind and Hardy sums, *Advan. Stud. Contemp. Math.*, **12** (2006), 237-246.
19. Y. Simsek, Hardy character sums related to Eisenstein series and theta functions, *Advan. Stud. Contemp. Math.*, **12** (2006), 39-53.
20. R. Sitaramachandra Rao, Dedekind and Hardy sums, *Acta Arith.*, **48(4)** (1978), 325-340.
21. E. T. Whittaker and G. N. Watson, *A course of Modern Analysis*, 4th. edition, Cambridge, 1962.
22. W. Zhang, On the mean values of Dedekind Sums, *J. Theorie Nombres Bordeaux*, **8** (1996), 429-442.
23. H. M. Srivastava and J. Choi, *Series associated by the zeta and related functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
24. H. M. Srivastava, T. Kim and Y. Simsek, q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series, *Russ. J. Math. Phys.*, **12(2)** (2005), 241-268.

Yilmaz Simsek
Department of Mathematics,
Faculty of Arts and Science,
University of Akdeniz,
07058 Antalya,
Turkey
E-mail: simsek@akdeniz.edu.tr