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STAR MATCHING AND DISTANCE TWO LABELLING

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Abstract. This paper first introduces a new graph parameter. Let t be a positive integer. A t-star-matching of a graph G is a collection of mutually vertex disjoint subgraphs $K_{1,i}$ of G with $1 \le i \le t$. The t-star-matching number, denoted by $SM_t(G)$, is the maximum number of vertices covered by a t-star-matching of G. Clearly $SM_1(G)/2$ is the edge independence number of G.

An $L(2,1)$-labelling of a graph G is an assignment of nonnegative integers to the vertices of G such that vertices at distance at most two get different numbers and adjacent vertices get numbers which are at least two apart. The $L(2,1)$-labelling number of a graph G is the minimum range of labels over all $L(2,1)$-labellings. If we require the assignment to be one-to-one, then similarly as above we can define the $L'(2,1)$-labelling and the $L'(2,1)$-labelling number of a graph G. Given a graph G, the path covering number of G, denoted by $p_v(G)$, is the smallest number of vertex-disjoint paths covering $V(G)$. By G^c we denote the complement graph of G.

In this paper, we design a polynomial time algorithm to compute $SM_t(G)$ for any graph G and any integer $t \geq 2$ and studies the properties of t-star-matchings of a graph G. For any graph G, we determine the path covering numbers of $(\mu(G))^c$ and $(G \times \hat{K}_2)^c$ in terms of $SM_4(G^c)$, and the $L'(2,1)$-labelling umbers of $\mu(G)$ and $G \times \hat{K}_2$ in terms of $SM_4(G^c)$, where $\mu(G)$ is the Mycielskian of G and $G \times \hat{K}_2$ is the direct product of G and $\hat{K}_2$ ($\hat{K}_2$ is a graph obtained from K_2 by adding a loop on one of its vertices). Our results imply that the path covering numbers of $(\mu(G))^c$ and $(G \times \hat{K}_2)^c$, the $L'(2,1)$-labelling umbers of $\mu(G)$ and $G \times \hat{K}_2$ can be computed in polynomial time for any graph G. So, for any graph G, it is polynomial-time solvable to determine whether $(\mu(G))^c$ and $(G \times \hat{K}_2)^c$ has a Hamiltonian path. And consequently, for any graph $G = (V, E)$, it is polynomially solvable to determine whether $\lambda(\mu(G)) \leq s$ for each $s \geq |V(\mu(G))|$ and $\lambda(G \times \hat{K}_2) \leq s$ for each $s \geq |V(G \times \hat{K}_2)|$. Using these results, we easily determine $L(2,1)$-labelling numbers and $L'(2,1)$-labelling numbers of several classes of graphs.

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1. Introduction

Given a graph G and an integer $t(\geq 1)$. A *t-star-matching* is a collection of disjoint subgraphs of G in which each subgraph is isomorphic to $K_{1,i}$ for some $i \in [1, t]$, where $K_{1,i}$ is an i-star. Let Γ be a t-star-matching of a graph G. For a vertex v of G, if there exits some $H \in \Gamma$ such that $v \in V(H)$ then v is said to be Γ-*saturated*, otherwise v is said to be Γ-*unsaturated*. A t-star-matching Γ is maximum if the number of Γ-saturated vertices is maximum among all t-star-matchings of G. The *t-star-matching number* of a graph G, denoted by $SM_t(G)$, is the number of vertices in G saturated by a maximum t-star-matching of G. Obviously, when $t = 1$, $SM_t(G)/2$ is precisely the edge independence number of G. Thus we can view the t-star-matching (the t-star-matching number) of a graph be a generalization of the matching (the edge independence number) of a graph. As far as we know, this kind of generalization had never been defined before. The maximum 1-star-matching (the maximum matching) of any simple graph G can be found in $O(n^3)$ (where $n = |V(G)|$) by Edmonds's Cardinality Matching Algorithm [5]. The current best know known algorithm (Micali and Vazirani [18]) for this problem has a running time $O(n^{\frac{1}{2}}m)$, where $n = |V(G)|$ and $m = |E(G)|$. In the next section, we shall give an algorithm to find the maximum t-star-matching for any fixed $t \geq 2$ with running time $O(nm)$. Thus we assume $t \geq 2$ throughout this paper.

In a kind of channel assignment problem, in order to avoid interference, "close" transmitters are required to receive different channels and "very close" transmitters are required to receive channels that are at least two channel apart. Griggs and Yeh in [21, 14] formulated this problem in graphs, in which radio transmitters are represented by vertices, two vertices are "very close" if they are adjacent in the graph and "close" if they are at distance 2 in the graph. More precisely, an $L(2,1)$-labelling f of a graph G is an assignment f of nonnegative integers to the vertices of G such that $|f(u)-f(v)| \geq 2$ if $uv \in E(G)$, and $|f(u)-f(v)| \geq 1$ if $d_G(u,v) = 2$, where $d_G(u,v)$ is the length (number of edges) of a shortest path between u and v in G. Given a graph G, for an $L(2,1)$-labelling f of G, elements of the image of f are called labels, and we define the span of f, denoted by $span(f)$, to be the difference between the maximum and minimum vertex labels of f. Without loss of generality we shall assume that the minimum label of $L(2,1)$-labellings of G is always 0. Then the span of f is the maximum vertex label. The $L(2,1)$-*labelling number*, denoted by $\lambda(G)$, is the minimum span over all $L(2,1)$-labellings of G. If $span(f) = \lambda(G)$, then we say that f is a λ-labelling of G. If we require the function f to be one-to-one, then similarly as above we can define the $L'(2,1)$-*labelling* and the $L'(2,1)$-*labelling number* (denoted by $\lambda'(G)$) of a graph G. We refer the reader to surveys on $L(2,1)$-labelling and its generalization $L(h,k)$-labelling [2, 13].

Given a graph G, the *path covering number* of G, denoted by $p_v(G)$, is the smallest number of vertex-disjoint paths covering $V(G)$. By G^c we denote the complement graph of G. The following result was proved by Georges et al. [12].

Theorem 1.1. *Given a graph G on n vertices, then*

$$\lambda(G)\begin{cases}\le n-1, & \text{if } \ p_v(G^c)=1,\\ = n+p_v(G^c)-2, & \text{if } \ p_v(G^c)\ge 2.\end{cases}$$

From the proof of the above theorem [12], we know that actually one can have a one-to-one $L(2,1)$-labelling of G with span $n+p_v(G^c)-2$ when $p_v(G^c)\ge 2$ and $n-1$ when $p_v(G^c)=1$. Thus this theorem is equivalent to the following theorem.

Theorem 1.2. *Given a graph G on n vertices, then*

$$\lambda'_{2,1}(G)=n+p_v(G^c)-2.$$

To decide whether $\lambda_{2,1}(G)\le |V(G)|$ for diameter 2 graphs is NP-complete [14]. The problem remains NP-complete if we ask whether there exists an $L(2,1)$-labelling of span at most s, where s is a fixed constant ≥ 4, while it is polynomial if $s\le 3$ [7]. The problems of finding the $L(2,1)$-labelling number of graphs with diameter 2 [14, 21], planar graphs [1, 11], bipartite graphs [1], split and chordal graphs [1] are all NP-hard.

It is polynomially solvable to decide whether there exists an $L(2,1)$-labelling of span at most s for each tree and each given s [4]. The same is true for each p-almost tree and each given integer s [7]. (A p-almost tree is a connected graph G with $|V(G)|+p-1$ edges.) It was proved [4] that there is a linear time algorithm to compute $\lambda(G)$, $\lambda'(G)$, and $p_v(G)$ for a cograph G. It is proved [20] that $\lambda'(G)$ for a bipartite graph G can be computed in polynomial time. In this paper we shall provide two new classes of graphs G for which $\lambda'(G)$ can be computed in polynomial time.

In 1955, Mycielski [19] introduced a graph transformation. For a graph $G=(V,E)$, the *Mycielskian* of G, $\mu(G)$, is defined to be the graph with vertex set $V_0\cup V_1\cup\{u\}$, where $V_i=\{v^i : v\in V\}$ $(i=0,1)$, and edge set $\{x^0y^0 : xy\in E\}\cup\{x^0y^1 : xy\in E\}\cup\{uv^1 : v^1\in V_1\}$. Please see Figure 1 for an illustration of this definition. Mycielskians of graphs have many interesting properties concerning several kinds of graph colorings as well as other parameters of graphs and had been studied extensively, see [3, 6, 8, 9, 10, 15, 16, 17].

Given two graphs G and H, the *direct product* of G and H is the graph $G\times H$ with vertex set $V(G)\times V(H)$ in which two vertices (x,y) and (x',y') are adjacent if and only if $xx'\in E(G)$ and $yy'\in E(H)$. Let $\hat{K}_2$ denote the graph obtained from K_2 by adding a loop on one of its vertices.

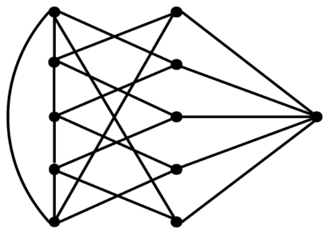

Fig. 1. The Mycielskian of C_5.

The next section gives a polynomial time algorithm to compute $SM_t(G)$ for any graph G and any integer $t \geq 2$ and studies the properties of t-star-matchings of a graph G. Section 3 determines, for any graph G, the path covering numbers of $(\mu(G))^c$ and $(G \times \hat{K}_2)^c$ in terms of $SM_4(G^c)$, and the $L'(2,1)$-labelling umbers of $\mu(G)$ and $G \times \hat{K}_2$ in terms of $SM_4(G^c)$. Our results imply that $p_v((\mu(G))^c)$, $\lambda'(\mu(G))$ and $p_v((G \times \hat{K}_2)^c)$, $\lambda'(G \times \hat{K}_2)$ can be computed in polynomial time for any graph G. So, for any graph G, it is polynomial-time solvable to determine whether $(\mu(G))^c$ and $(G \times \hat{K}_2)^c$ has a Hamiltonian path. And consequently, for any graph $G = (V, E)$, it is polynomially solvable to determine whether $\lambda(\mu(G)) \leq s$ for each $s \geq |V(\mu(G))|$ and $\lambda(G \times \hat{K}_2) \leq s$ for each $s \geq |V(G \times \hat{K}_2)|$. Using these results, we easily determine $L(2,1)$-labelling numbers and $L'(2,1)$-labelling numbers of $\mu(G)$ and $G \times \hat{K}_2$ for graphs G being K_n and $K_p \vee K_q^c$.

For nonnegative integers a and b with $a \leq b$, let $[a, b]$ denote the set $\{a, a+1, a+2, \ldots, b-1, b\}$.

2. t-STAR-MATCHING

Given a graph G and an integer $t \geq 2$. Suppose Γ is a t-star-matching of G. Let U denote the set of all Γ-unsaturated vertices. Denote by S the set of all vertices saturated by subgraphs $K_{1,i}$ with $1 \leq i \leq t-1$ in Γ. The vertex of degree i in $K_{1,i}$ is called the root of $K_{1,i}$ and is denoted by $r(K_{1,i})$. The vertices of degree 1 in $K_{1,i}$ are called leaves of $K_{1,i}$. If $i = 1$ then both ends of $K_{1,1}$ can be viewed as both roots and leaves. As we shall see, this will not affect our algorithms and proofs. Let R denote the set of roots of all subgraphs $K_{1,t}$ in Γ and F the set of leaves of all subgraphs $K_{1,t}$ in Γ.

A Γ-*augmenting path* is a path in G of the form $ur_1f_1r_2f_2 \ldots r_kf_kv$ with $u \in U$, $v \in S \cup F$, and for each $i \in [1, k]$, $r_i \in R$ and f_i a leaf in F adjacent to the root r_i. Please see Figure 2 for an illustration. An edge between U and $U \cup S \cup F$ is called a Γ-augmenting edge. And we shall consider a Γ-augmenting edge as a special Γ-augmenting path. A Γ-augmenting path is called *simple* if v is not contained in subgraphs $K_{1,t}$ with roots $r_1, r_2, \ldots, r_{k-1}$. A Γ-augmenting edge is also called a simple Γ-augmenting path. It is obvious that if G has a Γ-augmenting path then G has a simple Γ-augmenting path.

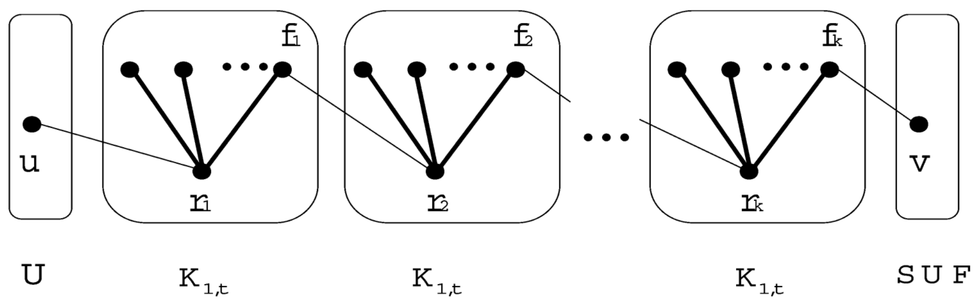

Fig. 2. A Γ-augmenting path.

The following Maximum t-star-matching Algorithm starts with any t-star-matching Γ. Each time it searches for a Γ-augmenting path and augments it along the path until no Γ-augmenting path exits. We actually only search for a simple Γ-augmenting path. The way to search a simple Γ-augmenting path is described in Simple Γ-augmenting Path Algorithm.

Let H be a subgraph of G, v a vertex of G, and xy an edge of G. We denote by $H - v$ the graph obtained from H by deleting the vertex v. $H + xy$ denotes the graph with vertex set $V(H) \cup \{x, y\}$ and edge set $E(H) \cup \{xy\}$. We simply write $(H - v) + xy$ as $H - v + xy$.

Maximum t-star-matching Algorithm

Input: A graph G and an integer $t \geq 2$.

Output: Maximum t-star-matching Γ of G.

Step 0. Choose a t-star-matching Γ of G. (Γ may be empty.) Identify the corresponding sets U, S, F and R.

Step 1. Find a simple Γ-augmenting path P of G (Apply the Simple Γ-augmenting Path Algorithm.) **If** G has no Γ-augmenting path, **then stop**. (Γ is a maximum t-star-matching of G.) **Else** let P be the Γ-augmenting path.

If P is a Γ-augmenting edge uv with $u \in U$ and $v \in U \cup S \cup F$ **then go to** Step 2, **else go to** Step 3.

Step 2. **If** v is in U **then** set $\Gamma := \Gamma \cup \{uv\}$, $U := U \setminus \{u, v\}$, $S := S \cup \{u, v\}$, $F := F$, $R := R$, and **go to** Step 1.

Else if v is in $S \cup F$ and is a root of some H in Γ, **then** set $\Gamma := (\Gamma \setminus \{H\}) \cup \{H + uv\}$.

If H is a $K_{1,t-1}$ of Γ **then** set $U := U \setminus \{u\}$, $S := S \setminus V(H)$, $F := F \cup (V(H) \setminus \{v\}) \cup \{u\}$, $R := R \cup \{v\}$, and **go to** Step 1. **Else** set $U := U \setminus \{u\}$, $S := S \cup \{u\}$, $F := F$, $R := R$, and **go to** Step 1.

Else if v is in $S \cup F$ and is a leaf of some H in Γ, **then** set $\Gamma := (\Gamma \setminus \{H\}) \cup \{H - v, uv\}$.

If H is a $K_{1,t}$ of Γ **then** set $U := U \setminus \{u\}$, $S := S \cup V(H) \cup \{u, v\}$, $F := F \setminus V(H)$, $R := R \setminus \{r(H)\}$, and **go to** Step 1. **Else** set $U := U \setminus \{u\}$, $S := S \cup \{u\}$, $F := F$, $R := R$, and **go to** Step 1.

Step 3. Let P be the nontrivial simple Γ-augmenting path of the form $ur_1f_1r_2f_2 \ldots r_kf_kv$ with $u \in U$, $v \in S \cup F$, $r_i \in R$ and $f_i \in F$ for $i \in [1, k]$. Let $H_1, H_2, \ldots, H_k$ be the subgraphs $K_{1,t}$ in Γ with roots $r_1, r_2, \ldots, r_k$ and let H be the element of Γ that saturates the vertex v. (Notice that if H is H_k then v is a leaf of H_k.)

If v is the root of H and thus $H \neq H_k$ **then** set

$$\Gamma := (\Gamma \setminus \{H_1, H_2, \ldots, H_k, H\}) \cup \{H_1 - f_1 + ur_1, H_2 - f_2 + f_1r_2, \ldots, H_k - f_k + f_{k-1}r_k, H + f_kv\}.$$

If H is a $K_{1,t-1}$ of Γ **then** set $U := U \setminus \{u\}$, $S := S \setminus V(H)$, $F := F \cup (V(H) \setminus \{v\}) \cup \{u\}$, $R := R \cup \{v\}$, and **go to** Step 1. **Else** set $U := U \setminus \{u\}$, $S := S \cup \{f_k\}$, $F := (F \cup \{u\}) \setminus \{f_k\}$, $R := R$, and **go to** Step 1.

Else if v is a leaf of H and $H \neq H_k$ **then** set

$$\Gamma := (\Gamma \setminus \{H_1, H_2, \ldots, H_k, H\}) \cup \{H_1 - f_1 + ur_1, H_2 - f_2 + f_1r_2, \ldots, H_k - f_k + f_{k-1}r_k, H - v, f_kv\}.$$

If H is a $K_{1,t}$ of Γ **then** set $U := U \setminus \{u\}$, $S := S \cup V(H) \cup \{f_k\}$, $F := (F \cup \{u\}) \setminus (V(H) \cup \{f_k\})$, $R := R \setminus \{r(H)\}$, and **go to** Step 1. **Else** set $U := U \setminus \{u\}$, $S := S \cup \{f_k\}$, $F := (F \cup \{u\}) \setminus \{f_k\}$, $R := R$, and **go to** Step 1.

Else if v is a leaf of H and $H = H_k$ **then** set

$$\Gamma := (\Gamma \setminus \{H_1, H_2, \ldots, H_k, H\}) \cup \{H_1 - f_1 + ur_1, H_2 - f_2 + f_1r_2, \ldots, H_k \setminus \{f_k, v\} + f_{k-1}r_k, f_kv\}.$$

Set $U := U \setminus \{u\}$, $S := (S \cup V(H)) \cup \{f_{k-1}\}$, $F := (F \cup \{u\}) \setminus (V(H) \cup \{f_{k-1}\})$, $R := R \setminus \{r(H)\}$, and **go to** Step 1.

Given a graph G, an integer $t \geq 2$, and a t-star matching Γ of G, the following Simple Γ-augmenting Path Algorithm will find a simple Γ-augmenting path in G or declare that none exits. In the following algorithm, U, S, R, F are the sets as we defined in the beginning of this section.

A Γ-*alternating path* is a path in G of the form $ur_1f_1r_2f_2\ldots r_kf_k$ with $u \in U$, and for each $i \in [1,k]$, $r_i \in R$ and f_i a leaf in F adjacent to the root r_i. We shall denote by B the set of vertices and edges the algorithm has passed through currently. $\tilde{R}$ denotes the set of roots in B. $\hat{R}$ denotes the set of new roots adjacent in G to some non-root vertex of B. Denote by $\hat{F}$ the set of leaves adjacent to some vertex of $\hat{R}$ in Γ. Note that we always have $\hat{R} \subseteq R$ and $\hat{F} \subseteq F$.

For a subset W of $V(G)$, denote by $N_G(W)$ the set of vertices in $V(G) \setminus W$ that have a neighbor in W.

Simple Γ-augmenting Path Algorithm

Input: A graph G, an integer $t \geq 2$, a t-star matching Γ of G, and the corresponding sets U,S,F,R.

Output: A simple Γ-augmenting path (or report that none exits).

Step 0. **If** there is an edge uv between U and $U \cup S \cup F$ **then stop** (uv is a Γ-augmenting edge), **else go to** Step 1.

Step 1. Set $B := U$, $\tilde{R} := \varnothing$, and $\hat{R} := N_G(U)$.

Step 2. Set $\hat{F} := N_\Gamma(\hat{R})$.

If G has no edge between $\hat{F}$ and $F \cup S$ **then go to** Step 3, **else** use a backtracking process to find a simple Γ-augmenting path, and **stop**.

Step 3. Set $B := B \cup \hat{R} \cup \hat{F} \cup E_G(B, \hat{R}) \cup E_G(\hat{R}, \hat{F})$, $\tilde{R} := \tilde{R} \cup \hat{R}$, and $\hat{R} := N_G(\hat{F}) \setminus \tilde{R}$.

If $\hat{R} \neq \varnothing$ **then go to** Step 2, **else stop** and report that there is no Γ-augmenting path in G.

Note that if G has a Γ-augmenting paths then it also has simple Γ-augmenting paths, and it is obvious that the Simple Γ-augmenting Path Algorithm will surely find out one of the simple Γ-augmenting.

Let $n = |V(G)|$ and $m = |E(G)|$. Given a t-star-matching Γ of G, we may describe it by marking each vertex according to which $K_{1,i}$ it belongs to and whether it is a root or not. Thus Step 0 in the Maximum t-star-matching Algorithm takes at most $O(n)$ time steps. We call for the Simple Γ-augmenting Path Algorithm at most n times and make augmenting at most n times. It is easy to see that each augmenting takes at most $O(n)$ time steps. The running time of the Simple Γ-augmenting Path

Algorithm depends on the implementation of certain routines. If one using some vertex labeling technique in Step 0 it will complete this process within $O(m)$ time steps. Step 2 and Step 3 may repeat many times, however, as in Step 0, by labeling each vertex the algorithm has passed through properly, the time steps it takes in total will also within $O(m)$. Therefore the overall running time of the Maximum t-star-matching Algorithm is at most $O(nm)$.

Next we show that the output Γ of the Maximum t-star-matching Algorithm is a maximum t-star-matching of G.

Theorem 2.1. *A t-star-matching Γ of G is maximum if and only if there is no Γ-augmenting path in G.*

Proof. The necessity is obvious. So we only show the sufficiency. Suppose Γ is a t-star-matching of G such that G contains no Γ-augmenting path. Note that, in this case, the Simple Γ-augmenting Path Algorithm stops with $\hat{R} = \varnothing$. Recall that U is the set of vertices unsaturated by Γ. When the algorithm stops, B is the set of vertices and edges covered by all Γ-alternating paths and $\tilde{R}$ is the set of roots in B of subgraphs $K_{1,t}$ in Γ. Let $\tilde{F}$ denote the set of leaves in B of subgraphs $K_{1,t}$ in Γ. Clearly we have $B = U \cup \tilde{R} \cup \tilde{F} \cup E_G(U \cup \tilde{F}, \tilde{R})$. In the beginning, $\hat{R} = N_G(U)$. Since there is no Γ-augmenting path in G, $\hat{R} \subseteq R$. Then, in Step 2, we obtain the set $\hat{F}$. It is clear that $|\hat{F}| = t|\hat{R}|$. Afterwards each time after we get a nonempty vertex set $\hat{R}$ in Step 3 we shall go back to Step 2 and get the new $\hat{F}$ and we always have $|\hat{F}| = t|\hat{R}|$. Thus when the algorithm ends we have $|\tilde{F}| = t|\tilde{R}|$. Since there is no Γ-augmenting path in G, $U \cup \tilde{F}$ is independent in G and $N_G(U \cup \tilde{F}) = \tilde{R}$. It follows that at least $|U|$ vertices of $U \cup \tilde{F}$ can not be saturated by any t-star-matching of G. Hence Γ is maximum. ■

When the Simple Γ-augmenting Path Algorithm stops with the report that there is no Γ-augmenting path in G, we actually arrive at the state described in the proof of Theorem 2.1. We conclude that at least $|U|$ vertices of $U \cup \tilde{F}$ cannot be saturated by any t-star-matching of G and so Γ is maximum. Thus there is no Γ-augmenting path in G. So the Simple Γ-augmenting Path Algorithm gives a correct report. And the Maximum t-star-matching Algorithm stops only when the Simple Γ-augmenting Path Algorithm reports that there is no Γ-augmenting path in G, thus the Maximum t-star-matching Algorithm outputs a maximum t-star-matching of G.

The following theorem follows immediately from Theorem 2.1 and the above discussions.

Theorem 2.2. *Given any graph G of order n and an integer $t \geq 2$ as inputs, the Maximum t-star-matching Algorithm will find a maximum t-star-matching of G within $O(nm)$ time steps.*

Theorem 2.3. *Suppose $G = G(V, E)$ is a graph and V' is a subset of V. Let t be an integer at least 2. Then G has a t-star-matching saturating all vertices of V' if and only if for any independent set S included in V' there holds $|N_G(S)| \geq \lceil |S|/t \rceil$.*

Proof. If G has a t-star-matching saturating all vertices of V', then for any independent set S in V', since any vertex in V can match at most t vertices of S, we have $|N_G(S)| \geq \lceil |S|/t \rceil$.

Now assume that for any independent set $S \subseteq V'$ there holds $|N_G(S)| \geq \lceil |S|/t \rceil$. We shall show that G has a t-star-matching saturating all vertices of V'. Suppose to the contrary that G has no t-star-matching saturating all vertices of V'. Let Γ be the t-star-matching of G such that the number of saturated vertices in V' is maximum. Denote by U the set of Γ-unsaturated vertices in V'. Apparently G contains no Γ-augmenting path starting from some vertex of U. Let B be the set of vertices covered by all Γ-alternating paths starting from U. Let R be the set of roots of Γ that are contained in B and F the set of leaves of Γ that are contained in B. Clearly $U \cup F$ is an independent set. By the definition of B, $N_G(U \cup F) = R$ and $|F| = t|R|$. It follows that $|N_G(U \cup F)| < \lceil |U \cup F|/t \rceil$, which contradicts the assumption of the theorem. ■

Theorem 2.3 can be viewed as an extension of P. Hall's theorem.

A t-star-matching of a graph G is said to be *perfect* if it saturates all vertices of G.

Corollary 2.4. *Let G be a graph and $t \geq 2$ an integer. Then G has a perfect t-star-matching if and only if for any independent set S of G there holds $|N_G(S)| \geq \lceil |S|/t \rceil$.*

If $t = 1$ then Theorem 2.3 and Corollary 2.4 are not true. Consider a 5-cycle, it is obvious that each independent set of a 5-cycle has its neighborhood larger than itself, however there is no perfect 1-star-matching. But for bipartite graphs, the corresponding result is true for $t = 1$.

Corollary 2.5. *Suppose $G = G(X, Y, E)$ is a bipartite graph. Let t be an integer at least 1. Then G has a t-star-matching saturating all vertices of X if and only if for any $S \subseteq X$ there holds $|N_G(S)| \geq \lceil |S|/t \rceil$.*

3. λ' AND λ OF $\mu(G)$ AND $G \times \hat{K}_2$

We first determine the path covering numbers of $(\mu(G))^c$ in terms of $SM_4(G^c)$.

Theorem 3.1. *For any graph G on n vertices, we have*

$$p_v((\mu(G))^c) = \begin{cases} 1, & \text{if } SM_4(G^c) \geq n-4; \\ \lceil (n - SM_4(G^c) - 2)/2 \rceil, & \text{if } SM_4(G^c) < n-4. \end{cases}$$

Proof. Recall that the graph $\mu(G)$ has vertex set $V_0 \cup V_1 \cup \{u\}$. V_0 is corresponding to the vertex set of G and the neighborhood of u in $\mu(G)$ is V_1. For simplicity, by G' we denote the graph $(\mu(G))^c$. Then it is easy to see that $G'[V_0]$ is G^c, $G'[V_1]$ is a complete graph on n vertices, the neighborhood of u in G' is V_0, v_i^0 is adjacent to v_i^1 in G' for $i = 1, 2, \ldots, n$, and $v_i^0 v_j^1$ $(i \neq j)$ is an edge of G' if and only if $v_i v_j$ is nonadjacent in G. For each subset A of $V(G)$, we denote by $V_0(A)$ and $V_1(A)$ the subsets of $V(G')$ corresponding to A in V_0 and in V_1 respectively.

Let $q = n - SM_4(G^c)$. We first show that if $q > 4$ then $p_v(G') \leq \lceil (q-2)/2 \rceil$ and if $q \leq 4$ then $p_v(G') = 1$. Suppose Γ is a maximum 4-star-matching of G^c $(= G'[V_0])$. We first observe that for each H in Γ there exits a path in $G'[V_0(H) \cup V_1(H)]$ covering all vertices of $V_0(H) \cup V_1(H)$ with both its initial and end vertices in $V_1(H)$. This is illustrated in Figure 3.

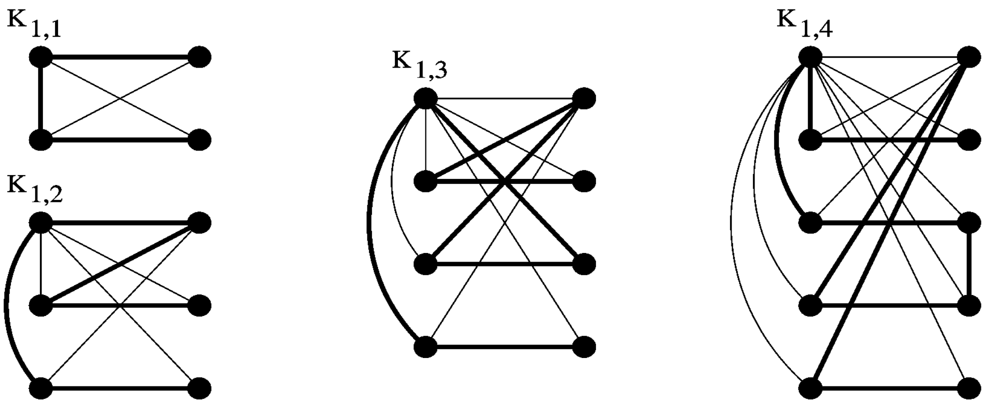

Fig. 3. Paths covering $V_0(H) \cup V_1(H)$ for $H = K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}$.

As $G'[V_1]$ is a complete graph, by connecting all these paths one can find a path P in G' which starts from and ends in V_1 and covers all Γ-saturated vertices in V_0 and their "twins" in V_1. Without loss of generality, let $v_1^0, v_2^0, \ldots, v_q^0$ be Γ-unsaturated vertices in V_0. If q is even and $q \geq 6$, then $v_2^0 v_2^1 P v_1^1 v_1^0 u v_4^0 v_4^1 v_3^1 v_3^0$, $v_5^0 v_5^1 v_6^1 v_6^0, \cdots, v_{q-1}^0 v_{q-1}^1 v_q^1 v_q^0$ is a path covering of G' with $(q-2)/2$ paths. If q is odd and $q \geq 5$, then we may similarly obtain a path covering of G' with $\lceil (q-2)/2 \rceil$ paths. And when $q \leq 4$, it is easy to see that G' has a Hamiltonian path, hence $p_v(G') = 1$.

To complete the proof of the theorem, it remains to show that if $q > 4$ then $p_v(G') \geq \lceil (q-2)/2 \rceil$. Let Γ be a maximum 4-star-matching of G^c. As in the proof of Theorem 2.1, let U be the set of Γ-unsaturated vertices (clearly $|U| = q$) and B the set of vertices covered by all Γ-alternating paths. Let $\tilde{R}$ be the set of roots in B of subgraphs $K_{1,4}$ in Γ and $\tilde{F}$ the set of leaves in B of subgraphs $K_{1,4}$ in Γ. Clearly $B = U \cup \tilde{R} \cup \tilde{F}$ and $|\tilde{F}| = 4|\tilde{R}|$. Since G^c has no Γ-augmenting edge and Γ-augmenting path, $U \cup \tilde{F}$ is independent in G^c and $N_{G^c}(U \cup \tilde{F}) = \tilde{R}$.

Let $\mathcal{P}$ be any path covering of G'. For each vertex x^0 in $V_0(U \cup \tilde{F})$, it is easy to see that $N_{G'}(x^0) \subseteq \{u, x^1\} \cup V_0(\tilde{R}) \cup V_1(\tilde{R})$. Note that, for each vertex w of $\{u\} \cup V_0(\tilde{R}) \cup V_1(\tilde{R})$, on the path P of $\mathcal{P}$ that covers vertex w, there are at most two vertices of $V_0(U \cup \tilde{F})$ which are adjacent to w. Since $|\tilde{F}| = 4|\tilde{R}|$, we conclude that at least $|U| - 2$ vertices of $V_0(U \cup \tilde{F})$ are end vertices of paths in $\mathcal{P}$. This implies that $|\mathcal{P}| \geq \lceil (|U| - 2)/2 \rceil = \lceil (q-2)/2 \rceil$ and the theorem follows. ■

The following two theorems follow from Theorems 3.1, 1.1 and 1.2 immediately.

Theorem 3.2. *For any graph G on n vertices, we have*

$$\lambda'(\mu(G)) = 2n + \max\{0, \lceil (n - SM_4(G^c) - 2)/2 \rceil - 1\}.$$

Theorem 3.3. *For any graph G on n vertices, we have*

$$\lambda(\mu(G)) \begin{cases} \leq 2n, & \text{if } SM_4(G^c) \geq n-4; \\ = 2n + \lceil (n - SM_4(G^c) - 2)/2 \rceil - 1, & \text{if } SM_4(G^c) < n-4. \end{cases}$$

Similar to Theorems 3.1, 3.3 and 3.2, we have the following three theorems.

Theorem 3.4. *For any graph G on n vertices, we have*

$$p_v((G \times \hat{K}_2)^c) = \begin{cases} 1, & \text{if } SM_4(G^c) \geq n-2; \\ \lceil (n - SM_4(G^c))/2 \rceil, & \text{if } SM_4(G^c) < n-2. \end{cases}$$

Theorem 3.5. *For any graph G on n vertices, we have*

$$\lambda'(G \times \hat{K}_2) = 2n - 1 + \max\{0, \lceil (n - SM_4(G^c))/2 \rceil - 1\}.$$

Theorem 3.6. *For any graph G on n vertices, we have*

$$\lambda(G \times \hat{K}_2) \begin{cases} \leq 2n - 1, & \text{if } SM_4(G^c) \geq n-2; \\ = 2n + \lceil (n - SM_4(G^c))/2 \rceil - 2, & \text{if } SM_4(G^c) < n-2. \end{cases}$$

Theorems 2.2, 3.1 and 3.4 imply that $p_v((\mu(G))^c)$ and $p_v((G \times \hat{K}_2)^c)$ can be computed in polynomial time for any graph G. Thus, for any graph G, it is polynomial-time solvable to determine whether $(\mu(G))^c$ and $(G \times \hat{K}_2)^c$ has a Hamiltonian path. Theorems 2.2, 3.1, 3.2, 3.4 and 3.5 imply that, for any graph G, $\lambda'(\mu(G))$ and $\lambda'(G \times \hat{K}_2)$ can be computed in polynomial time. And consequently, for any graph $G = (V, E)$, it is polynomially solvable to determine whether $\lambda(\mu(G)) \leq s$ for each $s \geq |V(\mu(G))|$ and $\lambda(G \times \hat{K}_2) \leq s$ for each $s \geq |V(G \times \hat{K}_2)|$.

Finally, we give some simple applications of the above theorems.

It is not difficult to see that if G is a graph of diameter at most 2 then the diameter of $\mu(G)$ is 2.

Corollary 3.7. *Suppose $n \geq 2$ is an integer. Then $p_v((\mu(K_n))^c) = \lceil (n-2)/2 \rceil$ if $n > 4$ and $p_v((\mu(K_n))^c) = 1$ if $n \leq 4$. And $\lambda'(\mu(K_n)) = \lambda(\mu(K_n)) = 2n + \lceil (n-2)/2 \rceil - 1$ if $n \geq 3$, and $\lambda(\mu(K_2)) = 4$.*

Proof. If $n \geq 5$, then $SM_4(K_n^c) = 0 < n - 4$ and the corollary follows from Theorems 3.1, 3.2 and 3.3. So we only need to deal with the cases when $n = 2, 3$, and 4.

Since $SM_4(K_n^c) = 0$, by Theorem 3.1, $p_v((\mu(K_n)^c) = 1$ for $n \leq 4$. It follows from Theorems 1.1 and 1.2 that $\lambda(\mu(K_n)) \leq \lambda'(\mu(K_n)) \leq 2n$ for $n \leq 4$. On the other hand, since $\mu(K_n)$ is of diameter 2, $\lambda(\mu(K_n)) \geq 2n$. Therefore $\lambda'(\mu(K_n)) = \lambda(\mu(K_n)) = 2n = 2n + \lceil (n-2)/2 \rceil - 1$ for $n = 3, 4$. It is clear that $\lambda'(\mu(K_2)) = \lambda(\mu(K_2)) = \lambda(C_5) = 4$. ∎

Similarly, by Theorem 3.6, we have the following corollary.

Corollary 3.8. *Suppose $n \geq 2$ is an integer. Then $p_v((K_n \times \hat{K}_2)^c) = \lceil n/2 \rceil$. And $\lambda'(K_n \times \hat{K}_2) = \lambda(K_n \times \hat{K}_2) = 2n + \lceil n/2 \rceil - 2$.*

Suppose G and H are two graphs. Let the *join* of G and H be the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$.

Corollary 3.9. *Suppose $n \geq 1$ and $m \geq 2$ are two integers. Let $G = K_n + K_m^c$. Then*

$$p_v((\mu(G))^c) = \begin{cases} 1, & \text{if } n \leq 4; \\ \lceil (n-2)/2 \rceil, & \text{if } n > 4. \end{cases}$$

And

$$\lambda'(\mu(G)) = \lambda(\mu(G)) = 2(n + m) + \max\{0, \lceil (n-2)/2 \rceil - 1\}.$$

Proof. Since G^c is the disjoint union of K_n^c and K_m $(m \geq 2)$, we have $SM_4(G^c) = m$. Thus, by Theorem 3.1, $p_v((\mu(G))^c)$ equals 1 if $n \leq 4$ and $\lceil (n-2)/2 \rceil$ if

$n > 4$. It is straightforward to check that $\mu(G)$ is of diameter 2. Now the corollary follows from Theorems and . ■

Corollary 3.9. *Suppose $n \geq 2$ and $m \geq 2$ are two integers. Let $G = K_n + K_m^c$. Then $p_v((G \times \hat{K}_2)^c) = \lceil n/2 \rceil$. And $\lambda'(G \times \hat{K}_2) = \lambda(G \times \hat{K}_2) = 2(n+m) + \lceil n/2 \rceil - 2$.*

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