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WEAK CONVERGENCE THEOREM BY A MODIFIED EXTRAGRADIENT METHOD FOR NONEXPANSIVE MAPPINGS AND MONOTONE MAPPINGS

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Abstract. In this paper, we introduce a modified extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone and Lipschitz continuous mapping. Our modified extragradient method is a variant of the so-called extragradient method. We obtain a weak convergent theorem for two sequences generated by this modified extragradient method.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. A mapping $A : C \to H$ is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C.$$

The variational inequality problem is the problem of finding $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

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The set of solutions of the variational inequality problem is denoted by Ω. A mapping $A : C \to H$ is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \quad \forall u, v \in C;$$

see Refs. 1-2. It is obvious that each α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous. A mapping $S : C \to C$ is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|, \quad \forall u, v \in C;$$

see Ref. 3. We denote by $F(S)$ the set of fixed points of S. For finding an element of $F(S) \cap \Omega$ under the assumption that a set $C \subset H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is α-inverse-strongly-monotone, Takahashi and Toyoda (Ref. 4) considered the following iterative scheme:

$$(I) \qquad \begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \ \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap \Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated by (I) converges weakly to some point of $F(S) \cap \Omega$. On the other hand, to solve the variational inequality problem in the finite-dimensional Euclidean space $\mathcal{R}^n$ under the assumption that a set $C \subset \mathcal{R}^n$ is nonempty, closed and convex, a mapping $A : C \to \mathcal{R}^n$ is monotone and k-Lipschitz continuous and Ω is nonempty, Korpelevich (Ref. 5) first introduced the following so-called extragradient method:

$$(II) \qquad \begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A \bar{x}_n), \ \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 1/k)$. He showed that the sequences $\{x_n\}$ and $\{\bar{x}_n\}$ generated by this extragradient method converge to the same point $z \in \Omega$. Recently, motivated by Korpelevich's extragradient method, Nadezhkina and Takahashi (Ref. 10) constructed an iterative scheme to find an element of $F(S) \cap \Omega$ and presented the following weak convergence result.

Theorem 1.1. *See Theorem 3.1 in Ref. 10. Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a monotone and k-Lipschitz continuous mapping and $S : C \to C$ a nonexpansive mapping such that $F(S) \cap \Omega \neq$*

$\emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by

$$(1) \qquad \begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega$, where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega} x_n$.

Inspired by Nadezhkina and Takahashi's results (Ref. 10), the authors (Ref. 12) introduced another iterative scheme for finding an element of $F(S) \cap \Omega$ and obtained the following strong convergence theorem.

Theorem 1.2. *See Theorem 3.1 in Ref. 12. Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a monotone and k-Lipschitz continuous mapping and $S : C \to C$ a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions: (a) $\{\lambda_n k\} \subset (0, 1-\delta)$ for some $\delta \in (0, 1)$, and (b) $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $P_{F(S)\cap\Omega}(x_0)$ provided $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

In this paper, we consider a modified extragradient method which is a variant of the extragradient method, i.e.,

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C[(1-\beta_n)(x_n - \lambda_n A x_n) + \beta_n P_C(x_n - \lambda_n A x_n)], \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0\,, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1), \{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, 1/k)$ such that $\displaystyle\sum_{n=0}^{\infty} \beta_n^2 < \infty$

It is shown that $\{x_n\}$ and $\{y_n\}$ generated by the above scheme converge weakly to the same point $z \in F(S) \cap \Omega$, where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega} x_n$. It is easy to see that if $\beta_n = 0$ for all $n \geq 0$, then this iterative scheme reduces to (1). Our main result is the improvement and extension of Nadezhkina and Takahashi's result in Ref. 10.

Throughout the rest of this paper, we denote by "$\to$" and "$\rightharpoonup$" the strong convergence and the weak convergence, respectively.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. It is known that

$$\|\lambda x+(1-\lambda)y\|^2 = \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H, \lambda \in [0, 1].$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called the metric projection of H onto C. Then P_C is a nonexpansive mapping of H onto C characterized by the following properties (see Ref. 3 for more details): $P_C x \in C$ and for all $x \in H, y \in C$,

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{2}$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2. \tag{3}$$

Let $A : C \to H$ be a mapping. It is easy to see from (2) that the following implications hold:

$$\bar{x} \in \Omega \iff \bar{x} = P_C(\bar{x} - \lambda A\bar{x}), \ \forall \lambda > 0. \tag{4}$$

Note that H satisfies the Opial property (Ref. 6): for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ we have $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. A monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for all $(y, g) \in G(T)$, then $f \in Tx$. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-y, w\rangle \geq 0,\ \forall y \in C\}$. Define a set-valued mapping $T : H \to 2^H$ by

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$; see Ref. 7.

The following lemmas will be used in the sequel.

Lemma 2.1. *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \geq 0$, and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that*

$$\limsup_{n\to\infty} \|v_n\| \leq c,\ \limsup_{n\to\infty} \|w_n\| \leq c \text{ and } \lim_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n)w_n\| = c$$

for some $c \geq 0$. Then $\lim_{n\to\infty} \|v_n - w_n\| = 0$.

Lemma 2.2. *Let H be a real Hilbert space and let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad \forall n \geq 0.$$

Then the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

Remark 2.1. Lamma 2.1 was proved by Schu (Ref. 8) in a uniformly convex Banach space and Lamma 2.2 was proved by Takahashi and Toyoda (Ref. 4).

Lemma 2.3. *Demiclosedness Principle. See Ref. 3. Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If $F(S) \neq \emptyset$, then $I - S$ is demiclosed, that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I-S)x_n\}$ converges strongly to some y, it follows that $(I-S)x = y$. Here I is the identity operator of H.*

Lemma 2.4. *See Ref. 11. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

3. Weak Convergence Theorem

In this section, we use a modified extragradient method to find a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone and Lipschitz continuous mapping in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a monotone and k-Lipschitz continuous mapping and $S : C \to C$ a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C[(1-\beta_n)(x_n - \lambda_n A x_n) + \beta_n P_C(x_n - \lambda_n A x_n)], \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0\,, \end{cases}$$

where $\{\alpha_n\} \subset [a, b]$, for some $a, b \in (0, 1)$, $\{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [c, d]$, for some $c, d \in (0, 1/k)$, such that

(*i*) $\sum_{n=0}^{\infty} \beta_n^2 < \infty$;

(*ii*) $\{Ax_n\}$ *is bounded.*

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega$, where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega} x_n$.

Proof. We first claim that $\{x_n\}$ is bounded. Indeed, put $t_n = P_C(x_n - \lambda_n A y_n)$ for all $n \geq 0$. Let $x^* \in F(S) \cap \Omega$. Then $x^* = P_C(x^* - \lambda_n A x^*)$. Taking $x = x_n - \lambda_n A y_n$ and $y = x^*$ in (3), we obtain

$$\begin{aligned} \|t_n - x^*\|^2 &\leq \|x_n - \lambda_n A y_n - x^*\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - x^*\|^2 - 2\lambda_n \langle A y_n, x_n - x^* \rangle + \lambda_n^2 \|A y_n\|^2 \\ &\quad - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, x_n - t_n \rangle - \lambda_n^2 \|A y_n\|^2 \\ &= \|x_n - x^*\|^2 + 2\lambda_n \langle A y_n, x^* - t_n \rangle - \|x_n - t_n\|^2 \\ &= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - 2\lambda_n \langle A y_n - A x^*, y_n - x^* \rangle \\ &\quad - 2\lambda_n \langle A x^*, y_n - x^* \rangle + 2\lambda_n \langle A y_n, y_n - t_n \rangle \\ &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n \rangle. \end{aligned} \tag{5}$$

Now, observe that

$$
\begin{aligned}
&\langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), P_C(x_n - \lambda_n A x_n) - y_n\rangle \\
&\le \|x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n)\| \|P_C(x_n - \lambda_n A x_n) - y_n\| \\
&\le \{\lambda_n \|A x_n\| + \|x_n - P_C(x_n - \lambda_n A x_n)\|\} \|x_n - \lambda_n A x_n \\
&\quad -[(1-\beta_n)(x_n - \lambda_n A x_n) + \beta_n P_C(x_n - \lambda_n A x_n)]\| \\
&= \beta_n \{\lambda_n \|A x_n\| + \|P_C x_n - P_C(x_n - \lambda_n A x_n)\|\} \|x_n - \lambda_n A x_n \\
&\quad -P_C(x_n - \lambda_n A x_n)\| \\
&\le 2\beta_n \lambda_n \|A x_n\| \{\lambda_n \|A x_n\| + \|P_C x_n - P_C(x_n - \lambda_n A x_n)\|\} \\
&\le 2\beta_n \lambda_n \|A x_n\| \{2\lambda_n \|A x_n\|\} \\
&= 4\beta_n \lambda_n^2 \|A x_n\|^2.
\end{aligned}
$$

Further, from (2) we have

$$
\begin{aligned}
&\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle \\
&= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&= \langle (1-\beta_n)(x_n - \lambda_n A x_n) + \beta_n P_C(x_n - \lambda_n A x_n) - y_n, t_n - y_n\rangle \\
&\quad +\beta_n \langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - y_n\rangle \\
&\quad +\langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&\le \beta_n \langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - y_n\rangle \\
&\quad +\langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&= \beta_n \langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - P_C(x_n - \lambda_n A x_n)\rangle \\
&\quad +\beta_n \langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), P_C(x_n - \lambda_n A x_n) - y_n\rangle \\
&\quad +\langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&\le \beta_n \langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), P_C(x_n - \lambda_n A x_n) - y_n\rangle \\
&\quad +\langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&\le 4\beta_n^2 \lambda_n^2 \|A x_n\|^2 + \lambda_n k \|x_n - y_n\| \|t_n - y_n\| \\
&\le 4\beta_n^2 \lambda_n^2 \|A x_n\|^2 + \frac{1}{2}\lambda_n^2 k^2 \|x_n - y_n\|^2 + \frac{1}{2}\|y_n - t_n\|^2.
\end{aligned}
$$

Thus, we deduce that for all $n \geq 0$

$$\begin{aligned}
\|t_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad +8\beta_n^2\lambda_n^2\|Ax_n\|^2 + \lambda_n^2 k^2\|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\
&= \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + 8\beta_n^2\lambda_n^2\|Ax_n\|^2 \\
&\leq \|x_n - x^*\|^2 + 8\beta_n^2\lambda_n^2\|Ax_n\|^2,
\end{aligned} \tag{6}$$

and hence

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n x_n + (1-\alpha_n)St_n - x^*\|^2 \\
&= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(St_n - x^*)\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|St_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|t_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{\|x_n - x^*\|^2 \\
&\quad +(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + 8\beta_n^2\lambda_n^2\|Ax_n\|^2\} \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)\{(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\
&\quad +8\beta_n^2\lambda_n^2\|Ax_n\|^2\} \\
&\leq \|x_n - x^*\|^2 + 8\beta_n^2\lambda_n^2\|Ax_n\|^2.
\end{aligned} \tag{7}$$

Note that $\{Ax_n\}$ is bounded and $\sum_{n=0}^{\infty}\beta_n^2$ is convergent. Therefore, according to Lemma 2.4,

$$\ell = \lim_{n\to\infty}\|x_n - x^*\|$$

exists and hence the sequences $\{x_n\}$ and $\{t_n\}$ are bounded. By (7),

$$\begin{aligned}
(1-\alpha_n)(1-\lambda_n^2 k^2)\|x_n - y_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad +8(1-\alpha_n)\beta_n^2\lambda_n^2\|Ax_n\|^2.
\end{aligned}$$

So we have for all $n \geq 0$,

$$\begin{aligned}
\|x_n - y_n\|^2 &\leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) \\
&\quad +\frac{8\beta_n^2\lambda_n^2}{1-\lambda_n^2 k^2}\|Ax_n\|^2.
\end{aligned}$$

Since $\{\alpha_n\} \subset (a,b) \subset (0,1)$ and $\{\lambda_n\} \subset (c,d) \subset (0,1/k)$, we have

$$x_n - y_n \to 0 \quad \text{as } n \to \infty.$$

On the other hand we obtain

$$
\begin{aligned}
&\|y_n - t_n\|^2 \\
&= \|P_C[(1-\beta_n)(x_n - \lambda_n Ax_n) + \beta_n P_C(x_n - \lambda_n Ax_n)] - P_C(x_n - \lambda_n Ay_n)\|^2 \\
&\leq \|(1-\beta_n)(x_n - \lambda_n Ax_n) + \beta_n P_C(x_n - \lambda_n Ax_n) - (x_n - \lambda_n Ay_n)\|^2 \\
&= \|(1-\beta_n)\lambda_n(Ay_n - Ax_n) + \beta_n[P_C(x_n - \lambda_n Ax_n) - (x_n - \lambda_n Ay_n)]\|^2 \\
&\leq (1-\beta_n)\lambda_n^2\|Ay_n - Ax_n\|^2 + \beta_n\|P_C(x_n - \lambda_n Ax_n) - (x_n - \lambda_n Ay_n)\|^2 \\
&\leq \lambda_n^2 k^2\|y_n - x_n\|^2 + \beta_n\{\|P_C(x_n - \lambda_n Ax_n) - P_C x_n\| + \lambda_n\|Ay_n\|\}^2 \\
&\leq \lambda_n^2 k^2\|y_n - x_n\|^2 + \beta_n(\lambda_n\|Ax_n\| + \lambda_n\|Ay_n\|)^2 \\
&= \lambda_n^2 k^2\|y_n - x_n\|^2 + \beta_n\lambda_n^2(\|Ax_n\| + \|Ay_n\|)^2 \\
&\leq \frac{\lambda_n^2 k^2}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) \\
&\quad + \frac{8\beta_n^2\lambda_n^4 k^2}{1-\lambda_n^2 k^2}\|Ax_n\|^2 + \beta_n\lambda_n^2(\|Ax_n\| + \|Ay_n\|)^2.
\end{aligned}
$$

Since $\sum_{n=0}^{\infty}\beta_n^2 < \infty$, we have $\lim_{n\to\infty}\beta_n = 0$ and so

$$y_n - t_n \to 0 \quad \text{as } n \to \infty.$$

From

$$\|x_n - t_n\| \leq \|x_n - y_n\| + \|y_n - t_n\|,$$

we also have

$$x_n - t_n \to 0 \quad \text{as } n \to \infty.$$

The Lipschitz continuity od A implies that

$$Ay_n - At_n \to 0 \quad \text{as } n \to \infty.$$

Since $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some z.

We claim that $z \in F(S) \cap \Omega$. We shall prove that $z \in \Omega$. Since $x_n - t_n \to 0$ and $y_n - t_n \to 0$, we have $t_{n_i} \rightharpoonup z$ and $y_{n_i} \rightharpoonup z$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v, w) \in G(T)$ so that

$$w \in Tv = Av + N_C v$$

and hence $w - Av \in N_C v$. This shows that

$$\langle v - u, w - Av\rangle \geq 0, \quad \forall u \in C.$$

On the other hand, from

$$t_n = P_C(x_n - \lambda_n A y_n) \text{ and } v \in C,$$

we have

$$\langle x_n - \lambda_n A y_n - t_n, t_n - v\rangle \geq 0,$$

and hence

$$\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + Ay_n\rangle \geq 0.$$

Since

$$w - Av \in N_C v \text{ and } t_{n_i} \in C,$$

we have

$$\begin{aligned}\langle v - t_{n_i}, w\rangle &\geq \langle v - t_{n_i}, Av\rangle \\ &\geq \langle v - t_{n_i}, Av\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle \\ &\quad - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle.\end{aligned}$$

Hence

$$\langle v - z, w\rangle \geq 0 \quad \text{as } n_i \to \infty.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in \Omega$.

Next, we will show that $z \in F(S)$. Indeed, let $x^* \in F(S) \cap \Omega$. By (6), for all $n \geq 0$,

$$\|St_n - x^*\|^2 \leq \|t_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 8\beta_n^2\lambda_n^2\|Ax_n\|^2,$$

which implies that

$$\limsup_{n\to\infty} \|St_n - x^*\| \leq \ell.$$

Moreover,

$$\lim_{n\to\infty} \|\alpha_n(x_n - x^*) + (1 - \alpha_n)(St_n - x^*)\| = \lim_{n\to\infty} \|x_{n+1} - x^*\| = \ell.$$

By Lemma 2.1,

$$\lim_{n\to\infty} \|St_n - x_n\| = 0.$$

Since

$$\|Sx_n - x_n\| \le \|Sx_n - St_n\| + \|St_n - x_n\| \le \|x_n - t_n\| + \|St_n - x_n\|,$$

we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

By Lemma 2.3, $I - S$ is demiclosed. Since $x_{n_i} \rightharpoonup z$, it follows that $z \in F(S)$.

Let $\{x_{m_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{m_j} \rightharpoonup z'$. Then $z' \in F(S)\cap\Omega$. We will show that $z = z'$. Assume that $z \neq z'$. From the Opial condition (Ref. 6) we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z\| &= \liminf_{n\to\infty} \|x_{n_i} - z\| < \liminf_{i\to\infty} \|x_{n_i} - z'\| \\ &= \lim_{n\to\infty} \|x_n - z'\| = \liminf_{j\to\infty} \|x_{m_j} - z'\| \\ &< \liminf_{j\to\infty} \|x_{m_j} - z\| = \lim_{n\to\infty} \|x_n - z\|.\end{aligned}$$

This is a contradiction. Consequently, we have $z = z'$. This assures that

$$x_n \rightharpoonup z \in F(S) \cap \Omega.$$

Since $x_n - y_n \to 0$ as $n \to \infty$, we have

$$y_n \rightharpoonup z \in F(S) \cap \Omega.$$

Now, let $u_n = P_{F(S)\cap\Omega} x_n$. By (5) and the monotonicity of A, for $u = F(S) \cap \Omega$, we have $\|t_n - u\| \le \|x_n - u\|$ and so

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n(x_n - u) + (1-\alpha_n)[SP_C(x_n - \lambda_n Ay_n) - u]\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (\alpha_n)\|t_n - u\|^2 \\ &\le \|x_n - u\|^2.\end{aligned}$$

By Lemma 2.2, $\{u_n\}$ converges strongly to some $z_0 \in F(S) \cap \Omega$. Since $\langle z - u_n, u_n - x_n\rangle \ge 0$, it follows that $\langle z - z_0, z_0 - z\rangle \ge 0$, and hence $z = z_0$. This completes the proof of Theorem 3.1. ■

Remark 3.1. We note that in Theorem 3.1, if $\beta_n = 0$ for all $n \ge 0$, it follows from (7) that

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2, \quad \forall n \ge 0.$$

Thus $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Hence $\{x_n\}$ is bounded and so is $\{Ax_n\}$. In this case, we can remove the boundedness of $\{Ax_n\}$. Consequently, Nadezhkina and Takahashi's Theorem 3.1 (Ref. 10) follows immediately from our Theorem 3.1.

4. Applications

In this section, we give two applications of Theorem 3.1.

Theorem 4.1. *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone k-Lipschitz continuous mapping and let $S : H \to H$ be a nonexpansive mapping such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 &= x \in H, \\ y_n &= x_n - \lambda_n A x_n, \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n) S(x_n - \lambda_n A y_n), \ \forall n \geq 0\,, \end{cases}$$

where $\{\alpha_n\} \subset [a,b]$, for some $a, b \in (0,1)$, and $\{\lambda_n\} \subset [c,d]$, fore some $c, d \in (0, 1/k)$. Then $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap A^{-1}0$, where $z = \lim_{n\to\infty} P_{F(S)\cap A^{-1}0} x_n$.

Proof. It is obvious that $A^{-1}0 = \Omega$ and $P_H = I$ is the identity mapping of H. Put $\beta_n = 0$ for all $n \geq 0$. Then we have

$$\begin{aligned} y_n &= P_H[(1-\beta_n)(x_n - \lambda_n A x_n) + \beta_n P_H(x_n - \lambda_n A x_n)] \\ &= x_n - \lambda_n A x_n, \end{aligned}$$

and

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n) S P_H(x_n - \lambda_n A y_n) \\ &= \alpha_n x_n + (1-\alpha_n) S(x_n - \lambda_n A y_n). \end{aligned}$$

Note that inequality (7) yields

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2.$$

This implies that $\{x_n\}$ is bounded and so is $\{Ax_n\}$. Hence by Theorem 3.1 we obtain the desired result. ■

Remark 4.1. Notice that $F(S) \cap A^{-1}0$ is contained in the set of solutions of the variational inequality problem VI($F(S), A$). See also Yamada (Ref. 9) for the case when $A : H \to H$ is a strongly monotone and Lipschitz continuous mapping and $S : H \to H$ is a nonexpansive mapping.

Theorem 4.2. *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone k-Lipschitz continuous mapping and $B : H \to 2^H$ be a maximal monotone mapping* *such that* $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. *Let* J_r^B *be the resolvent of* B *for each* $r > 0$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences generated by*

$$\begin{cases} x_0 &= x \in H, \\ y_n &= x_n - \lambda_n A x_n, \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n) J_r^B (x_n - \lambda_n A y_n), \ \forall n \geq 0\,, \end{cases}$$

where $\{\alpha_n\} \subset [a, b]$, *for some* $a, b \in (0, 1)$, *and* $\{\lambda_n\} \subset [c, d]$, *fore some* $c, d \in (0, 1/k)$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to the same point* $z \in A^{-1}0 \cap B^{-1}0$, *where* $z = \lim_{n\to\infty} P_{A^{-1}0 \cap B^{-1}0} x_n$.

Proof. We have $A^{-1}0 = \Omega$ and $F(J_r^B) = B^{-1}0$. Putting $P_H = I$, by Theorem 4.1 we obtain the desired result. ∎

Remark 4.2. Theorems 4.1 and 4.2 are essentially Nadezhkina and Takahashi's Theorems 4.1 and 4.2 (Ref. 10), respectively.

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