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# SOME GENERALIZED KY FAN'S INEQUALITIES

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**Abstract.** In this paper, we generalize Ky Fan's minimax inequality to vector-valued function with values in a topological vector space acting on the product of two other topological vector spaces which are connected by another function. In these results, the concavity or convexity on a function is transferred to another function. And a sufficient condition for the existence of solution for a variational inclusion is given.

## 1. Introduction

Let $X$ and $Y$ be nonempty sets and $f : X \times Y \longrightarrow R$ be a function. The minimax theorem implies that the equality

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{x\in X} \inf_{y\in Y} f(x,y)$$

holds under certain conditions.

The minimax inequalities are special forms of minimax theorem. In 1972, Ky Fan [1] proved the following minimax inequality and discussed its geometrical form and applications to fixed point theory.

### Ky Fan's Inequality

Let $X$ be a compact convex subset in a topological vector space $E$ and let $f : X \times X \longrightarrow R$ be a function such that

(1) $f(x, \cdot)$ is lower semicontinuous on $X$ for every $x \in X$;

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(2) $f(\cdot, y)$ is quasiconcave for every $y \in X$. Then

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x).$$

By relaxing the compactness, the closedness or the convexity, many generalizations of Ky Fan's Inequality were given ([2]-[6]) and numerous applications of this inequality were obtained. Also, by introducing varieties of characterization on the convexity of the set-valued mappings ([7]-[11], [13]), many minimax theorems involving scalar functions have been extended to minimax theorems for set-valued mappings ([8]-[12]).

Inspired and motivated by these works, in this paper, we give new generalizations of Ky Fan's minimax inequality for set-valued mappings from the following aspects: (1) the set-valued mappings act on the product space $X \times Y$ of two topological vector spaces $X$ and $Y$ connected by a mapping $\varphi$; (2) the concavity or convexity on a mapping is transferred to another function; (3) the range of the set-valued mapping is extended from normed space to topological vector space. Finally, we give a sufficient condition on the existence of solution for a variational inclusion .

## 2. Preliminaries

**Definition 2.1.** Let $X$ and $Y$ be topological spaces, $S : Y \longrightarrow 2^X$ be a set-valued mapping with nonempty values. Then $S$ is said to be

(i) upper semicontinuous (usc) at $y \in Y$ if for any neighborhood $U$ of $S(y)$, there exists a neighborhood $V$ of $y$ such that we have $S(y') \subset U$ for every $y' \in V$ .

(ii) lower semicontinuous (lsc) at $y \in Y$ if for any $x \in S(y)$ and for any sequence of elements $\{y_n\}$ in $Y$ converging to $y$, there exists a sequence of elements $x_n \in S(y_n)$ converging to $x$.

(iii) upper(resp. lower) semicontinuous on $Y$ if $S$ is upper(resp. lower) semicontinuous at every point $y \in Y$.

**Proposition 2.1.** [13, Proposition 1.4.4] *Let $X$ and $Y$ be topological spaces, and let $S : Y \longrightarrow 2^X$ be a set-valued mapping with nonempty values. Then*

(1) *$S$ is upper semicontinuous on $Y$ if and only if for any closed subset $M$ of $X$, the inverse image of $M$*

$$S^{-1}(M) = \{y \in Y | S(y) \cap M \neq \emptyset\}$$

*is closed;*

(2) *$S$ is lower semicontinuous on $Y$ if and only if for any closed subset $M$ of $X$, the core of $M$*

$$S^{+1}(M) = \{y \in Y | S(y) \subset M\}$$

*is closed.*

**Definition 2.2.** [13, pp. 57] Let $Y$ be a convex subset of a vector spaces $G$ and let $S{:}Y \to 2^G$ be a set-valued mapping. $S$ is said to be convex on $Y$ (resp. concave on $Y$) if for all $y_1, y_2 \in Y$ and $\lambda \in [0,1]$,

$$\lambda S(y_1) + (1-\lambda)S(x_2) \subset (resp., \supset) S(\lambda y_1 + (1-\lambda)y_2).$$

**Proposition 2.2.** *Let $Y$ be a convex subset of a vector spaces $G$ and let $S : Y \longrightarrow 2^G$ be a set-valued mapping. Then $S$ is convex on $Y$ (resp., concave on $Y$) if and only if for all $n \geq 2$ and for all $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ and for all $y_1, y_2, \ldots, y_n \in Y$,*

$$\sum_{i=1}^n \lambda_i S(y_i) \subset (resp., \supset) S(\sum_{i=1}^n \lambda_i y_i).$$

**Definition 2.3.** Let $Y$ be a convex subset of a vector space and let $X$ be a convex subset of a vector space with an order relation $\leq$. A mapping $\varphi : Y \longrightarrow X$ is said to be convex (resp., concave) if for any $\lambda \in [0,1]$ and $y_1, y_2 \in Y$,

$$\varphi(\lambda y_1 + (1-\lambda)y_2) \leq (resp., \geq) \lambda\varphi(y_1) + (1-\lambda)\varphi(y_2).$$

**Definition 2.4.** Let $(Y, \leq)$ be an ordered topological vector space and let $X$ be a nonempty set. A set-valued mapping $S : Y \longrightarrow 2^X$ is said to be monotone increasing (resp., decreasing) if for any $y_1 \leq y_2$,

$$S(y_1) \subset (resp., \supset) S(y_2).$$

## 3. The Main Results

The following lemma is one of the most fundamental result in nonlinear analysis.

**Ky Fan Lemma**

Let $Y$ be a nonempty subset of a Hausdorff topological vector space $G$. If $S : Y \longrightarrow 2^G$ is a set-valued mapping with closed values, and has the following properties:

(i) there exists $y_0 \in Y$ such that $S(y_0)$ is compact;

(ii) $S$ is a $KKM$ set-valued map (i.e., for each finite set $\{y_1, y_2, \ldots, y_n\}$ in $Y$, the convex hull of $\{y_1, y_2, \ldots, y_n\}$, $\text{conv}\{y_1, y_2, \ldots, y_n\} \subset \cup_{i=1}^{n} S(y_i)$), then

$$\cap_{y \in Y} S(y) \neq \emptyset.$$

As a generalization of Ky Fan Lemma, Cheng [14] gave the following result.

**Lemma 3.1.** ([14]). *Let $E, G$ be Hausdorff topological vector spaces. If $Y \subset G$, $\varphi : Y \longrightarrow E$ is a mapping, and $S : Y \longrightarrow 2^E$ is a nonempty closed-valued mapping such that*

($i$) *there exists $y_0 \in Y$ such that $S(y_0)$ is compact;*

($ii$) *$conv\{\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_n)\} \subset \cup_{i=1}^{n} S(y_i)$ for each finite set $\{y_1, y_2, \ldots, y_n\}$ in $Y$, then*

$$\cap_{y \in Y} S(y) \neq \emptyset.$$

Our main results can be formulated as follows.

**Theorem 3.1.** *Let $E, G, Z$ be Hausdorff topological vector spaces where $E$ is endowed with an order relation $\leq$. Assume $X$ is a nonempty compact convex subset of $E$, $Y$ is a convex subset of $G$ and $M$ is a nonempty closed subset of $Z$ with $Z \backslash M$ is convex. Let $\varphi{:}Y \rightarrow X$ be a convex (resp. concave) mapping and let $F{:}X \times Y \rightarrow 2^Z$ be a set-valued mapping with the following properties:*

($i$) *$F(\cdot, y)$ is lower semicontinuous on $X$ for all $y$ in $Y$.*

($ii$) *there exists a set-valued mapping $H{:}X \times Y \rightarrow 2^Z$ such that*

  ($a$) *$H(\varphi(y), y) \subset M$ for all $y \in Y$,*

  ($b$) *$H(x, y) \subset M$ implies that $F(x, y) \subset M$ for all $x \in X$ and $y \in Y$,*

  ($c$) *$H(x, \cdot)$ is convex on $Y$ for all $x \in X$,*

  ($d$) *$H(\cdot, y)$ is monotone decreasing (resp. increasing) on $X$ for all $y \in Y$.*

Then there exists $x_0 \in X$ such that $F(x_0, y) \subset M$ for all $y \in Y$.

*Proof.* For all $y \in Y$, let $S(y) = \{x \in X : F(x, y) \subset M\}$.By ($a$) and ($b$) in (ii), for all $y \in Y$, $S(y) \neq \emptyset$, since $\varphi(y) \in S(y)$. By (i) and Proposition 2.1, for all $y \in Y$, $S(y)$ is closed in $X$, therefore, $S(y)$ is compact. It remains to prove that (ii) in Lemma 3.1 holds.

Suppose no. Then there exists $\{y_1, y_2, \ldots, y_n\}$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset R$, $\lambda_i \geq 0 (i = 1, 2, \ldots, n)$, $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$\sum_{i=1}^{n} \lambda_i \varphi(y_i) \notin S(y_j),\ j = 1, 2, \ldots, n.$$

That is

$$F(\sum_{i=1}^{n} \lambda_i \varphi(y_i), y_j) \not\subset M,\ j = 1, 2, \ldots, n.$$

By (ii)(b),

$$H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), y_j) \not\subset M,\ j = 1, 2, \ldots, n.$$

Hence

$$H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), y_j) \cap (Z \setminus M) \neq \emptyset,\ \forall j = 1, 2, \ldots, n.$$

Since $Z \backslash M$ is convex set and $H(x, \cdot)$ is convex on $Y$, by Proposition 2.2, it follows that

$$\begin{aligned} \emptyset &\neq \sum_{i=1}^{n} \lambda_i H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), y_i) \cap (Z \setminus M) \\ &\subset H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), \sum_{i=1}^{n} \lambda_i y_i) \cap (Z \setminus M). \end{aligned} \tag{1}$$

Since $\varphi$ is convex (resp., concave) and $H(\cdot, y)$ is monotone decreasing (resp., increasing) on $X$, we have

$$H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), \sum_{i=1}^{n} \lambda_i y_i) \subset H(\varphi(\sum_{i=1}^{n} \lambda_i y_i), \sum_{i=1}^{n} \lambda_i y_i). \tag{2}$$

By (ii)(a) and (1),(2), we obtain

$$\begin{aligned} \emptyset &\neq \sum_{i=1}^{n} \lambda_i H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), y_i) \cap (Z \setminus M) \\ &\subset H(\varphi(\sum_{i=1}^{n} \lambda_i y_i), \sum_{i=1}^{n} \lambda_i y_i) \cap (Z \setminus M) \subset M \cap (Z \setminus M) = \emptyset. \end{aligned}$$

This contradiction shows that (ii) of Lemma 3.1 holds. Therefore $\bigcap_{y \in Y} S(y) \neq \emptyset$, i.e., there exists an element $x_0 \in X$ such that $F(x_0, y) \subset M$ for all $y \in Y$. ■

**Corollary 3.1.** *Let $E, G, Z$ be Hausdorff topological vector spaces where $E$ is endowed with an order relation $\leq$. Assume that $X$ is a nonempty compact convex subset of $E$, $Y$ is a convex subset of $G$, and $M$ is a nonempty closed subset of $Z$ with $Z \backslash M$ is convex. Let $\varphi : Y \longrightarrow X$ be a convex (resp., concave) mapping, and $F : X \times Y \longrightarrow 2^Z$ be a set-valued mapping with the following properties:*

*(i) $F(\cdot, y)$ is lower semi-continuous on $X$ for all $y \in Y$,*

$(ii)$ *$F(x, \cdot)$ is convex on $Y$ for all $x \in X$,*

$(iii)$ *$F(\cdot, y)$ is monotone decreasing (resp., monotone increasing) on $X$ for all $y \in Y$,*

$(iv)$ *$F(\varphi(y), y) \subset M$ for all $y \in Y$.*

Then there exists $x_0 \in X$ such that $F(x_0, y) \subset M$ for all $y \in Y$.

**Remark 3.1.** From the proof of Theorem 3.1, if the mapping $\varphi$ is linear on convexity coefficient (that means $\varphi(\lambda y_1 + (1-\lambda)y_2) = \lambda\varphi(y_1) + (1-\lambda)\varphi(y_2)$ for all $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$), then the monotonicity of $H$ and the order structure on $E$ are not needed in Theorem 3.1 and Corollary 3.1.

**Theorem 3.2.** *Let $E, G, Z$ be Hausdorff topological vector spaces. Assume that $X$ is a nonempty compact convex subset of $E$, $Y$ is a convex subset of $G$, and $M$ is a nonempty closed subset of $Z$ with $Z \backslash M$ is convex. Let $\varphi : Y \to X$ be a linear mapping on convexity coefficient and let $F : X \times Y \longrightarrow 2^{Z}$ be a set-valued mapping with the following properties:*

(*i*) *$F(\cdot, y)$ is lower semi-continuous on $X$ for all $y \in Y$,*

(*ii*) *$F(x, \cdot)$ is convex on $Y$ for all $x \in X$,*

(*iii*) *$F(\varphi(y), y) \subset M$ for all $y \in Y$.*

*Then there exists $x_0 \in X$ such that $F(x_0, y) \subset M$ for all $y \in Y$.*

**Theorem 3.3.** *Let $E, G, Z$ be Hausdorff topological vector spaces. Assume that $X$ is a nonempty convex compact subset of $E$, $Y$ is a subset of $G$ and $M$ is a nonempty closed subset of $Z$. Let $\varphi : Y \to X$ be a mapping and let $F : X \times Y \longrightarrow 2^Z$ be a set-valued mapping with the following properties:*

(*i*) *$F(\cdot, y)$ is lower semicontinuous on $X$ for all $y \in Y$,*

(*ii*) *for any finite set $\{y_1, y_2, \ldots, y_n\}$ in $Y$, $conv\{\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_n)\} \subset \bigcup_{i=1}^{n}\{x \in X : F(x, y_i) \subset M\}$.*

*Then there exists $x_0 \in X$ such that $F(x_0, y) \subset M$ for all $y \in Y$.*

*Proof.* Define the set-valued mapping $S : Y \longrightarrow 2^X$ by

$$S(y) = \{x \in X : F(x, y) \subset M\},\ y \in Y.$$

It follows that $S(y) \neq \emptyset$ for all $y \in Y$, since $F(\varphi(y), y) \subset M$ by (ii). Taking into account that $M$ is closed and the assumption(i), it follows from Proposition 2.1 that $S$ has closed values. Since $X$ is a compact set, $S(y)$ is compact for every $y \in Y$. It is easy to see that the above assumption (ii) implies condition (ii) in Lemma 3.1.

Consequently, the set-valued mapping $S$ defined above meets with conditions in Lemma 3.1, and hence $\cap_{y\in X}S(y) \neq \emptyset$ and this implies the conclusion. ■

**Remark 3.2.** When weakening slightly the condition (i) in Theorem 3.2 and Theorem 3.3 to:

(i) $S(y) = \{x \in X : F(x, y) \subset M\}$ is a closed-valued mapping for all $y \in Y$,

Theorem 3.2 and Theorem 3.3 still hold.

From Theorem 3.3, we obtain the following generalization of Ky Fan's minimax inequality.

**Corollary 3.2.** *Let $X, Y$ be nonempty compact convex subset of Hausdorff topological vector spaces $E$ and $G$ respectively. If $\varphi : Y \longrightarrow X$ is a linear mapping on convexity coefficient, and $f : X \times Y \longrightarrow \mathbb{R}$ is a mapping satisfying:*

*(i)* *$f(\cdot, y)$ is lower semicontinuous on $X$ for all $y \in Y$,*

*(ii)* *$f(x, \cdot)$ is concave on $Y$ for all $x \in X$,*

*then there exists $x_0 \in X$ such that*

$$\sup_{y\in Y} f(x_0, y) \leq \sup_{y\in Y} f(\varphi(y), y).$$

*Proof.* Let $m = \sup_{y\in Y} f(\varphi(y), y)$. If $m = +\infty$, take any $x_0$. Consider $m < +\infty$. Let $Z = R$, $F = f$ and $M = (-\infty, m]$. Then $f(\cdot, y)$ is lsc on $X$ for all $y \in Y$ implies that $f$ has closed lower level sets. Hence $S(y) = \{x \in X : F(x, y) \subset M\}$ is closed-valued for all $y \in Y$. It remains to prove that (ii) in Theorem 3.3 holds. Suppose no. Then there exists a finite set $\{y_1, y_2, \ldots, y_n\} \subset Y$ and $x_0 \in conv\{\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_n)\}$ such that for every $i = 1, 2, \cdots, n$, $f(x_0, y_i) > m$ where $x_0 = \sum_{i=1}^{n} \alpha_i\varphi(y_i)$, $\alpha_i$ are positive numbers with $\sum_{i=1}^{n} \alpha_i = 1$. Let $y_0 = \sum_{i=1}^{n} \alpha_i y_i$. Then $x_0 = \varphi(y_0)$. Since $f(x, \cdot)$ is concave for every $x \in X$, we have $f(\varphi(y_0), y_0) > m$ and this is a contradiction. ■

**Theorem 3.4.** *Let $E, G, Z$ be Hausdorff topological vector spaces where $E$ is endowed with an order relation $\leq$. Assume that $X$ is a nonempty compact convex subset of $E$, $Y$ is a convex subset of $G$, and $M$ is a nonempty convex open subset of $Z$. Let $\varphi : Y \longrightarrow X$ be a convex (resp., concave) mapping, and let $F : X \times Y \longrightarrow 2^Z$ be a set-valued mapping with the following properties:*

*(i)* *$F(\cdot, y)$ is upper semi-continuous on $X$ for all $y \in Y$;*

*(ii)* *there exists a set-valued mapping $H : X \times Y \longrightarrow 2^Z$ such that*

$(a)$ *$H(\varphi(y), y) \cap (Z \setminus M) \neq \emptyset$ for all $y \in Y$;*

$(b)$ *for all $x \in X$ and $y \in Y$, $H(x, y) \not\subset M$ implies that $F(x, y) \not\subset M$,*

$(c)$ *$H(x, \cdot)$ is concave on $Y$ for all $x \in X$;*

$(d)$ *$H(\cdot, y)$ is monotone increasing (resp., decreasing) on $X$ for all $y \in Y$.*

*Then there exists $x_0 \in X$ such that $F(x_0, y) \cap (Z \setminus M) \neq \emptyset$ for all $y \in Y$.*

*Proof.* For all $y \in Y$, let $S(y) = \{x \in X : F(x, y) \cap (Z \setminus M) \neq \emptyset\}$.From $(a)$ and $(b)$ in (ii), it follows that $S(y) \neq \emptyset$ since $\varphi(y) \in S(y)$. From (i) and proposition 2.1, for all $y \in Y$, $S(y)$ is a closed subset of $X$ hence is compact by the compactness of $X$. Therefore, (i) of Lemma 3.1 is satisfied. It remains to prove that (ii) in Lemma 3.1 is true. Suppose that there exist $y_1, y_2, \ldots, y_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$\sum_{i=1}^{n} \lambda_i \varphi(y_i) \notin S(y_j),\ j = 1, 2, \ldots, n.$$

By the definition of $S(y)$

$$F(\sum_{i=1}^{n} \lambda_i \varphi(y_i), y_j) \subset M,\ j = 1, 2, \ldots, n.$$

By (ii) (b),

$$H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), y_j) \subset M,\ j = 1, 2, \ldots, n. \tag{3}$$

From (3) and the convexity of $M$, it follows that

$$\sum_{j=1}^{n} \lambda_j H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), y_i) \subset M. \tag{4}$$

Since $H(x, \cdot)$ is concave on $Y$, $\varphi : Y \longrightarrow X$ is convex (resp., concave), and $H(\cdot, y)$ is monotone increasing (resp., monotone decreasing) on $X$, we have

$$\begin{aligned} H(\varphi(\textstyle\sum_{i=1}^{n} \lambda_i y_i), \sum_{j=1}^{n} \lambda_j y_j) &\subset H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), \sum_{j=1}^{n} \lambda_j y_j) \\ &\subset \sum_{j=1}^{n} \lambda_j H(\sum_{i=1}^{n} \lambda_i \varphi(y_i), y_j) \subset M, \end{aligned}$$

which contradicts to (ii)(a). This completes the proof. ■

**Corollary 3.3.** *Let $E, G, Z$ be Hausdorff topological vector spaces where $E$ is endowed with an order relation $\leq$. Assume that $X$ is a nonempty compact convex subset of $E$, $Y$ is a convex subset of $G$, and $M$ is a nonempty open convex subset of $Z$. Let $\varphi : Y \longrightarrow X$ be a convex (resp., concave) mapping and $F : X \times Y \longrightarrow 2^Z$ be a set-valued mapping satisfying:*

$(i)$ *$F(\cdot, y)$ is upper semicontinuous on $X$ for all $y \in Y$;*

$(ii)$ *$F(x, \cdot)$ is concave on $Y$ for all $x \in X$;*

$(iii)$ *$F(\cdot, y)$ is monotone increasing (resp., monotone decreasing) on $X$ for all $y \in y$;*

$(iv)$ *$F(\varphi(y), y) \cap (Z \setminus M) \neq \emptyset$ for all $y \in Y$.*

*Then there exists $x_0 \in X$ such that $F(x_0, y) \cap (Z \setminus M) \neq \emptyset$ for all $y \in Y$.*

**Remark 3.3.** Similar to Remark 3.1, the conclusions of Theorem 3.4 and Corollary 3.3 hold if $\varphi : Y \longrightarrow X$ is a linear mapping on convexity coefficient and the monotonicity of $H$ and the order structure of $E$ are not needed.

**Theorem 3.5.** *Let $E, G, Z$ be Hausdorff topological vector spaces. Assume that $X$ is a nonempty compact convex subset of $E$, $Y$ is a convex subset of $G$, and $M$ is an open convex subset of $Z$. If $\varphi : Y \longrightarrow X$ is a linear mapping on convexity coefficient and $F : X \times Y \longrightarrow 2^Z$ is a set-valued mapping satisfying:*

$(i)$ *$F(\cdot, y)$ is upper semicontinuous on $X$ for all $y \in Y$;*

$(ii)$ *$F(x, \cdot)$ is concave on $Y$ for all $x \in X$;*

$(iii)$ *$F(\varphi(y), y) \cap (Z \setminus M) \neq \emptyset$ for all $y \in Y$,*

*then there exists $x_0 \in X$ such that $F(x_0, y) \cap (Z \setminus M) \neq \emptyset$ for all $y \in Y$.*

In the studies of minimax theory, it is an important topic that how to weaken the compactness, linearity of the spaces and convexity of functions. By weakening slightly the compactness of $X$ in Theorem 3.1, we have the following conclusion.

**Theorem 3.6.** *Let $E, G, Z$ be Hausdorff topological vector spaces where $E$ is endowed with an order relation $\leq$. Assume that $X$ is a convex subset of $E$, $Y$ is a convex subset of $G$, and $M$ is a nonempty closed subset of $Z$ with $Z \setminus M$ is convex. Let $\varphi : Y \longrightarrow X$ be a convex (resp., concave) continuous mapping, and $F : X \times Y \longrightarrow 2^Z$ be a set-valued mapping satisfying:*

$(i)$ *$F(\cdot, y)$ is lower semicontinuous on $X$ for all $y \in Y$;*

$(ii)$ *there exists a set-valued mapping $H : X \times Y \longrightarrow 2^Z$ such that*

$(a)$ *$H(\varphi(y), y) \subset M$ for all $y \in Y$;*

($b$) *For all* $x \in X$ *and* $y \in Y$*,* $H(x, y) \subset M$ *implies* $F(x, y) \subset M$*;*

($c$) $H(x, \cdot)$ *is convex on* $Y$ *for all* $x \in X$*;*

($d$) $H(\cdot, y)$ *is monotone decreasing (resp., monotone increasing) on* $X$ *for all* $y \in Y$*;.*

($iii$) *there exists a compact subset* $Y_0$ *of* $Y$ *and an element* $y_0 \in Y_0$ *such that* $F(x, y_0) \cap (Z \setminus M) \neq \emptyset$ *for all* $x \in X \setminus \varphi(Y_0)$*.*

*Then there exists* $x_0 \in X$ *such that* $F(x_0, y) \subset M$ *for all* $y \in Y$*.*

*Proof.* Let $S(y) = \{x \in X : F(x, y) \subset M\}$ for all $y \in Y$. Then $S(y) \neq \emptyset$. We claim that $S(y_0) \subset \varphi(Y_0)$. In fact, supposing there exists an element $x \in S(y_0)$ such that $x \notin \varphi(Y_0)$. Then $F(x, y_0) \subset M$ that contradicts to (iii). Now, $\varphi(Y_0)$ is compact since $Y_0$ is compact and $\varphi$ is continuous. Hence $S(y_0)$ is compact. The rest of the proof is the same as the proof of Theorem 3.1. ■

**Remark 3.4.** Different from Theorem 3.1 and 3.4, the mapping $\varphi$ in Theorem 3.6 must be continuous.

**Corollary 3.4.** *Let* $E, G, Z$ *be Hausdorff topological vector spaces where* $E$ *is endowed with an order relation* $\leq$*. Assume that* $X$ *is a convex subset of* $E$*,* $Y$ *is a convex subset of* $G$*, and* $M$ *is a nonempty closed subset of* $Z$ *with* $Z \setminus M$ *is convex. Let* $\varphi : Y \longrightarrow X$ *be convex (resp., concave) continuous mapping, and* $F : X \times Y \longrightarrow 2^Z$ *be a set-valued mapping satisfying*

($i$) $F(\cdot, y)$ *is lower semicontinuous on* $X$ *for all* $y \in Y$*;*

($ii$) $F(x, \cdot)$ *is convex on* $Y$ *for all* $x \in X$*;*

($iii$) $F(\cdot, y)$ *is monotone decreasing (resp., monotone increasing) on* $X$ *for all* $y \in Y$*;*

($iv$) $F(\varphi(y), y) \subset M$ *for all* $y \in Y$*;*

($v$) *there exist a compact subset* $Y_0$ *of* $Y$ *and an element* $y_0 \in Y_0$ *such that* $F(x, y_0) \cap (Z \setminus M) \neq \emptyset$ *for all* $x \in X \setminus \varphi(Y_0)$*.*

Then there exists $x_0 \in X$ such that $F(x_0, y) \subset M$ for all $y \in Y$.

**Theorem 3.7.** *Let* $E, G, Z$ *be Hausdorff topological vector spaces. Assume that* $X$ *is a convex subset of* $E$*,* $Y$ *is a convex subset of* $G$*, and* $M$ *is a closed subset of* $Z$ *with* $Z \setminus M$ *is convex. Let the mapping* $\varphi : Y \longrightarrow X$ *be continuous and linear on convexity coefficient, and let* $F : X \times Y \longrightarrow 2^Z$ *be a set-valued mapping satisfying:*

($i$) $F(\cdot, y)$ *is lower semicontinuous on* $X$ *for all* $y \in Y$*;*

($ii$) $F(x, \cdot)$ *is convex on* $Y$ *for all* $x \in X$*;*

$(iii)$ $F(\varphi(y), y) \subset M$ *for all* $y \in Y$;

$(iv)$ *there exist a compact subset* $Y_0$ *of* $Y$ *and* $y_0 \in Y_0$ *such that* $F(x, y_0) \cap (Z \setminus M) \neq \emptyset$ *for all* $x \in X \setminus \varphi(Y_0)$.

Then there exists $x_0 \in X$ such that $F(x_0, y) \subset M$ for all $y \in Y$.

## 4. An Application

In this section, we prove the existence of solution for a variational inclusion.

**Theorem 4.1.** *Let* $E, G, Z$ *be real normed spaces and let* $\mathcal{B}$ *denote the space of all bounded linear operators from* $E$ *to* $Z$. *Assume* $X$ *is a compact convex subset of* $E$, *and* $M$ *is an open convex subset of* $Z$ *with* $0 \notin M$. *Let* $\varphi : G \to E$ *be a continuous linear mapping,* $Y = \varphi^{-1}(X)$, *and let* $T : X \to 2^{\mathcal{B}}$ *be an upper semi-continuous set-valued mapping with* $card\ T(x) < +\infty$ *for all* $x \in X$. *For* $x, u \in X$, *let* $T(x)(u) = \bigcup_{A \in T(x)} A(u)$. *Then there exists* $x_0 \in X$ *such that*

$$T(x_0)(x_0 - \varphi(y)) \cap (Z \setminus M) \neq \emptyset$$

*for all* $y \in Y$.

*Proof.* Define $F : X \times Y \longrightarrow 2^Z$ by $F(x, y) = T(x)(x - \varphi(y))$ for $(x, y) \in X \times Y$.We verify the hypotheses of Theorem 3.5 hold.

In order to verify (i), fixed $x \in X$ and $y \in Y$, let $U$ be a neighborhood of $F(x, y)$. Since $card\ T(x) < +\infty$, there exists $\varepsilon > 0$ such that for all $A \in T(x)$

$$O_Z(A(x - \varphi(y)), \varepsilon) \subset U, \tag{5}$$

where $O_Z(A(x-\varphi(y)), \varepsilon)$ is the open ball in $Z$ with center $A(x-\varphi(y))$ and radius $\varepsilon$. Let

$$\varepsilon_1 = min(\frac{\varepsilon}{3(\|\varphi(y)\| + 1)}, \frac{\varepsilon}{3(\|x\| + 1)}).$$

Since $T$ is upper semicontinuous on $X$, for $\cup_{A \in T(x)} O_{\mathcal{B}}(A, \varepsilon_1)$ ( a neighborhood of $T(x)$ in $\mathcal{B}$) , there exists $r^* > 0$ such that for all $w \in O_X(x, r^*)$

$$T(w) \subset \cup_{A \in T(x)} O_{\mathcal{B}}(A, \varepsilon_1). \tag{6}$$

Since $card\ T(x) < +\infty$, $C \triangleq \sup\{\|A\| : A \in T(x)\} < +\infty$. Let

$$r = min(\frac{\varepsilon}{3(C + 1)}, r^*, 1).$$

We claim that for all $x' \in O_X(x, r)$, $F(x', y) \subset U$ which implies that $F(\cdot, y)$ is usc on $X$. Let $x' \in O_X(x, r)$ and let $A_1 \in T(x')$. From (6) and the fact that $r \leq r^*$, we have $T(x') \subset \bigcup_{A \in T(x)} O_{\mathcal{B}}(A, \varepsilon_1)$.Therefore, there exists $A_0 \in T(x)$ such that $A_1 \in O_{\mathcal{B}}(A_0, \varepsilon_1)$. It is easy to check that

$$\|A_1(x' - \varphi(y)) - A_0(x - \varphi(y))\| < \varepsilon.$$

Therefore $A_1(x' - \varphi(y)) \in \cup_{A \in T(x)} O_Z(A(x - \varphi(y)), \varepsilon)$ for all $A_1 \in T(x')$. Hence $F(x', y) \subset U$ by (5).

In order to verify (ii), let $A \in T(x)$, $y_1, y_2 \in Y$, and $\lambda \in [0, 1]$. Since $A$ is linear,

$$A(x - \lambda\varphi(y_1) - (1 - \lambda)\varphi(y_2)) = \lambda A(x - \varphi(y_1))$$
$$+(1 - \lambda)A(x - \varphi(y_2)) \in \lambda T(x)(x - \varphi(y_1)) + (1 - \lambda)T(x)(x - \varphi(y_2)).$$

From this and linearity of $\varphi$, $F(x, \lambda y_1 + (1 - \lambda)y_2) \subset \lambda F(x, y_1) + (1 - \lambda)F(x, y_2)$. Hence $F(x, \cdot)$ is concave on $X$. Finally, since $0 \in Z \setminus M$ and

$$F(\varphi(y), y) = T(\varphi(y))(\varphi(y) - \varphi(y)) = T(\varphi(y))(0) = \cup_{A \in T(\varphi(y))} A(0) = 0,$$

the condition (iii) is satisfied.

Therefore, applying Theorem 3.5, we can see that there exists $x_0 \in X$ such that $F(x_0, y) \cap (Z \setminus M) \neq \emptyset$ for all $y \in Y$,which is exactly the desired conclusion. ■

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