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CONVERGENCE OF THE g-NAVIER-STOKES EQUATIONS

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Abstract. The 2D g-Navier-Stokes equations have the following form,

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega$$

with the continuity equation

$$\nabla\cdot(g\mathbf{u}) = 0, \quad \text{in } \Omega,$$

where g is a smooth real valued function. We get the Navier-Stokes equations, for $g = 1$. In this paper, we investigate solutions $\{\mathbf{u}_g, p_g\}$ of the g-Navier-Stokes equations, as $g \to 1$ in some suitable spaces.

1. Introduction

We consider the 2-dimensional g-Navier-Stokes equations, for periodic boundary conditions on the domain $\Omega = (0,1)\times(0,1)$,

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega\times(0,T), \tag{1.1}$$

$$\nabla\cdot(gu) = 0 \quad \text{in} \quad \Omega\times(0,T). \tag{1.2}$$

Here ν and f are given, and the velocity u and the pressure p are the unknowns. For the details of the derivation of the g-Navier-Stokes equations, one can refer [5]. We assume that $g(\mathbf{x}) \in C^\infty_{per}(\Omega)$ and $0 < m \le g(x,y) \le M$, for all $(x,y)\in\Omega$. Now, we define the Hilbert space $L^2_{per}(\Omega, g) = L^2_{per}(\Omega, R^2, g)$ as the set $L^2_{per}(\Omega)$ with the scalar product and the norm,

$$< \mathbf{u},\mathbf{v} >_g = \int_\Omega (\mathbf{u}\cdot\mathbf{v})\ g\ d\mathbf{x} \quad \text{and} \quad \|\ \mathbf{u}\ \|^2_g = < \mathbf{u},\mathbf{u} >_g .$$

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Similarly, we define $H^1_{per}(\Omega, g)$ as the set $H^1_{per}(\Omega)$ under the norm,

$$\| \mathbf{u} \|_{H^1(\Omega,g)} = [\langle \mathbf{u}, \mathbf{u} \rangle_g + \sum_{i=1}^{2} \langle D_i\mathbf{u}, D_i\mathbf{u} \rangle_g]^{\frac{1}{2}}.$$

For periodic boundary conditions, we use;

$$\begin{aligned} H_g &= CL_{L^2_{per}(\Omega,g)}\{\mathbf{u} \in C^\infty_{per}(\Omega) \ : \ \nabla \cdot g\mathbf{u} = 0, \ \int_\Omega \mathbf{u}\ d\mathbf{x} = \mathbf{0}\} \\ V_g &= \{\mathbf{u} \in H^1_{per}(\Omega, g) \ : \ \nabla \cdot g\mathbf{u} = 0, \ \int_\Omega \mathbf{u}\ d\mathbf{x} = \mathbf{0}\} \\ Q &= CL_{L^2_{per}(\Omega,g)}\{\nabla\phi : \phi \in C^1_{per}(\bar{\Omega}, R)\}, \end{aligned}$$

where H_g is endowed with the scalar product and the norm in $L^2_{per}(\Omega, g)$, and V_g is the space with the scalar product and the norm given by

$$< \mathbf{u}, \mathbf{v} >_{V_g} = \int_\Omega (D_i\mathbf{u} \cdot D_i\mathbf{v})\ g\ d\mathbf{x} \quad \text{and} \quad \| \mathbf{u} \|^2_{V_g} =< \mathbf{u}, \mathbf{u} >_{V_g} . \tag{1.3}$$

Also, for a given $\mathbf{v} \in L^2_{per}(\Omega, g)$, one obtains

$$\mathbf{v} = \mathbf{u} + \frac{\mathbf{k}}{g} + \nabla p, \ \text{for } \mathbf{u} \in H_g, \ \nabla p \in Q, \ \mathbf{k} = \frac{1}{\int_\Omega \frac{1}{g}\ d\mathbf{x}} \int_\Omega \mathbf{v}\ d\mathbf{x} \tag{1.4}$$

and a orthogonal projection $P_g : L^2_{per}(\Omega, g) \mapsto H_g$, as $P_g\mathbf{v} = \mathbf{u}$. Then we have $Q \subset H_g^\perp$. One note that the space Q does not depend on g.

For a linear operator, we consider $A_g\mathbf{u} = P_g(-\Delta_g\mathbf{u})$ where

$$-\Delta_g\mathbf{u} = -\frac{1}{g}(\nabla \cdot g\nabla)\mathbf{u} = -\Delta\mathbf{u} - \frac{1}{g}(\nabla g \cdot \nabla)\mathbf{u}.$$

For $\mathbf{u} \in \mathcal{D}(A_g) = V_g \cap H^2(\Omega)$, we have

$$\langle A_g^{\frac{1}{2}}\mathbf{u}, A_g^{\frac{1}{2}}\mathbf{u} \rangle_g = \langle A_g\mathbf{u}, \mathbf{u} \rangle_g = \langle P_g[-\frac{1}{g}(\nabla \cdot g\nabla)\mathbf{u}], \mathbf{u} \rangle_g = \int_\Omega (\nabla\mathbf{u} \cdot \nabla\mathbf{u}) g\ d\mathbf{x},$$

which implies

$$\| A_g^{\frac{1}{2}}\mathbf{u} \|_g^2 = \| \nabla\mathbf{u} \|^2_g = \| \mathbf{u} \|^2_{V_g}, \quad \text{for} \quad \mathbf{u} \in V_g. \tag{1.5}$$

In addition, for $\mathbf{u} \in \mathcal{D}(A_g^\alpha)$ and $0 \le \alpha \le 1$, we have some positive constant $\tilde{\delta} = \tilde{\delta}(\alpha, m, M)$ such that

$$\lambda_1^{2\alpha} \| \mathbf{u} \|^2_g \le \| A_g^\alpha\mathbf{u} \|^2_g, \ \text{ and } \ \| \mathbf{u} \|_{H^{2\alpha}(\Omega,g)} \le \tilde{\delta} \ \| A_g^\alpha\mathbf{u} \|_g, \tag{1.6}$$

where λ_1 is the first eigenvalue of A_g.

We take the orthogonal projection P_g into (1.1) to get

$$\frac{d\mathbf{u}}{dt} + A_g\mathbf{u} + B_g(\mathbf{u},\mathbf{u}) = \mathbf{q} \qquad \text{on} \qquad H_g, \tag{1.7}$$

where $A_g\mathbf{u} = P_g(-\Delta_g\mathbf{u})$, $B_g(\mathbf{u},\mathbf{u}) = P_g(\mathbf{u}\cdot\nabla)\mathbf{u}$, $\mathbf{q} = P_g[\mathbf{f} - \frac{1}{g}(\nabla g\cdot\nabla)\mathbf{u}]$.

For the g-Navier-Stokes equations, one can also refer [7-9]. With $g = 1$ in (1.1)-(1.2), we get the 2-dimensional Navier-Stokes equations,

$$\frac{\partial\mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p \;=\; \mathbf{f} \;\text{ in }\; \Omega\times(0,T), \tag{1.8}$$

$$\nabla\cdot\mathbf{v} = \;0 \;\text{ in }\; \Omega\times(0,T). \tag{1.9}$$

One can refer [1, 2, 3, 4, 10, 11] and [12] for the Navier-Stokes equations.

In this paper, we will prove that a solution $\{\mathbf{u}_g, p_g\}$ of (1.1)-(1.2) with initial condition $\mathbf{u}_g(0)$ converges to a solution $\{\mathbf{v}, p\}$ of (1.8)-(1.9) with initial condition $P_1\mathbf{u}_g(0)$ in the following sense: for a weak solution

$$\begin{aligned} &\mathbf{u}_g \to \mathbf{v} \;\text{ in }\; L^2(0,T;H^1(\Omega)), \;\text{ in }\; L^\infty(0,T;L^2(\Omega)),\\ &\nabla p_g \to \nabla p \;\text{ in }\; H^{-1}(\Omega\times(0,T)), \end{aligned}$$

where $0 < T < \infty$, as $g \to 1$ in $W^{1,\infty}(\Omega)$, and for a strong solution

$$\begin{aligned} &\mathbf{u}_g \to \mathbf{v} \;\text{ in }\; L^2(0,T;H^2(\Omega)), \;\text{ in }\; L^\infty(0,T;H^1(\Omega)),\\ &\nabla p_g \to \nabla p \;\text{ in }\; L^2(\Omega\times(0,T)), \end{aligned}$$

where $0 < T < \infty$, as $g \to 1$ in $W^{2,\infty}(\Omega)$.

2. Preliminaries

In this section we will introduce useful lemmas in [5] and [6]. We define a trilinear form

$$b_g(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{2}\int_\Omega \mathbf{u}_i(D_i\mathbf{v}_j)\mathbf{w}_j g\,dx$$

where $\mathbf{u},\mathbf{v},\mathbf{w}$ lie in appropriate subspaces of $L^2_{per}(\Omega,g)$. Then one obtains $b(\mathbf{u},\mathbf{v},\mathbf{w})$ = $-b(\mathbf{u},\mathbf{w},\mathbf{v})$ so that $b_g(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$ for sufficient smooth functions $\mathbf{u},\mathbf{v},\mathbf{w}\in H_g$. Moreover, we have the following estimates.

Lemma 2.1. *Let* $\alpha_i, i = 1, 2, 3$ *be nonnegative real numbers that satisfy*

$$\alpha_1 + \alpha_2 + \alpha_3 \geq 1$$

and the vector $(\alpha_1, \alpha_2, \alpha_3)$ *is not equal to* $(1, 0, 0)$*, nor* $(0, 1, 0)$*, nor* $(0, 0, 1)$*. Then there are positive constants* γ_i=$\gamma_i(m, M, \alpha_1, \alpha_2, \alpha_3, \Omega)$*, for* $i = 1, 2$ *such that*

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \gamma_1 \parallel \mathbf{u} \parallel_{H^{\alpha_1}} \parallel \mathbf{v} \parallel_{H^{(\alpha_2+1)}} \parallel \mathbf{w} \parallel_{H^{\alpha_3}}$$

where $\mathbf{u} \in H^{\alpha_1}$*,* $\mathbf{v} \in H^{\alpha_2+1}$ *and* $\mathbf{w} \in H^{\alpha_3}$*, and*

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \gamma_2 \parallel A_g^{\frac{\alpha_1}{2}} \mathbf{u} \parallel_g \parallel A_g^{\frac{(\alpha_2+1)}{2}} \mathbf{v} \parallel_g \parallel A_g^{\frac{\alpha_3}{2}} \mathbf{w} \parallel_g,$$

for all $\mathbf{u} \in V_g^{\alpha_1}$*,* $\mathbf{v} \in V_g^{(\alpha_2+1)}$ *and* $\mathbf{w} \in V_g^{\alpha_3}$*.*

We define that

$$\parallel \mathbf{f} \parallel_{2,2}^2 = \int_0^\infty \parallel \mathbf{f}(t) \parallel_g^2 \, dt.$$

Lemma 2.1. *We assume that* $\parallel \nabla g \parallel_\infty^2 < \frac{m^3\pi^2}{M}$ *and* $\mathbf{f} \in L^2(0, \infty; L^2(\Omega, g))$*. Let* $\mathbf{u} = \mathbf{u}(t)$ *be a weak solution of (1.7) on* $[0, T)$ *with initial condition* $\mathbf{u}_0$*. Then the followings hold:*

(i) *For* $\mathbf{u}_0 \in H_g$*, one has*

$$\parallel \mathbf{u}(t) \parallel_g^2 \leq e^{-\alpha_1 t} \parallel \mathbf{u}_0 \parallel_g^2 + \alpha_2 \parallel \mathbf{f} \parallel_{2,2}^2, \tag{2.1}$$

for all $0 \leq t < T$ *and*

$$\int_{t_1}^t \parallel A_g^{\frac{1}{2}} \mathbf{u}(s) \parallel_g^2 \, ds \leq 2 \parallel \mathbf{u}(t_1) \parallel_g^2 + 2\alpha_2 \parallel \mathbf{f} \parallel_{2,2}^2,$$

for $0 \leq t_1 \leq t \leq T$*.*

(ii) *For* $\mathbf{u}_0 \in V_g$*, there exist constants,* $r_1 = r_1(m, M, \mathbf{f})$*,* $r_2 = r_2(m, M, \mathbf{f})$ *and* $L_1 = L_1(m, M, \mathbf{f})$*(*$L_1$ *does not depend on* $\mathbf{u}_0$*) such that for* $0 \leq t < T$*,*

$$\parallel A_g^{\frac{1}{2}} \mathbf{u}(t) \parallel_g^2 \leq r_1 \left(1 + \parallel A_g^{\frac{1}{2}} \mathbf{u}_0 \parallel_g^2\right) e^{-\alpha_1 t} + L_1. \tag{2.2}$$

One should recall that we denote by H_1, V_1, P_1, A_1 instead of H_g, V_g, P_g, A_g for the constant function $g = 1$.

Lemma 2.3. *Assume that* $\nabla p \in Q$ *and* $p \in H^3(\Omega)$. *Then we have*

$$\begin{aligned} P_g[\frac{d}{dt}(\nabla p(t))] &= \frac{d}{dt}P_g[\nabla p(t)] = 0 \\ P_g[-\Delta(\nabla p(t))] &= P_g[\nabla(-\Delta p(t))] = 0 \\ P_g[(\nabla p(t)\cdot\nabla)\nabla p(t)] &= P_g[\nabla(\frac{1}{2}(\nabla p(t)\cdot\nabla p(t)))] = 0. \end{aligned}$$

Lemma 2.4. *We have* $P_1P_g(\mathbf{v}) = \mathbf{v}$ *for* $\mathbf{v} \in H_1$ *and* $P_gP_1(\mathbf{u}) = \mathbf{u}$ *for* $\mathbf{u} \in H_g$.

Lemma 2.5. *For given* $\mathbf{u} \in H_g$, *we can write as*

$$\mathbf{u} = \mathbf{v} + \nabla p, \quad \text{for } \mathbf{v} \in H_1,\ \nabla p \in Q \tag{2.3}$$

and there exist constants $c_3 = c_3(m, M)$ *and* $c_4 = c_4(m, M)$ *such that*

$$\| \Delta p \| \le c_3 \| \nabla g \|_\infty \| \mathbf{u} \|, \quad \| p \|_{H^2(\Omega)} \le c_4 \| \nabla g \|_\infty \| \mathbf{u} \|. \tag{2.4}$$

In addition, we have $c_5 = c_5(m, M)$ *and* $c_6 = c_6(m, M)$ *such that*

$$\| \Delta p \| \le c_5 \| \nabla g \|_\infty \| \mathbf{v} \|, \quad \| p \|_{H^2(\Omega)} \le c_6 \| \nabla g \|_\infty \| \mathbf{v} \|. \tag{2.5}$$

Lemma 2.6. *We assume that* $\int_\Omega \frac{1}{g}\, d\mathbf{x} = 1$. *Then, for* $\mathbf{u} \in L^2(\Omega)$ *we have*

$$P_1P_g\mathbf{u} = P_1\mathbf{u} - P_1(\frac{\mathbf{k}}{g}), \tag{2.6}$$

where $\mathbf{k} = \int_\Omega \mathbf{u}\, d\mathbf{x}$. *As a result,* $P_1P_g\mathbf{u} = P_1\mathbf{u}$ *if* $\int_\Omega \mathbf{u}\, d\mathbf{x} = 0$.
Furthermore, for $\mathbf{u} \in L^2(\Omega)$ *and* $\mathbf{w} \in H_1$ *we have*

$$|\langle P_1P_g\mathbf{u}, \mathbf{w}\rangle| \le |\langle \mathbf{u}, \mathbf{w}\rangle| + \frac{1}{m}\| \mathbf{k} \| \, \| \mathbf{w} \|. \tag{2.7}$$

Next, we want to see the relationship between the norms in H_g and H_1 as well as in V_g and V_1.

Lemma 2.7. *Let* $\mathbf{u} \in H_g$ *with* $\mathbf{u} = \mathbf{v} + \nabla p$, *for* $\mathbf{v} \in H_1$, $\nabla p \in Q$.
Then the followings hold;

(*1*) *We have*

$$\frac{1}{M} \| \mathbf{u} \|_g^2 \le \| \mathbf{v} \|^2 \le \frac{1}{m} \| \mathbf{u} \|_g^2. \tag{2.8}$$

(2) *For* $\mathbf{u} \in V_g$, *we have*

$$\mathbf{u} = \mathbf{v} \ + \ \nabla p, \qquad \mathbf{v} \in V_1, \ \nabla p \in Q,$$

and

$$\| \nabla \mathbf{u} \|^2 = \| \nabla \mathbf{v} \|^2 + \| \nabla(\nabla q) \|^2.$$

In addition, if $\| \nabla g \|_\infty^2 < \frac{m^3\pi^2}{M}$ *then we have*

$$l_1 \ \| A_g^{\frac{1}{2}} \mathbf{u} \|_g^2 \le \| A_1^{\frac{1}{2}} \mathbf{v} \|^2 \le \frac{1}{m} \ \| A_g^{\frac{1}{2}} \mathbf{u} \|_g^2, \tag{2.9}$$

where

$$l_1 = l_1(g) = \frac{4\pi^2}{M \ (4\pi^2 + c_6^2 \| \nabla g \|_\infty^2)}.$$

(3) *For* $\mathbf{u} \in \mathcal{D}(A_g)$, *we have*

$$\mathbf{u} = \mathbf{v} + \nabla p, \qquad \mathbf{v} \in \mathcal{D}(A_1), \ \nabla p \in Q.$$

In addition, if $\| \nabla g \|_\infty^2 < \frac{m^3\pi^2}{M}$ *then we have*

$$l_2 \ \| A_g \mathbf{u} \|_g^2 \le \| A_1 \mathbf{v} \|^2 \le l_3 \ \| A_g \mathbf{u} \|_g^2,$$

where

$$l_2 = l_2(g) = \frac{4\pi^4 m^2}{M \left(2\pi^2 m + 2\pi \| \nabla g \|_\infty + c_6 \| \nabla g \|_\infty^2\right)^2}.$$

and

$$l_3 = l_3(g) = \frac{(m\sqrt{\lambda_1^g} + 2\| \nabla g \|_\infty)^2}{m^3 \lambda_1^g},$$

λ_1^g *is the smallest eigenvalue of* A_g.

3. Main Theorems

In this section we assume $\int_\Omega \frac{1}{g} \, d\mathbf{x} = 1$ for simple calculations.

3.1. Weak Solutions

Let us define the set Λ_w with the metric inherited from $W^{1,\infty}(\Omega)$ as $g \in \Lambda_w$ if

(1) $g(\mathbf{x}) \in C_{per}^\infty(\Omega)$ with $0 < m \le g(x, y) \le M$, for all $(x, y) \in \Omega$.

(2) $\| g \|_{W^{1,\infty}}^2 < \frac{m^3\pi^2}{M}$.

Theorem 3.1. *Assume that* $g \in \Lambda_w$ *and* $\mathbf{f} \in L^2(0,\infty; L^2(\Omega, g))$ *with* $\int_\Omega \mathbf{f}\ d\mathbf{x} = 0$. *Let* $(\mathbf{u}_g(t), p_g(t))$ *be a weak solution of* $(1.1)-(1.2)$ *with* $\mathbf{u}_0 = \mathbf{u}_g(0) \in H_g$. *And* $(\mathbf{v}(t), p(t))$ *be a weak solution of* $(1.8)-(1.9)$ *with* $\mathbf{v}(0) = P_1\mathbf{u}_0 \in H_1$. *Then we have*

$$\mathbf{u}_g \to \mathbf{v} \ \ in \ \ L^2(0,T;H^1(\Omega)),\ in \ \ L^\infty(0,T;L^2(\Omega)), \tag{3.1}$$

$$\nabla p_g \to \nabla p \ \ in \ \ H^{-1}(\mathcal{Q}), \tag{3.2}$$

for $\mathcal{Q} = \Omega \times (0,T)$ *and for* $0 < T < \infty$, *as* $\| \nabla g \|_\infty \to 0$.

Proof. For $\mathbf{u}_g \in H_g$, we have $\mathbf{v}_g \in H_1$ and $\nabla q_g \in Q$ such that $\mathbf{u}_g = \mathbf{v}_g + \nabla q_g$. Since $\mathbf{u}_g(t)$ is a strong solution of equations $(1.1)-(1.2)$ for $t \geq t_0 > 0$, by lemma and lemma , we obtain

$$\frac{d\mathbf{v}_g}{dt} + A_1\mathbf{v}_g + P_1(\mathbf{v}_g\cdot\nabla)\mathbf{v}_g + P_1(\mathbf{v}_g\cdot\nabla)\nabla q_g + P_1P_g(\nabla q_g\cdot\nabla)\mathbf{v}_g = P_1\mathbf{f}, \tag{3.3}$$

for all $t \geq t_0 > 0$. Let $\mathbf{v}_g - \mathbf{v} = \mathbf{w}$ then we get

$$\frac{d\mathbf{w}}{dt} + A_1\mathbf{w} + P_1(\mathbf{v}_g\cdot\nabla)\mathbf{w} + P_1(\mathbf{w}\cdot\nabla)\mathbf{v} + P_1(\mathbf{v}_g\cdot\nabla)\nabla q_g + P_1P_g(\nabla q_g\cdot\nabla)\mathbf{v}_g = 0 \tag{3.4}$$

for $t \geq t_0 > 0$. So, we have

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\| \mathbf{w} \|^2 + \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 &\leq |\langle(\mathbf{w}\cdot\nabla)\mathbf{v}, \mathbf{w}\rangle| + |\langle(\mathbf{v}_g\cdot\nabla)\nabla q_g, \mathbf{w}\rangle| \\ &\quad + |\langle P_1P_g(\nabla q_g\cdot\nabla)\mathbf{v}_g, \mathbf{w}\rangle| \\ &= |I| + |II| + |III|, \text{ for } t \geq t_0 > 0. \end{aligned} \tag{3.5}$$

First, we obtain

$$\begin{aligned} |I| &= |\langle(\mathbf{w}\cdot\nabla)\mathbf{v}, \mathbf{w}\rangle| \leq 2\| \mathbf{w} \| \, \| \nabla\mathbf{w} \| \, \| \nabla\mathbf{v} \| \\ &\leq \frac{1}{4}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + 4\| A_1^{\frac{1}{2}}\mathbf{v} \|^2 \| \mathbf{w} \|^2. \end{aligned} \tag{3.6}$$

Also, by lemma , (1.6), (2.1), (2.4) and the Young inequality, we get

$$\begin{aligned} |II| &= |\langle(\mathbf{v}_g\cdot\nabla)\nabla q_g, \mathbf{w}\rangle| \leq \gamma_1\| \mathbf{v}_g \|_{H^1}\| q_g \|_{H^2}\| \mathbf{w} \|_{H^1} \\ &\leq \frac{1}{4}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + c_7\| \nabla g \|_\infty^2\| A_1^{\frac{1}{2}}\mathbf{v}_g \|^2 \end{aligned} \tag{3.7}$$

for some constant $c_7 = c_7(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$. Similar to $|II|$, by (2.7) we get

$$\begin{aligned} |III| &= |\langle P_1P_g(\nabla q_g\cdot\nabla)\mathbf{v}_g, \mathbf{w}\rangle| \leq |\langle(\nabla q_g\cdot\nabla)\mathbf{v}_g, \mathbf{w}\rangle| + \frac{1}{m}\| \mathbf{k} \| \, \| \mathbf{w} \| \\ &\leq \frac{1}{4}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + c_8\| \nabla g \|_\infty^2\| A_1^{\frac{1}{2}}\mathbf{v}_g \|^2 + \frac{1}{m}\| \mathbf{k} \| \, \| \mathbf{w} \| \end{aligned} \tag{3.8}$$

for some constant $c_8 = c_8(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$, where $\mathbf{k} = \int_\Omega (\nabla q_g \cdot \nabla)\mathbf{v}_g \, d\mathbf{x}$.

Since we have

$$\| \mathbf{k} \| = | \int_\Omega (\nabla q_g \cdot \nabla)\mathbf{v}_g \, d\mathbf{x}| \leq \| \nabla q_g \| \, \| \nabla \mathbf{v}_g \|,$$

by (1.5), (2.5) and the Young inequality, we obtain

$$|III| \leq \frac{1}{4}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + \frac{1}{2}\| A_1^{\frac{1}{2}}\mathbf{v}_g \|^2 \| \mathbf{w} \|^2 + c_9 \| \nabla g \|_\infty^2 \| A_1^{\frac{1}{2}}\mathbf{v}_g \|^2, \tag{3.9}$$

for some constant $c_9 = c_9(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$.

Therefore, from (3.5), (3.6), (3.7) and (3.9) we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\| \mathbf{w} \|^2 + \frac{1}{4}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 \; &\leq \; (4\| A_1^{\frac{1}{2}}\mathbf{v} \|^2 + \frac{1}{2}\| A_1^{\frac{1}{2}}\mathbf{v}_g \|^2)\| \mathbf{w} \|^2 \\ &+ \; (c_7 + c_9)\| \nabla g \|_\infty^2 \| A_1^{\frac{1}{2}}\mathbf{v}_g \|^2, \end{aligned}$$

for all $t \geq t_0 > 0$. So, we can rewrite as

$$\frac{d}{dt}\| \mathbf{w} \|^2 \leq \beta_5(t)\| \mathbf{w} \|^2 + \beta_6(t)$$

where

$$\begin{aligned} \beta_5(t) \; &= \; 8\| A_1^{\frac{1}{2}}\mathbf{v}(t) \|^2 + \| A_1^{\frac{1}{2}}\mathbf{v}_g(t) \|^2 \\ \beta_6(t) \; &= \; 2 \; (c_7 + c_9)\| \nabla g \|_\infty^2 \| A_1^{\frac{1}{2}}\mathbf{v}_g(t) \|^2. \end{aligned}$$

By the Gronwall inequality and taking $\lim_{t_0 \to 0}$ we obtain

$$\| \mathbf{w}(t) \|^2 \leq e^{\int_0^t \beta_5(s)ds} \left[\| \mathbf{w}(0) \|^2 + \int_0^t \beta_6(t)ds \right], \tag{3.11}$$

for all $t > 0$. One note that by the classical theory of the Navier-Stokes equations, there exist constant $c_{10} = c_{10}(\| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$ such that for all $0 < t \leq T$,

$$\int_0^t \| A_1^{\frac{1}{2}}\mathbf{v}(s) \|^2 ds \leq c_{10}. \tag{3.12}$$

Also, with $g \in \Lambda_w$, by lemma and lemma we have some positive constant $c_{11} = c_{11}(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$ such that for all $0 < t \leq T$,

$$\int_0^t \| A_1^{\frac{1}{2}}\mathbf{v}_g(s) \|^2 ds \leq \frac{1}{m}\int_0^t \| A_g^{\frac{1}{2}}\mathbf{u}_g(s) \|^2 ds \leq c_{11}. \tag{3.13}$$

Since $\| \mathbf{w}(0) \|^2 = 0$, we have some constant $c_{12} = c_{12}(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$ such that

$$\| \mathbf{w}(t) \|^2 \le c_{12} \| \nabla g \|_\infty^2, \text{ for all } 0 < t < T. \tag{3.14}$$

So, by (2.1), (2.4) and (3.14), we get

$$\begin{aligned} \| \mathbf{u}_g(t) - \mathbf{v}(t) \|^2 &= \| \mathbf{v}_g(t) - \mathbf{v}(t) \|^2 + \| \mathbf{u}_g(t) - \mathbf{v}_g(t) \|^2 \\ &= \| \mathbf{w}(t) \|^2 + \| \nabla q_g(t) \|^2 \\ &\le c_{12} \| \nabla g \|_\infty^2 + c_4^2 \| \nabla g \|_\infty^2 \| \mathbf{u}_g(t) \|^2 \le c_{13} \| \nabla g \|_\infty^2, \end{aligned}$$

for some positive constant $c_{13} = c_{13}(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$ and for all $0 < t < T$. It means that

$$\| \mathbf{u}_g - \mathbf{v} \|_{L^\infty(0,T;L^2(\Omega))}^2 := \operatorname{ess} \sup_{0<t<T} \| \mathbf{u}_g - \mathbf{v} \|^2 \le c_{13} \| \nabla g \|_\infty^2 \to 0,$$

as $g \to 1$ in $W^{1,\infty}(\Omega)$.

Next, to prove the first part of (3.1), we take the integral from t_0 to T and take $\lim_{t_0 \to 0}$ both sides of (3.10). Then, by (3.10), (3.12), (3.13) and (3.14), we obtain

$$\int_0^T \| A_1^{\frac{1}{2}} \mathbf{w}(s) \|^2 ds \le (16c_{10}c_{12} + 2c_{11}c_{12} + 4c_7c_{11} + 4c_9c_{11}) \| \nabla g \|_\infty^2 + 2 \| \mathbf{w}(0) \|^2.$$

Since $\| \mathbf{w}(0) \|^2 = 0$, we have

$$\int_0^T \| A_1^{\frac{1}{2}} \mathbf{w}(s) \|^2 ds \le c_{14} \| \nabla g \|_\infty^2, \tag{3.15}$$

for some constant $c_{14} = c_{14}(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$.

Therefore, we obtain from (1.6), (2.5), (3.13) and (3.15) that

$$\begin{aligned} \int_0^T \| \mathbf{u}_g - \mathbf{v} \|_{H^1}^2 ds &\le \int_0^T \| \mathbf{u}_g - \mathbf{v}_g + \mathbf{v}_g - \mathbf{v} \|_{H^1}^2 ds \\ &\le 2 \int_0^T \left(\| \mathbf{u}_g - \mathbf{v}_g \|_{H^1}^2 + \| \mathbf{v}_g - \mathbf{v} \|_{H^1}^2 \right) ds \\ &\le 2 \int_0^T \left(\| \nabla q_g \|_{H^1}^2 + \| \mathbf{w} \|_{H^1}^2 \right) ds \\ &\le 2 \int_0^T \left(\| q_g \|_{H^2}^2 + \tilde{\delta}^2 \| A_1^{\frac{1}{2}} \mathbf{w} \|^2 \right) ds \\ &\le 2 \int_0^T \left(c_6^2 \| \nabla g \|_\infty^2 \| \mathbf{v}_g \|^2 + \tilde{\delta}^2 \| A_1^{\frac{1}{2}} \mathbf{w} \|^2 \right) ds \\ &\le 2(c_6^2 c_{11} + c_{14} \tilde{\delta}^2) \| \nabla g \|_\infty^2 \end{aligned}$$

which goes to zero as $\| \nabla g \| \to 0$.

At last, to prove (3.2), one note that for all $\mathbf{w} \in V_1$, we obtain $\frac{\mathbf{w}}{g} \in V_g$. So, we obtain

$$\begin{aligned} &\langle \mathbf{u}_g', \mathbf{w}\rangle + \langle \Delta \mathbf{u}_g, \mathbf{w}\rangle + \langle -(\mathbf{u}_g \cdot \nabla)\mathbf{u}_g, \mathbf{w}\rangle - \langle \mathbf{f}, \mathbf{w}\rangle \\ =\ &\langle \mathbf{u}_g', \frac{\mathbf{w}}{g}\rangle_g + \langle \Delta \mathbf{u}_g, \frac{\mathbf{w}}{g}\rangle_g + \langle -(\mathbf{u}_g \cdot \nabla)\mathbf{u}_g, \frac{\mathbf{w}}{g}\rangle_g - \langle \mathbf{f}, \frac{\mathbf{w}}{g}\rangle_g = 0. \end{aligned}$$

Therefore, by proposition 1.1 in chapter I of Temam[11], we have suitable $\nabla p_g \in Q$ such that

$$\nabla p_g = \mathbf{f} - \mathbf{u}_g' + \Delta \mathbf{u}_g - (\mathbf{u}_g \cdot \nabla)\mathbf{u}_g. \tag{3.16}$$

Also, by classical theory of the Navier-Stokes equations, we have

$$\nabla p = \mathbf{f} - \mathbf{v}' + \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{3.17}$$

Hence, to prove (3.2), we claim for any $\mathbf{w} \in H^1(\mathcal{Q})$

$$\begin{aligned} &|\int_0^T \langle \nabla p_g - \nabla p, \mathbf{w}(t)\rangle\ dt| \le |\int_0^T \langle \mathbf{u}_g' - \mathbf{v}', \mathbf{w}(t)\rangle\ dt| \\ &+ |\int_0^T \langle \Delta \mathbf{u}_g - \Delta \mathbf{v}, \mathbf{w}(t)\rangle\ dt| + |\int_0^T \langle (\mathbf{u}_g \cdot \nabla)\mathbf{u}_g - (\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{w}(t)\rangle\ dt| \\ &= |I| + |II| + |III| \le C(g) \| \mathbf{w} \|_{H^1(\mathcal{Q})} \to 0, \end{aligned} \tag{3.18}$$

as $\| \nabla g \|_\infty \to 0$, where $C(g)$ is some constant which depends on g.

First, by using the integration by parts and (3.1), we obtain

$$\begin{aligned} |II| &= |\int_0^T \langle -\Delta(\mathbf{u}_g - \mathbf{v}), \mathbf{w}(t)\rangle dt| = \int_0^T |\langle \nabla(\mathbf{u}_g - \mathbf{v}), \nabla \mathbf{w}(t)\rangle| dt \\ &\le \left(\int_0^T \| \mathbf{u}_g - \mathbf{v} \|_{H^1}^2 dt \right)^{\frac{1}{2}} \| \mathbf{w} \|_{H^1(\mathcal{Q})} \to 0, \end{aligned} \tag{3.19}$$

for any $\mathbf{w} \in H^1_{per}(\mathcal{Q})$, as $\| \nabla g \|_\infty \to 0$.

Also, since $\mathbf{v} \in L^2(0,T;V_1)$ and $\mathbf{u}_g \in L^2(0,T;V_g)$, by (3.1) we obtain

$$
\begin{aligned}
&|III| \\
&= |\int_0^T \langle (\mathbf{u}_g\cdot\nabla)\mathbf{u}_g - (\mathbf{v}\cdot\nabla)\mathbf{v}, \mathbf{w}(t)\rangle \, dt| \\
&= |\int_0^T \langle ((\mathbf{u}_g-\mathbf{v})\cdot\nabla)\mathbf{u}_g, \mathbf{w}(t)\rangle \, dt| + |\int_0^T \langle (\mathbf{v}\cdot\nabla)(\mathbf{u}_g-\mathbf{v}), \mathbf{w}(t)\rangle \, dt| \\
&\le \| \mathbf{w}(t) \|_{H^1(Q)} \left(\int_0^T \| \mathbf{u}_g - \mathbf{v} \|_{H^1}^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \| \mathbf{u}_g \|_{H^1}^2 dt\right)^{\frac{1}{2}} \\
&+ \| \mathbf{w}(t) \|_{H^1(Q)} \left(\int_0^T \| \mathbf{u}_g - \mathbf{v} \|_{H^1}^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \| \mathbf{v} \|_{H^1}^2 dt\right)^{\frac{1}{2}} \to 0,
\end{aligned} \tag{3.20}
$$

for any $\mathbf{w} \in H^1_{per}(\mathcal{Q})$, as $\| \nabla g \|_\infty \to 0$.

Next, one should note that we can assume $\mathbf{w}(T) = 0$, because the set of $\mathbf{w}(t) \in H^1_{per}(\mathcal{Q})$ with $\mathbf{w}(T) = 0$ is dense in the space $H^1_{per}(\mathcal{Q})$. So, by the integration by parts, we have

$$
\begin{aligned}
&|I| = |\int_0^T \langle \frac{\partial}{\partial t}(\mathbf{u}_g - \mathbf{v}), \mathbf{w}(t)\rangle \, dt| \\
&\le |\langle (\mathbf{u}_g(0) - \mathbf{v}(0)), \mathbf{w}(0)\rangle| + |\int_0^T \langle \mathbf{u}_g - \mathbf{v}, \frac{\partial}{\partial t}\mathbf{w}(t)\rangle \, dt| \\
&\le \| \mathbf{u}_g(0) - \mathbf{v}(0) \| \, \| \mathbf{w}(0) \| + \| \mathbf{w}(t) \|_{H^1(Q)} \left(\int_0^T \| \mathbf{u}_g - \mathbf{v} \|^2 dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.21}
$$

Since $P_1\mathbf{u}_g(0) = \mathbf{v}(0)$, as $\| \nabla g \|_\infty \to 0$, we have

$$
\| \mathbf{u}_g(0) - \mathbf{v}(0) \| = \| \mathbf{u}_g(0) - P_1\mathbf{u}_g(0) \| \le c_6 \| \nabla g \|_\infty \| \mathbf{v}(0) \| \to 0. \tag{3.22}
$$

Also, by (3.1), the second term of (3.21) also goes to 0 as $\| \nabla g \|_\infty \to 0$. So, from (3.21) and (3.22), $|I|$ goes to zero as $\| \nabla g \|_\infty \to 0$.

Therefore, by (3.18), (3.19), (3.20) and (3.21), we complete the proof of (3.2)■

3.2. Strong Solutions

Let us define the set Λ_s with the metric inherited from $W^{2,\infty}(\Omega)$ as $g \in \Lambda_s$, if $g \in \Lambda_w$ and $\| g \|_{W^{2,\infty}} \le M_0$ for some constant M_0.

Before we prove main theorem we will prove the following useful lemmas by using equation (3.3).

Lemma 3.2. *Assume that $g \in \Lambda_s$ and $\mathbf{f} \in L^2(0,\infty; L^2(\Omega, g))$ with $\int_\Omega \mathbf{f} \, d\mathbf{x} = 0$. Let $\mathbf{u}_g = \mathbf{v}_g + \nabla q_g$ be a strong solution of (1.1) – (1.2) with $\mathbf{u}_0 = \mathbf{u}_g(0) \in V_g$.*

Then there exists some constant $c_{15} = c_{15}(m, M, M_0, \| A_1^{\frac{1}{2}}\mathbf{v}_g(0) \|, \| \mathbf{f} \|_{2,2})$ *such that*

$$\| A_1^{\frac{1}{2}}\mathbf{v}_g(t) \|^2 \leq c_{15}, \tag{3.23}$$

for all $0 \leq t < T$.

Proof. By taking the scalar product with $A_1\mathbf{v}_g$ to the equation (3.3) we obtain

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\| A_1^{\frac{1}{2}}\mathbf{v}_g \|^2 + \| A_1\mathbf{v}_g \|^2 &\leq |\langle P_1P_g(\nabla q_g \cdot \nabla)\mathbf{v}_g, A_1\mathbf{v}_g\rangle| \\ &\quad + |\langle(\mathbf{v}_g \cdot \nabla)\nabla q_g, A_1\mathbf{v}_g\rangle| + |\langle \mathbf{f}, A_1\mathbf{v}_g\rangle| \\ &= |I| + |II| + |III|, \end{aligned} \tag{3.24}$$

because $\langle(\mathbf{v}_g \cdot \nabla)\mathbf{v}_g, A_1\mathbf{v}_g\rangle = 0$. From (1.6) and (2.9), Note

$$\| q_g \|_{H^3}^2 \leq \frac{\tilde{\delta}^2\delta_0^2M_0^2}{l_1}\| A_1^{\frac{1}{2}}\mathbf{v}_g \|^2, \tag{3.25}$$

for some positive constant $\delta_0 = \delta_0(m, M, \alpha)$. So, by lemma , (1.6), (3.25) and the Young inequality, we have

$$\begin{aligned} |II| &= |\langle(\mathbf{v}_g \cdot \nabla)\nabla q_g, A_1\mathbf{v}_g\rangle| \leq \gamma_1 \| \mathbf{v}_g \|_{H^1}\| q_g \|_{H^3}\| A_1\mathbf{v}_g \| \\ &\leq \frac{1}{4}\| A_1\mathbf{v}_g \|^2 + \frac{\gamma_1^2\tilde{\delta}^4\delta_0^2M_0^2}{l_1}\| A_1^{\frac{1}{2}}\mathbf{v}_g \|^4. \end{aligned} \tag{3.26}$$

Also, by (2.7) we have

$$|I| = |\langle P_1P_g(\nabla q_g \cdot \nabla)\mathbf{v}_g, A_1\mathbf{v}_g\rangle| \leq |\langle(\nabla q_g \cdot \nabla)\mathbf{v}_g, A_1\mathbf{v}_g\rangle| + \frac{1}{m}\| \mathbf{k} \| \, \| A_1\mathbf{v}_g \|, \tag{3.27}$$

where $\mathbf{k} = \int_\Omega (\nabla q_g \cdot \nabla)\mathbf{v}_g \, d\mathbf{x}$. Similar to $|II|$, we obtain

$$\begin{aligned} |\langle(\nabla q_g \cdot \nabla)\mathbf{v}_g, A_1\mathbf{v}_g\rangle| &\leq \gamma_1\| q_g \|_{H^3} \| \mathbf{v}_g \|_{H^1}\| A_1\mathbf{v}_g \| \\ &\leq \frac{1}{4}\| A_1\mathbf{v}_g \|^2 + \frac{\gamma_1^2\tilde{\delta}^4\delta_0^2M_0^2}{l_1}\| A_1^{\frac{1}{2}}\mathbf{v}_g \|^4. \end{aligned} \tag{3.28}$$

Since

$$\| \mathbf{k} \| = |\int_\Omega (\nabla q_g \cdot \nabla)\mathbf{v}_g \, d\mathbf{x}| \leq \| \nabla q_g \| \, \| \nabla\mathbf{v}_g \|,$$

we have by (1.5), (2.5) and the Young inequality that

$$\begin{aligned} \frac{1}{m}\| \mathbf{k} \| \, \| A_1\mathbf{v}_g \| &\leq \frac{1}{m}\| \nabla q_g \| \, \| \nabla\mathbf{v}_g \| \, \| A_1\mathbf{v}_g \| \\ &\leq \frac{1}{4}\| A_1\mathbf{v}_g \|^2 + \frac{c_6^2M_0^2}{m^2} \| A_1^{\frac{1}{2}}\mathbf{v}_g \|^4. \end{aligned} \tag{3.29}$$

Therefore, by (3.27), (3.28) and (3.29) we have

$$|I| \leq \frac{1}{2} \| A_1 \mathbf{v}_g \|^2 + \left(\frac{\gamma_1^2 \tilde{\delta}^4 \delta_0^2}{l_1} + \frac{c_6^2}{m^2} \right) M_0^2 \| A_1^{\frac{1}{2}} \mathbf{v}_g \|^4 . \tag{3.30}$$

Also we have

$$|III| = |\langle \mathbf{f}, A_1 \mathbf{v}_g \rangle| \leq \frac{1}{8} \| A_1 \mathbf{v}_g \|^2 + 8 \| \mathbf{f} \|^2. \tag{3.31}$$

Hence, by (3.24), (3.26), (3.30) and (3.31) we obtain

$$\frac{d}{dt} \| A_1^{\frac{1}{2}} \mathbf{v}_g(t) \|^2 + \frac{1}{4} \| A_1 \mathbf{v}_g(t) \|^2 \leq \beta_7(t) \| A_1^{\frac{1}{2}} \mathbf{v}_g(t) \|^2 + \beta_8(t) \tag{3.32}$$

which implies

$$\frac{d}{dt} \| A_1^{\frac{1}{2}} \mathbf{v}_g(t) \|^2 \leq \beta_7(t) \| A_1^{\frac{1}{2}} \mathbf{v}_g(t) \|^2 + \beta_8(t), \quad 0 < t < T, \tag{3.33}$$

where

$$\begin{aligned} \beta_7 &= \left(\frac{4\gamma_1^2 \tilde{\delta}^4 \delta_0^2}{l_1} + \frac{2c_6^2}{m^2} \right) M_0^2 \| A_1^{\frac{1}{2}} \mathbf{v}_g(t) \|^2 \\ \beta_8 &= 16 \| \mathbf{f}(t) \|^2. \end{aligned} \tag{3.34}$$

Therefore, by (3.13), (3.33) and the Gronwall inequality, there exists a constant $c_{15} = c_{15}(m, M, M_0, \| A_1^{\frac{1}{2}} \mathbf{v}_g(0) \|, \| \mathbf{f} \|_{2,2})$ such that

$$\| A_1^{\frac{1}{2}} \mathbf{v}_g(t) \|^2 \leq e^{\int_0^T \beta_7(s) ds} \left[\| A_1^{\frac{1}{2}} \mathbf{v}_g(0) \|^2 + \int_0^T \beta_8(s) ds \right] \leq c_{15}$$

for all $0 \leq t < T$. ■

Lemma 3.3. *Assume that* $g \in \Lambda_s$ *and* $\mathbf{f} \in L^2(0, \infty; L^2(\Omega, g))$ *with* $\int_\Omega \mathbf{f} \, d\mathbf{x} = 0$. *Let* $\mathbf{u}_g = \mathbf{v}_g + \nabla q_g$ *be a strong solution of* (1.1) – (1.2) *with* $\mathbf{u}_0 = \mathbf{u}_g(0) \in V_g$. *Then there exists some constant* $c_{16} = c_{16}(m, M, M_0, \| A_1^{\frac{1}{2}} \mathbf{v}_g(0) \|, \| \mathbf{f} \|_{2,2})$ *such that*

$$\int_0^T \| A_1 \mathbf{v}_g \|^2 \, ds \leq c_{16}. \tag{3.35}$$

Proof. First we note from (3.23) and (3.34) that

$$\begin{aligned}\beta_7(t) &= \left(\frac{4\gamma_1^2\tilde{\delta}^4\delta_0^2}{l_1}+\frac{2c_6^2}{m^2}\right)M_0^2\| A_1^{\frac{1}{2}}\mathbf{v}_g(t)\|^2 \\ &\le c_{15}\left(\frac{4\gamma_1^2\tilde{\delta}^4\delta_0^2}{l_1}+\frac{2c_6^2}{m^2}\right)M_0^2\end{aligned} \tag{3.36}$$

for all $0 \le t < T$. So, by integrating from 0 to T both sides of (3.32) we obtain from (3.13) that

$$\begin{aligned}&\int_0^T \| A_1\mathbf{v}_g(s)\|^2 ds \\ &\le 4\| A_1^{\frac{1}{2}}\mathbf{v}_g(0)\|^2 + 4\int_0^T\left(\beta_7(s)\| A_1^{\frac{1}{2}}\mathbf{v}_g(s)\|^2+\beta_8(s)\right)ds \\ &\le 4\| A_1^{\frac{1}{2}}\mathbf{v}_g(0)\|^2 + 4c_{11}c_{15}\left(\frac{4\gamma_1^2\tilde{\delta}^4\delta_0^2}{l_1}+\frac{2c_6^2}{m^2}\right)M_0^2+64\|\mathbf{f}\|_{2,2}^2 \le c_{16},\end{aligned}$$

for some positive constant c_{16}. ■

Lemma 3.4. *For given* $\mathbf{u} \in L^2_{per}(\Omega)$ *we have*

$$\| P_g\mathbf{u} - P_1\mathbf{u}\| \le \frac{2}{m}\|\nabla g\|_\infty \|\mathbf{u}\| + \frac{\| 1-g\|_\infty}{m}\|\mathbf{k}\|, \tag{3.37}$$

where $\mathbf{k} = \int_\Omega \mathbf{u}\, d\mathbf{x}$.

Proof. For any $\mathbf{u} \in L^2_{per}(\Omega)$, we can write as

$$P_g\mathbf{u} + \nabla r_g + \frac{\mathbf{k}}{g} = \mathbf{u} = P_1\mathbf{u} + \nabla r_1 + \mathbf{k}, \quad \text{for } \nabla r_g,\ \nabla r_1 \in Q. \tag{3.38}$$

So, we have

$$\frac{1}{g}(\nabla\cdot g\nabla)r_g = \frac{1}{g}(\nabla\cdot g\mathbf{u}) = \nabla\cdot\mathbf{u} + \frac{\nabla g}{g}\cdot\mathbf{u} \text{ and } \Delta r_1 = \nabla\cdot\mathbf{u}.$$

Now, one note $\frac{1}{g}(\nabla\cdot g\nabla)r_g = \Delta r_g + (\frac{\nabla g}{g}\cdot\nabla)r_g$. Therefore, we get

$$\Delta r_1 - \Delta r_g = \frac{\nabla g}{g}\cdot\mathbf{u} - (\frac{\nabla g}{g}\cdot\nabla)r_g.$$

Hence, we have

$$\|\nabla r_1 - \nabla r_g\| \le \|\Delta(r_1 - r_g)\| \le \frac{2}{m}\|\nabla g\|_\infty \|\mathbf{u}\|.$$

So, we have from (3.38) that

$$\begin{aligned}\| P_1\mathbf{u} - P_g\mathbf{u} \| &\leq \| \nabla r_1 - \nabla r_g \| + \| \frac{\mathbf{k}}{g} - \mathbf{k} \| \\ &\leq \frac{2}{m}\| \nabla g \|_\infty \| \mathbf{u} \| + \frac{\| 1-g \|_\infty}{m} \| \mathbf{k} \|. \qquad \blacksquare\end{aligned}$$

Remark 3.5. Let $\mathbf{u} = \mathbf{v}+\nabla p$, for $\mathbf{u} \in H^\alpha(\Omega)$, $\mathbf{v} \in H_g$ and $\nabla p \in Q$. Then we have a constant $\delta_0 = \delta_0(m, M, \alpha)$ such that $\| p \|_{H^{\alpha+2}} \leq \delta_0 \| g \|_{\alpha+1,\infty} \| \mathbf{u} \|_{H^\alpha}$, where $\| g \|_{k,\infty} = \sum_{1\leq j\leq k} \| D^j g \|_\infty$.

Theorem 3.6. *Let* $g \in \Lambda_s$ *and* $\mathbf{f} \in L^2(0,\infty; L^2(\Omega, g))$ *with* $\int_\Omega \mathbf{f}\ d\mathbf{x} = 0$. *Let* $(\mathbf{u}_g(t), p_g(t))$ *be a strong solution of* (1.1) – (1.2) *with* $\mathbf{u}_0 = \mathbf{u}_g(0) \in V_g$. *And* $(\mathbf{v}(t), p(t))$ *be a strong solution of* (1.8) – (1.9) *with* $\mathbf{v}(0) = P_1\mathbf{u}_0 \in V_1$. *Then we have*

$$\mathbf{u}_g \to \mathbf{v} \ \ in \ \ L^\infty(0,T; H^1(\Omega)), \ \ in \ \ L^2(0,T; H^2(\Omega)) \tag{3.39}$$

$$\nabla p_g \to \nabla p \ \ in \ \ L^2(\mathcal{Q}), \tag{3.40}$$

for $\mathcal{Q} = \Omega \times (0,T)$ *and for* $0 < T < \infty$, *as* $\| g \|_{2,\infty} \to 0$

Proof. By taking the scalar product with $A_1\mathbf{w}$ to both sides of (3.4) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + \| A_1\mathbf{w} \|^2 \\ \leq\ & |\langle(\mathbf{v}_g\cdot\nabla)\mathbf{w}, A_1\mathbf{w}\rangle| + |\langle(\mathbf{w}\cdot\nabla)\mathbf{v}, A_1\mathbf{w}\rangle| \\ +\ & |\langle(\mathbf{v}_g\cdot\nabla)\nabla q_g, A_1\mathbf{w}\rangle| + |\langle P_1P_g(\nabla q_g\cdot\nabla)\mathbf{v}_g, A_1\mathbf{w}\rangle| \\ =\ & |I| + |II| + |III| + |IV|,\end{aligned} \tag{3.41}$$

for all $t \geq 0$. By lemma and the Young inequality we have

$$\begin{aligned}|I| &= |\langle(\mathbf{v}_g\cdot\nabla)\mathbf{w}, A_1\mathbf{w}\rangle| \leq \gamma_2\| A_1\mathbf{v}_g \| \| A_1^{\frac{1}{2}}\mathbf{w} \| \| A_1\mathbf{w} \| \\ &\leq \frac{1}{8}\| A_1\mathbf{w} \|^2 + 8\gamma_2^2\| A_1\mathbf{v}_g \|^2 \| A_1^{\frac{1}{2}}\mathbf{w} \|^2.\end{aligned} \tag{3.42}$$

Similar to $|I|$ we obtain

$$\begin{aligned}|II| &= |\langle(\mathbf{w}\cdot\nabla)\mathbf{v}, A_1\mathbf{w}\rangle| \leq \gamma_2\| A_1^{\frac{1}{2}}\mathbf{w} \| \| A_1\mathbf{v} \| \| A_1\mathbf{w} \| \\ &\leq \frac{1}{8}\| A_1\mathbf{w} \|^2 + 8\gamma_2^2\| A_1\mathbf{v} \|^2 \| A_1^{\frac{1}{2}}\mathbf{w} \|^2.\end{aligned} \tag{3.43}$$

Next, by using lemma , (1.6), (2.1), (2.4) and (2.8), there exists some constant c_{17} = $c_{17}(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$ such that

$$
\begin{aligned}
|III| = \; & |\langle(\mathbf{v}_g \cdot \nabla)\nabla q_g, A_1\mathbf{w}\rangle| \leq \gamma_1 \| \mathbf{v}_g \|_{H^2} \| q_g \|_{H^2} \| A_1\mathbf{w} \| \\
& \leq \frac{1}{8} \| A_1\mathbf{w} \|^2 + c_{17} \| \nabla g \|_\infty^2 \| A_1\mathbf{v}_g \|^2.
\end{aligned} \tag{3.44}
$$

By applying (2.7) we have

$$
|IV| = |\langle P_1 P_g(\nabla q_g \cdot \nabla)\mathbf{v}_g, A_1\mathbf{w}\rangle| \leq |\langle(\nabla q_g \cdot \nabla)\mathbf{v}_g, A_1\mathbf{w}\rangle| + \frac{1}{m} \| \mathbf{k} \| \, \| A_1\mathbf{w} \| \tag{3.45}
$$

where $\mathbf{k} = \int_\Omega (\nabla q_g \cdot \nabla)\mathbf{v}_g \, d\mathbf{x}$. Similar to $|III|$, we obtain

$$
\begin{aligned}
|\langle(\nabla q_g \cdot \nabla)\mathbf{v}_g, A_1\mathbf{w}\rangle| \; & \leq \gamma_1 \| q_g \|_{H^2} \| \mathbf{v}_g \|_{H^2} \; \| A_1\mathbf{w} \| \\
& \leq \frac{1}{8} \| A_1\mathbf{w} \|^2 + c_{17} \| \nabla g \|_\infty^2 \| A_1\mathbf{v}_g \|^2.
\end{aligned} \tag{3.46}
$$

Also, by (2.1), (2.4) and (2.8) we obtain

$$
\begin{aligned}
\frac{1}{m} \| \mathbf{k} \| \; \| A_1\mathbf{w} \| \; & \leq \frac{1}{m} \| q_g \|_{H^2} \; \| \nabla\mathbf{v}_g \| \; \| A_1\mathbf{w} \| \\
& \leq \frac{1}{8} \| A_1\mathbf{w} \|^2 + c_{18} \| \nabla g \|_\infty^2 \; \| A_1\mathbf{v}_g \|^2,
\end{aligned} \tag{3.47}
$$

for some constant $c_{18} = c_{18}(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$. So, from (3.45), (3.46) and (3.47) we have

$$
|IV| \leq \frac{1}{4} \| A_1\mathbf{w} \|^2 + (c_{17} + c_{18}) \| \nabla g \|_\infty^2 \; \| A_1\mathbf{v}_g \|^2. \tag{3.48}
$$

Therefore, from (3.41), (3.42), (3.43), (3.44) and (3.48), we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + \frac{3}{8} \| A_1\mathbf{w} \|^2 \; & \leq 8\gamma_2^2 \Big(\| A_1\mathbf{v}_g \|^2 + \| A_1\mathbf{v} \|^2 \Big) \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 \\
& + \; (2c_{17} + c_{18}) \| \nabla g \|_\infty^2 \| A_1\mathbf{v}_g \|^2,
\end{aligned} \tag{3.49}
$$

for all $t \geq 0$. So, we have

$$
\frac{d}{dt} \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 \leq \beta_9(t) \| A_1^{\frac{1}{2}}\mathbf{w} \|^2 + \beta_{10}(t), \text{ for all } t \geq 0,
$$

where

$$
\beta_9(t) - 16\gamma_2^2 \left(\| A_1\mathbf{v}_g(t) \|^2 + \| A_1\mathbf{v}(t) \|^2 \right) \tag{3.50}
$$

(3.51) $$\beta_{10}(t) = (4c_{17} + 2c_{18}) \| \nabla g \|_\infty^2 \| A_1 \mathbf{v}_g(t) \|^2.$$

By the Gronwall inequality, we get

(3.52) $$\| A_1^{\frac{1}{2}} \mathbf{w}(t) \|^2 \leq e^{\int_0^t \beta_9(s)ds} \left[\| A_1^{\frac{1}{2}} \mathbf{w}(0) \|^2 + \int_0^t \beta_{10}(s)ds \right],$$

for all $t \geq 0$. Now, by (3.35) and the classical theory of the Navier-Stokes equations for periodic boundary conditions, there exists $c_{19} = c_{19}(m, M, M_0, \| A_1^{\frac{1}{2}} \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$ such that

(3.53) $$\int_0^T \beta_9(s)ds = \int_0^T 16\gamma_2^2 \left(\| A_1 \mathbf{v}_g(s) \|^2 + \| A_1 \mathbf{v}(s) \|^2 \right) ds \leq c_{19}$$

and there exists $c_{20} = c_{20}(m, M, M_0, \| A_1^{\frac{1}{2}} \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$ such that

(3.54) $$\int_0^T \beta_{10}(s)ds = \int_0^T (4c_{17} + 2c_{18}) \| \nabla g \|_\infty^2 \| A_1 \mathbf{v}_g(s) \|^2 \, ds \leq c_{20} \| \nabla g \|_\infty^2.$$

Therefore, from (3.52), (3.53) and (3.54) we have

$$\| A_1^{\frac{1}{2}} \mathbf{w}(t) \|^2 \leq e^{c_{19}} \left[\| A_1^{\frac{1}{2}} \mathbf{w}(0) \|^2 + c_{20} \| \nabla g \|_\infty^2 \right], \text{for all} 0 \leq t < T$$

which implies

(3.55) $$\| \nabla(\mathbf{v}_g(t) - \mathbf{v}(t)) \|^2 = \| A_1^{\frac{1}{2}} \mathbf{w}(t) \|^2 \leq c_{20} \, e^{c_{19}} \, \| \nabla g \|_\infty^2,$$

because $\mathbf{w}(0) = 0$.

Next, by (2.1), (2.4) and (2.8), there exists constant $c_{21} = c_{21}(m, M, \| \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$ such that

(3.56) $$\| \nabla(\mathbf{u}_g - \mathbf{v}_g) \|^2 = \| \nabla(\nabla q_g) \|^2 \leq c_4 \| \nabla g \|_\infty^2 \| \mathbf{u}_g \|^2 \leq c_{21} \| \nabla g \|_\infty^2.$$

Since $\int_\Omega \mathbf{u}_g d\mathbf{x} = \int_\Omega \mathbf{v} d\mathbf{x} = 0$ and $\mathbf{u}_g, \mathbf{v} \in H^1_{per}(\Omega)$, we have

$$\| \mathbf{u}_g - \mathbf{v} \|_{H^1} \leq 2 \, \| \nabla(\mathbf{u}_g - \mathbf{v}) \|.$$

So, we obtain from (3.55) and (3.56)

$$\begin{aligned} \| \mathbf{u}_g(t) - \mathbf{v}(t) \|_{H^1}^2 &\leq 2 \, \| \nabla(\mathbf{u}_g - \mathbf{v}) \|^2 \leq 4(\, \| \nabla(\mathbf{u}_g - \mathbf{v}_g) \|^2 + \| \nabla(\mathbf{v}_g - \mathbf{v}) \|^2) \\ &\leq 4 \, (c_{21} + c_{20} \, e^{c_{19}}) \| \nabla g \|_\infty^2. \end{aligned}$$

Next, to prove second part of (3.39), we take the integral from 0 to T both sides of (3.49). Then, we obtain by (3.53), (3.54) and (3.55) that

$$\begin{aligned}\frac{3}{4}\int_0^T \| A_1\mathbf{w}(s) \|^2 \, ds &\le \int_0^T \beta_9(s) \, \| A_1^{\frac{1}{2}}\mathbf{w}(s) \|^2 \, ds + \int_0^T \beta_{10}(s) \, ds \\ &\le (c_{19}c_{20}e^{c_{19}} + c_{20}) \| \nabla g \|_\infty^2,\end{aligned}$$

because $\| A_1^{\frac{1}{2}}\mathbf{w}(0) \| = 0$. So, by (1.6), we obtain

$$\int_0^T \| \mathbf{w}(s) \|_{H^2}^2 \, ds \le \tilde{\delta}^2 \int_0^T \| A_1\mathbf{w}(s) \|^2 \le \frac{4\tilde{\delta}^2}{3} (c_{19}c_{20}e^{c_{19}} + c_{20}) \| \nabla g \|_\infty^2. \tag{3.57}$$

Also, we obtain due to lemma , (2.9) and remark that

$$\begin{aligned}&\int_0^T \| \mathbf{u}_g(s) - \mathbf{v}_g(s) \|_{H^2}^2 \, ds = \int_0^T \| \nabla q_g \|_{H^2}^2 \, ds \le \int_0^T \| q_g \|_{H^3}^2 \, ds \\ &\le \delta_0^2 \| g \|_{2,\infty}^2 \int_0^T \| \mathbf{u}_g \|_{H^1}^2 \, ds \le c \, \delta_0^2 \| g \|_{2,\infty}^2,\end{aligned} \tag{3.58}$$

for some constant $c = c(m, M, \| A_1^{\frac{1}{2}}\mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$. So, from (3.57) and (3.58), we get

$$\begin{aligned}&\int_0^T \| \mathbf{u}_g(s) - \mathbf{v}(s) \|_{H^2}^2 ds \\ &\le 2\left(\int_0^T \| \mathbf{u}_g(s) - \mathbf{v}_g(s) \|_{H^2}^2 ds + \int_0^T \| \mathbf{v}_g(s) - \mathbf{v}(s) \|_{H^2}^2 ds\right) \\ &= 2\left(\int_0^T \| \mathbf{u}_g(s) - \mathbf{v}_g(s) \|_{H^2}^2 ds + \int_0^T \| \mathbf{w}(s) \|_{H^2}^2 ds\right) \to 0\end{aligned}$$

which completes the proof of the second part in (3.39).

At last, to prove (3.40) one note by (3.16) and (3.17) that

$$\nabla p_g = \mathbf{f} - \mathbf{u}_g' - \Delta\mathbf{u}_g - (\mathbf{u}_g \cdot \nabla)\mathbf{u}_g \tag{3.59}$$

and

$$\nabla p = \mathbf{f} - \mathbf{v}' - \Delta\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{3.60}$$

By (3.39), we obtain

$$\int_0^T \| \Delta(\mathbf{u}_g - \mathbf{v}) \|^2 \, dt \le \int_0^T \| \mathbf{u}_g - \mathbf{v} \|_{H^2}^2 \, dt \to 0, \tag{3.61}$$

as $\| g \|_{2,\infty} \to 0$.

Also, by (3.39), the Hölder inequality and the Sobolev inequality, we obtain

$$
\begin{aligned}
&\textstyle\int_0^T \| (\mathbf{u}_g \cdot \nabla)\mathbf{u}_g - (\mathbf{v} \cdot \nabla)\mathbf{v} \|^2 \, dt \\
\le\ & 2 \int_0^T \| [(\mathbf{u}_g - \mathbf{v}) \cdot \nabla]\mathbf{u}_g \|^2 + \| (\mathbf{v} \cdot \nabla)(\mathbf{u}_g - \mathbf{v}) \|^2 \, dt \\
\le\ & 2 \int_0^T \| \mathbf{u}_g - \mathbf{v} \|_{H^2}^2 \, dt \left(\sup_{0 \le t < T} \| \mathbf{u}_g(t) \|_{H^1}^2 + \sup_{0 \le t < T} \| \mathbf{v}(t) \|_{H^1}^2 \right) \\
\le\ & 2c_{22} \int_0^T \| \mathbf{u}_g - \mathbf{v} \|_{H^2}^2 \, dt \to 0,
\end{aligned} \tag{3.62}
$$

for some constant $c_{22} = c_{22}(m, M, \| A_1^{\frac{1}{2}} \mathbf{v}_0 \|, \| \mathbf{f} \|)$, as $\| g \|_{2,\infty} \to 0$. By (2.9) note that for all $g \in \Lambda_s$,

$$
l_1 \ \| A_g^{\frac{1}{2}} \mathbf{u}_g(0) \| \le \ \| A_1^{\frac{1}{2}} \mathbf{v}_g(0) \| = \ \| A_1^{\frac{1}{2}} \mathbf{v}(0) \|.
$$

So, for all $g \in \Lambda_s$, we can have constant c_{22} depending on $\| A_1^{\frac{1}{2}} \mathbf{v}(0) \|$ rather than on $\| A_1^{\frac{1}{2}} \mathbf{u}_g(0) \|$. Next, we want to prove

$$
\int_0^T \| \mathbf{u}_g' - \mathbf{v}' \|^2 \, dt \to 0, \text{ as } g \to 1 \text{ in } W^{2,\infty}(\Omega).
$$

Before we do that, one should remind that $\mathbf{u}_g$ satisfies

$$
\mathbf{u}_g' = P_g \mathbf{f} - P_g(-\Delta \mathbf{u}_g) - P_g((\mathbf{u}_g \cdot \nabla)\mathbf{u}_g) \tag{3.63}
$$

and $\mathbf{v}$ satisfies

$$
\mathbf{v}' = P_1 \mathbf{f} - P_1(-\Delta \mathbf{v}) - P_1((\mathbf{v} \cdot \nabla)\mathbf{v}). \tag{3.64}
$$

Since $\int_\Omega \mathbf{f} \, d\mathbf{x} = 0$, by lemma , we obtain

$$
\begin{aligned}
\textstyle\int_0^T \| P_g \mathbf{f} - P_1 \mathbf{f} \|^2 \, dt &\le \int_0^T \frac{4}{m^2} \| \nabla g \|_\infty^2 \| \mathbf{f} \|^2 \, dt \\
&\le \frac{4}{m^2} \| \nabla g \|_\infty^2 \| \mathbf{f} \|_{2,2}^2 \to 0,
\end{aligned} \tag{3.65}
$$

as $\| g \|_{2,\infty} \to 0$. By lemma and lemma , we have $\mathbf{u}_g = \mathbf{v}_g + \nabla q_g$ and

$$
P_g(\Delta \mathbf{u}_g) = P_g(\Delta \mathbf{v}_g) \quad \text{and} \quad P_1(\Delta \mathbf{u}_g) = P_1(\Delta \mathbf{v}_g).
$$

So, we obtain due to lemma that

$$\begin{aligned}
& \| P_g(-\Delta \mathbf{u}_g) - P_1(-\Delta \mathbf{v}) \| \\
\leq & \| P_g(-\Delta \mathbf{u}_g) - P_1(-\Delta \mathbf{u}_g) \| + \| P_1(-\Delta \mathbf{u}_g) - P_1(-\Delta \mathbf{v}) \| \\
= & \| P_g(-\Delta \mathbf{v}_g) - P_1(-\Delta \mathbf{v}_g) \| + \| P_1(-\Delta \mathbf{u}_g) - P_1(-\Delta \mathbf{v}) \| \\
\leq & \frac{2}{m} \| \nabla g \|_\infty \| -\Delta \mathbf{v}_g \| + \| -\Delta(\mathbf{u}_g - \mathbf{v}) \| \\
\leq & \frac{2}{m} \| \nabla g \|_\infty \| \mathbf{v}_g \|_{H^2} + \| (\mathbf{u}_g - \mathbf{v}) \|_{H^2}
\end{aligned}$$

which implies

$$\begin{aligned}
& \int_0^T \| P_g(-\Delta \mathbf{u}_g) - P_1(-\Delta \mathbf{v}) \|^2 \, dt \\
\leq & \frac{4}{m^2} \| \nabla g \|_\infty^2 \int_0^T \| \mathbf{v}_g \|_{H^2}^2 \, dt + \int_0^T \| \mathbf{u}_g - \mathbf{v} \|_{H^2}^2 \, dt.
\end{aligned} \tag{3.66}$$

Therefore, by lemma , (1.6) and (3.39), (3.66) goes to zero as $\| g \|_{2,\infty} \to 0$.

Next, we get by lemma that

$$\begin{aligned}
& \| P_g(\mathbf{u}_g \cdot \nabla)\mathbf{u}_g - P_1(\mathbf{v} \cdot \nabla)\mathbf{v} \| \\
= & \| P_g(\mathbf{u}_g \cdot \nabla)\mathbf{u}_g - P_g(\mathbf{v} \cdot \nabla)\mathbf{v} \| + \| P_g(\mathbf{v} \cdot \nabla)\mathbf{v} - P_1(\mathbf{v} \cdot \nabla)\mathbf{v} \| \\
\leq & \| (\mathbf{u}_g \cdot \nabla)\mathbf{u}_g - (\mathbf{v} \cdot \nabla)\mathbf{v} \| + \frac{2}{m} \| \nabla g \|_\infty \| (\mathbf{v} \cdot \nabla)\mathbf{v} \|.
\end{aligned} \tag{3.67}$$

Also, by (3.62) we obtain

$$\int_0^T \| (\mathbf{u}_g \cdot \nabla)\mathbf{u}_g - (\mathbf{v} \cdot \nabla)\mathbf{v} \|^2 \, dt \leq 2c_{22} \int_0^T \| \mathbf{u}_g - \mathbf{v} \|_{H^2}^2 \, dt \to 0 \tag{3.68}$$

as $\| g \|_{2,\infty} \to 0$. Moreover, by the Hölder inequality, the Sobolev inequality and the classical theory of the Navier-Stokes equations, we obtain

$$\int_0^T \| (\mathbf{v} \cdot \nabla)\mathbf{v} \|^2 \, dt \leq c \int_0^T \| \mathbf{v} \|_{H^2}^2 \| \mathbf{v} \|_{H^1}^2 \, dt \leq c_{23} \tag{3.69}$$

for some constant $c_{23} = c_{23}(\| A_1^{\frac{1}{2}} \mathbf{v}_0 \|, \| \mathbf{f} \|_{2,2})$. Refer chapter 3 in Temma[12] for the details of (3.69). Therefore, from (3.67), (3.68) and (3.69), we have

$$\int_0^T \| P_g(\mathbf{u}_g \cdot \nabla)\mathbf{u}_g - P_1(\mathbf{v} \cdot \nabla)\mathbf{v} \|^2 \, dt \to 0, \text{ as } \| g \|_{2,\infty} \to 0. \tag{3.70}$$

So, from (3.63), (3.64), (3.65), (3.66) and (3.70) we obtain

$$\int_0^T \| \mathbf{u}_g' - \mathbf{v}' \|^2 \, dt \to 0, \quad \text{as } g \to 1 \text{ in } W^{2,\infty}(\Omega). \tag{3.71}$$

Hence, by (3.59), (3.60), (3.61), (3.62) and (3.71), we complete the proof of (3.40). ■

4. Dirichlet Problem

In this section, we consider for Dirichlet boundary conditions on bounded domain $\Omega \subset R^2$. We assume that g satisfies $g(\mathbf{x}) \in C^\infty(\Omega)$ and $0 < m \leq g(\mathbf{x}) \leq M$, for all $\mathbf{x} \in \Omega$. For a mathematical setting, we use

$$H_g = CL_{L^2(\Omega,g)}\{\mathbf{u} \in C_0^\infty(\Omega) \; ; \; \nabla \cdot g\mathbf{u} = 0\} \text{ and}$$

$$V_g = \{\mathbf{u} \in H_0^1(\Omega, g) \; ; \; \nabla \cdot g\mathbf{u} = 0\}.$$

Also, for a orthogonal projection, $P_g : L^2(\Omega, g) \mapsto H_g$, we define $P_g\mathbf{u} = \mathbf{v} \in H_g$ where $\mathbf{u} = \mathbf{v} + \nabla p$ and p is the solution of $\frac{1}{g}(\nabla \cdot g\nabla)p = \frac{1}{g}(\nabla \cdot g\mathbf{u})$.

For the Poincaré inequality, there exists some constant $c > 0$ such that for $\mathbf{u} \in V_g$,

$$\frac{1}{M}\| \nabla \mathbf{u} \|_g^2 \leq \| \nabla \mathbf{u} \|^2 \leq c \| \mathbf{u} \|^2 \leq cM \| \mathbf{u} \|_g^2.$$

Moreover, for lemma , we have better results,

$$P_1 P_g \mathbf{u} = P_1 \mathbf{u}, \quad \text{for all} \quad \mathbf{u} \in L^2(\Omega),$$

which implies

$$\langle P_1 P_g \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle, \text{ for } \mathbf{u} \in L^2(\Omega) \text{ and } \mathbf{w} \in H_1.$$

Finally, we can obtain similar results for main theorems.

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