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MEAN VALUE OF THE CHARACTER SUMS OVER INTERVAL $\left[1, \frac{q}{8}\right)$

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Abstract. Let $q > 8$ be an odd integer and p a prime with $p < q$ and $p \nmid q$. The main purpose of this paper is to study the mean value properties of the character sums over interval $\left[1, \frac{q}{8}\right)$ by using the mean value theorems of the Dirichlet L-functions, and give some interesting mean value formulae.

1. Introduction

Let $q \geq 3$ be an integer and χ be a Dirichlet character modulo q. Character sums

$$\sum_{a=N+1}^{N+H} \chi(a)$$

was investigated by many scholars, see [1-8]. About the mean value of this kind of character sums, D.A.Burgess [4] obtained the following estimate:

$$\sum_{n=1}^{k} \left| \sum_{m=1}^{h} \chi(n+m) \right|^2 < k \cdot h,$$

where h is any positive integer. In fact this inequality was conjectured by Norton [5], who obtained the weaker upper bound $\frac{9}{8}kh$ before Burgess. To higher moments, Burgess [6] specified the problem to the case of $q = p$, and summed the fourth power mean over all nonprincipal characters. That is, he proved

$$\sum_{\chi \neq \chi_0} \sum_{n=1}^{p} \left| \sum_{m=1}^{h} \chi(n+m) \right|^4 \leq 6p^2h^2.$$

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For general modulo q, he summed the mean value over all primitive characters and obtained (see [7])

$$\sum_{\chi \bmod q}^{*} \sum_{n=1}^{q} \left| \sum_{m=1}^{h} \chi(n+m) \right|^4 \leq 8\tau^7(q) q^2 h^2,$$

where $\sum\limits_{\chi \bmod q}^{*}$ denotes the summation over all primitive characters modulo q and $\tau(n)$ is the Dirichlet divisor function. About the $2k$-th power mean of the character sums, H.L.Montgomery and R.C.Vaughan [8] gave an upper estimate about it in 1979 as follows:

$$\sum_{\chi \neq \chi_0} M(\chi)^{2k} \ll_k \phi(q) q^k,$$

where $M(\chi) = \max\limits_{N} \left| \sum\limits_{n=1}^{N} \chi(n) \right|$. The author and Zhang studied the $2k$-th power mean of the character sums over the quarter interval and obtained a sharp asymptotic formula, see [9].

The present work deals with the higher power mean of the character sums over interval $\left[1, \frac{q}{8}\right)$. First we transformed the character sums into L-functions. Then using the mean value theorems of Dirichlet L-functions, we studied the mean value properties of the character sums, and obtained some sharper asymptotic formulae for them.

In this paper, we will use the following notations:

$\sum\limits_{\chi(-1)=1}^{*}$ denotes the summation over all primitive characters modulo q such that $\chi(-1) = 1$,

$J(q)$ denotes the number of all primitive characters modulo q,

$\prod\limits_{p|q}$ denotes the product over all prime divisors p of q,

$\prod\limits_{\substack{p \nmid q \\ p \equiv 3,5 (\bmod 8)}}$ denotes the product over all prime p with $p \nmid q$ and $p \equiv 3, 5 (\bmod 8)$,

$\tau_k(n)$ denotes the k-th divisor function (i.e., the number of solutions of the equation $n_1 n_2 \cdots n_k = n$ in positive integers $n_1, n_2, \cdots, n_k$),

χ_4 is the primitive character modulo 4,

χ_8 is the real primitive character modulo 8 with $\chi_8(-1) = -1$ and

χ_8' denotes the real primitive character modulo 8 with $\chi_8'(-1) = 1$.

Now we give the main conclusions.

Theorem 1. *Let $q > 8$ be an odd integer and $k \geq 2$ be a fixed integer. Then we have the asymptotic formula*

$$\sum_{\chi(-1)=1}^{*} \left| \sum_{a<\frac{q}{8}} \chi(a) \right|^{2k} = \frac{J(q)q^k}{2^{2k+1}\pi^{2k}} \sum_{i=0}^{k} 2^{k-i} \binom{k}{i}^2 C_i + O\left(q^{k+\epsilon}\right),$$

where ϵ is any fixed positive number, $\binom{k}{i} = \frac{k!}{i!(k-i)!}$, $C_i = \sum_{\substack{n=1\\(n,2q)=1}}^{\infty} \frac{r_i^2(n)}{n^2}$ is a constant related to i, $r_i(n) = \sum_{t|n} \tau_i(t)\chi_4(t)\tau_{k-i}\left(\frac{n}{t}\right)\chi_8\left(\frac{n}{t}\right)$ for $1 < i < k$ and $r_0(n) = \sqrt{2}\tau_k(n), r_k(n) = \tau_k(n)$.

From this theorem, we can conclude that Montgomery and Vaughan's estimate for the $2k$-th power mean of character sums is the best possible.

Theorem 2. *Let $q > 8$ be an odd integer. Then we have the asymptotic formula*

$$\sum_{\chi(-1)=1}^{*} \left| \sum_{a<\frac{q}{8}} \chi(a) \right|^{4} = \frac{q^2 J(q)}{384} \left[\prod_{p|q} \left(1 - \frac{1}{p^4}\right) \prod_{\substack{p\nmid q\\ p\equiv 1,7 (\bmod 8)}} \left(1 + \frac{2}{p^2-1}\right)^2 + \frac{27}{32} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} \right] + O(q^{2+\epsilon}).$$

Noting that $J(q) = \frac{\phi^2(q)}{q}$ if q is a square-full number, where $\phi(q)$ is the Euler function, we have the following:

Corollary 1. *Let $q > 8$ be a square-full number with $2 \nmid q$ and $k \geq 2$ be a fixed integer. Then we have the asymptotic formula*

$$\sum_{\chi(-1)=1}^{*} \left| \sum_{a<\frac{q}{8}} \chi(a) \right|^{2k} = \frac{\phi^2(q)q^{k-1}}{2^{2k+1}\pi^{2k}} \sum_{i=0}^{k} 2^{k-i} \binom{k}{i}^2 C_i + O\left(q^{k+\epsilon}\right).$$

Corollary 2. *Let $p > 8$ be a prime. Then we have the asymptotic formula*

$$\sum_{\chi(-1)=1} \left| \sum_{a<\frac{p}{8}} \chi(a) \right|^{4} = \frac{p^3}{384} \left[\frac{27}{32} + \prod_{\substack{p_1\neq p\\ p_1\equiv 1,7 (\bmod 8)}} \left(1 + \frac{2}{p_1^2-1}\right)^2 \right] + O\left(p^{2+\epsilon}\right),$$

where $\sum_{\chi(-1)=1}$ *denotes the summation over all non-principal characters modulo* p *such that* $\chi(-1)=1$.

2. Some Lemmas

To prove the theorems, we need the following lemmas.

Lemma 1. *Let* χ *be a primitive Dirichlet character modulo* q *with* $\chi(-1)=1$. *Then for any real number* $\lambda \in [0,1)$ *with* $\lambda \neq \frac{r}{q}$, *we have*

$$\sum_{a=1}^{[\lambda q]} \chi(a) = \frac{\tau(\chi)}{\pi} \sum_{n=1}^{\infty} \bar{\chi}(n) \frac{\sin 2\pi n\lambda}{n},$$

where $[x]$ *denotes the greatest integer less than or equal to* x, $\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right)$ *is the Gauss sum and* $e(y) = e^{2\pi i y}$.

Proof. See Section 3.1 of [10]. ■

Lemma 2. *Let* $q \geq 8$ *be an odd integer and* χ *be a primitive Dirichlet character modulo* q *such that* $\chi(-1)=1$. *Then we have*

$$\sum_{a=1}^{[\frac{q}{8}]} \chi(a) = \frac{\tau(\chi)}{2\pi}\left[\bar{\chi}(2) L(1, \bar{\chi}\chi_4) + \sqrt{2} L(1, \bar{\chi}\chi_8)\right],$$

where χ_4 *is the nonprincipal Dirichlet character modulo* 4, χ_8 *denotes the real primitive character modulo* 8 *such that* $\chi(-1) = -1$ *and* $L(s,\chi)$ *denotes the Dirichlet L-function corresponding to* χ.

Proof. It is clear that $\frac{1}{8} \neq \frac{r}{q}$ if q is an odd number. So from Lemma 1, we can write

$$\begin{aligned}
\sum_{a=1}^{[\frac{q}{8}]} \chi(a) &= \frac{\tau(\chi)}{\pi} \sum_{n=1}^{\infty} \bar{\chi}(n) \frac{\sin \frac{\pi n}{4}}{n} \\
&= \frac{\tau(\chi)}{\pi}\left[\frac{\bar{\chi}(2)}{2} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \sin \frac{\pi n}{2}}{n} + \sum_{n=1}^{\infty} \frac{\bar{\chi}(2n-1) \sin \frac{\pi(2n-1)}{4}}{2n-1}\right] \\
&= \frac{\tau(\chi)}{\pi}\left[\frac{\bar{\chi}(2)}{2} L(1, \bar{\chi}\chi_4) + \sum_{n=1}^{\infty} \frac{\bar{\chi}(2n-1) \sin \frac{\pi(2n-1)}{4}}{2n-1}\right] \\
&= \frac{\tau(\chi)}{\pi}\left[\frac{\bar{\chi}(2)}{2} L(1, \bar{\chi}\chi_4) + \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\chi_4^0(n)(-1)^{[\frac{n}{4}]}}{n}\right],
\end{aligned}$$

where χ_4^0 denotes the principal character modulo 4. It is easily to deduce that $(-1)^{\left[\frac{n}{4}\right]}$ is a multiplicative function and its least period is 8. Hence, $\chi_4^0(n)(-1)^{\left[\frac{n}{4}\right]} = \chi_8^0(n)(-1)^{\left[\frac{n}{4}\right]}$ is a real primitive character modulo 8, denoted it by $\chi_8(n)$. It is clear that $\chi_8(-1) = -1$. So we have

$$\sum_{a=1}^{\left[\frac{q}{8}\right]} \chi(a) = \frac{\tau(\chi)}{2\pi}\left[\bar{\chi}(2)L(1, \bar{\chi}\chi_4) + \sqrt{2}L(1, \bar{\chi}\chi_8)\right].$$

This proves Lemma 2. ■

Lemma 3. *Let q and r be integers with $q \geq 2$ and $(r, q) = 1$, χ be a Dirichlet character modulo q. Then we have the identities*

$$\sideset{}{^*}\sum_{\chi \bmod q} \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sideset{}{^*}\sum_{\chi \bmod q}$ denotes the summation over all primitive characters modulo q, and $J(q)$ denotes the number of all primitive characters modulo q.

Proof. (See Lemma 4 of reference [11]). ■

Lemma 4. *Let q be any odd integer with $q > 2$, χ be the Dirichlet character modulo q and χ_4 be the primitive character modulo 4. Then we have the asymptotic formula:*

$$\sideset{}{^*}\sum_{\chi(-1)=1} |L(1, \bar{\chi}\chi_4)|^{2k} = \frac{J(q)}{2} \sum_{\substack{n=1 \\ (n,2q)=1}}^{\infty} \frac{\tau_k^2(n)}{n^2} + O(q^{\epsilon}),$$

where $\sideset{}{^*}\sum_{\chi(-1)=1}$ denotes the summation over all even primitive characters modulo q.

Proof. For convenience, we put

$$A(y, \chi) = \sum_{N<n\leq y} \chi(n)\tau_k(n),$$

where N is a parameter with $q \leq N < q^{2^k+1}$. Then from Abel's identity we have

$$L^k(1, \bar{\chi}\chi_4) = \sum_{n=1}^{\infty} \frac{\bar{\chi}\chi_4(n)\tau_k(n)}{n} = \sum_{1\leq n\leq N} \frac{\bar{\chi}\chi_4(n)\tau_k(n)}{n} + \int_N^{\infty} \frac{A(y, \bar{\chi}\chi_4)}{y^2}dy.$$

Hence, we can write

$$
\begin{aligned}
&\sideset{}{^*}\sum_{\chi(-1)=1} |L(1,\bar{\chi}\chi_4)|^{2k} \\
&= \sideset{}{^*}\sum_{\chi(-1)=1} \left(\sum_{1\le n_1\le N} \frac{\bar{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1} + \int_N^\infty \frac{A(y,\bar{\chi}\chi_4)}{y^2}dy \right) \\
&\quad \times \left(\sum_{1\le n_2\le N} \frac{\chi\chi_4(n_2)\tau_k(n_2)}{n_2} + \int_N^\infty \frac{A(y,\chi\chi_4)}{y^2}dy \right) \\
&= \sideset{}{^*}\sum_{\chi(-1)=1} \left(\sum_{1\le n_1\le N} \frac{\bar{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1} \right) \left(\sum_{1\le n_2\le N} \frac{\chi\chi_4(n_2)\tau_k(n_2)}{n_2} \right) \\
&\quad + \sideset{}{^*}\sum_{\chi(-1)=1} \left(\sum_{1\le n_1\le N} \frac{\bar{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1} \right) \left(\int_N^\infty \frac{A(y,\chi\chi_4)}{y^2}dy \right) \\
&\quad + \sideset{}{^*}\sum_{\chi(-1)=1} \left(\sum_{1\le n_2\le N} \frac{\chi\chi_4(n_2)\tau_k(n_2)}{n_2} \right) \left(\int_N^\infty \frac{A(y,\bar{\chi}\chi_4)}{y^2}dy \right) \\
&\quad + \sideset{}{^*}\sum_{\chi(-1)=1} \left(\int_N^\infty \frac{A(y,\bar{\chi}\chi_4)}{y^2}dy \right) \left(\int_N^\infty \frac{A(y,\chi\chi_4)}{y^2}dy \right) \\
&:= M_1 + M_2 + M_3 + M_4.
\end{aligned} \tag{1}
$$

We shall calculate each term in the expression (1).

(i) From Lemma 3 we have

$$
\begin{aligned}
&\sideset{}{^*}\sum_{\chi(-1)=1} \chi\chi_4(a) = \sideset{}{^*}\sum_{\chi\chi_4(-1)=-1} \chi\chi_4(a) \\
&= \frac{1}{2} \sideset{}{^*}\sum_{\chi\chi_4 \bmod 4q} (1-\chi\chi_4(-1))\chi\chi_4(a) \\
&= \frac{1}{2} \sideset{}{^*}\sum_{\chi\chi_4 \bmod 4q} \chi\chi_4(a) - \frac{1}{2} \sideset{}{^*}\sum_{\chi\chi_4 \bmod 4q} \chi\chi_4(-a) \\
&= \frac{1}{2} \sum_{d|(4q,a-1)} \mu\left(\frac{4q}{d}\right)\phi(d) - \frac{1}{2} \sum_{d|(4q,a+1)} \mu\left(\frac{4q}{d}\right)\phi(d).
\end{aligned} \tag{2}
$$

Hence, from (2) we can write

$$
\begin{aligned}
M_1 &= \sum_{\chi(-1)=1}^{*} \left(\sum_{1\le n_1 \le N} \frac{\bar{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1} \right) \left(\sum_{1\le n_2 \le N} \frac{\chi\chi_4(n_2)\tau_k(n_2)}{n_2} \right) \\
&= \frac{1}{2} \sideset{}{'}\sum_{1\le n_1\le N} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \sum_{d|(4q, \bar{n}_1 n_2 - 1)} \mu\left(\frac{4q}{d}\right)\phi(d) \\
&\quad -\frac{1}{2} \sideset{}{'}\sum_{1\le n_1\le N} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \sum_{d|(4q, \bar{n}_1 n_2 + 1)} \mu\left(\frac{4q}{d}\right)\phi(d) \qquad (3) \\
&= \frac{1}{2} \sum_{d|4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ n_2 \equiv n_1 (\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \\
&\quad -\frac{1}{2} \sum_{d|4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ n_2 \equiv -n_1 (\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2},
\end{aligned}
$$

where $\sideset{}{'}\sum_{1\le n\le N}$ denotes the summation over n from 1 to N such that $(n, 2q) = 1$.

For convenience, we split the sum over n_1 or n_2 into four cases: i) $d \le n_1, n_2 \le N$; ii) $d \le n_1 \le N$ and $1 \le n_2 \le d-1$; iii) $1 \le n_1 \le d-1$ and $d \le n_2 \le N$; iv) $1 \le n_1, n_2 \le d-1$. So we have

$$
\begin{aligned}
&\sum_{d|4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{d\le n_1\le N \\ n_2 \equiv n_1 (\bmod d)}} \sideset{}{'}\sum_{d\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \\
&\ll \sum_{d|4q} \phi(d) \sum_{1\le r_1 \le \frac{N}{d}} \sum_{1\le r_2 \le \frac{N}{d}} \sideset{}{'}\sum_{\substack{l_1=1 \\ l_2 \equiv l_1 (\bmod d)}}^{d-1} \sideset{}{'}\sum_{l_2=1}^{d-1} \frac{\tau_k(r_1 d + l_1)\tau_k(r_2 d + l_2)}{(r_1 d + l_1)(r_2 d + l_2)} \\
&\ll \sum_{d|4q} \phi(d) \sum_{1\le r_1 \le \frac{N}{d}} \sum_{1\le r_2 \le \frac{N}{d}} \sideset{}{'}\sum_{l_1=1}^{d-1} \frac{[(r_1 d + l_1)(r_2 d + l_1)]^{\epsilon}}{(r_1 d + l_1)(r_2 d + l_1)} \\
&\ll \sum_{d|4q} \frac{\phi(d)}{d} \sum_{1\le r_1 \le \frac{N}{d}} \sum_{1\le r_2 \le \frac{N}{d}} \frac{[(r_1 d + 1)(r_2 d + 1)]^{\epsilon}}{r_1 r_2} \ll q^{\epsilon},
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{d|4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{d\le n_1\le N \\ n_2 \equiv n_1 (\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le d-1} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \\
&\ll \sum_{d|4q} \phi(d) \sum_{1\le r_1 \le \frac{N}{d}} \sum_{1\le n_2 \le d-1} (r_1 n_2 d)^{\epsilon - 1} \ll q^{\epsilon}
\end{aligned}
$$

and

$$\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le n_1\le d-1\\ n_2\equiv n_1(\mod d)}}\sideset{}{'}\sum_{d\le n_2\le N}\frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}$$
$$\ll\sum_{d|4q}\phi(d)\sum_{1\le n_1\le d-1}\sum_{1\le r_2\le\frac{N}{d}}(n_1r_2d)^{\epsilon-1}\ll q^{\epsilon},$$

where we have used the estimate $\tau_k(n)\ll n^{\epsilon}$.

For the case $1\le n_1,n_2\le d-1$, the solution of the congruence $n_2\equiv n_1($ mod $d)$ is $n_2=n_1$. Hence,

$$\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le n_1\le d-1\\ n_2\equiv n_1(\mod d)}}\sideset{}{'}\sum_{1\le n_2\le d-1}\frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}$$
$$=\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d)\sideset{}{'}\sum_{1\le n_2\le d-1}\frac{\tau_k^2(n_2)}{n_2^2}$$
$$=\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d)\sum_{\substack{n_2=1\\(n_2,2q)=1}}^{\infty}\frac{\tau_k^2(n_2)}{n_2^2}+O(q^{\epsilon}).$$

Since $(q,2)=1$,

$$J(4q)=\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d)=\left(\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\right)\left(\sum_{d|4}\mu\left(\frac{4}{d}\right)\phi(d)\right)$$
$$=\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)=J(q).$$

So we have

$$\frac{1}{2}\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le n_1\le N\\ n_2\equiv n_1(\mod d)}}\sideset{}{'}\sum_{1\le n_2\le N}\frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}$$
$$=\frac{J(q)}{2}\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{\tau_k^2(n)}{n^2}+O(q^{\epsilon}).\tag{4}$$

Similarly, we can also get the estimate

$$\frac{1}{2}\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le n_1\le N\\ n_2\equiv -n_1(\mod d)}}\sideset{}{'}\sum_{1\le n_2\le N}\frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}$$
$$=\frac{1}{2}\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le n_1\le N\\ n_2+n_1=d}}\sideset{}{'}\sum_{1\le n_2\le N}\frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}$$

$$+\frac{1}{2}\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d)\sum_{\substack{1\le n_1\le N\\ n_2+n_1=ld,}}{}' \sum_{\substack{1\le n_2\le N\\ l\ge 2}}{}' \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}$$

$$\ll \sum_{d|4q}\phi(d)\sum_{1\le n\le d-1}\frac{\tau_k(n)\tau_k(d-n)}{n(d-n)}$$

$$+\sum_{d|4q}\phi(d)\sum_{1\le n_1\le N}{}'\sum_{l=\left[\frac{n_1}{d}\right]+2}^{\left[\frac{N+n_1}{d}\right]}\frac{\tau_k(n_1)\tau_k(ld-n_1)}{ldn_1-n_1^2}$$

$$\ll \sum_{d|4q}\frac{\phi(d)}{d}\sum_{1\le n\le d-1}\frac{\tau_k(n)\tau_k(d-n)}{n} \tag{5}$$

$$+\sum_{d|4q}\frac{\phi(d)}{d}\sum_{1\le n_1\le N}{}'\sum_{l=\left[\frac{n_1}{d}\right]+2}^{\left[\frac{N+n_1}{d}\right]}\frac{n_1^{\epsilon}(ld-n_1)^{\epsilon}}{ln_1-\frac{n_1^2}{d}}$$

$$\ll q^{\epsilon}+\sum_{d|4q}\frac{\phi(d)}{d}\sum_{n_1=1}^{N}\sum_{l=1}^{N}\frac{n_1^{\epsilon}l^{\epsilon}}{ln_1}$$

$$\ll q^{\epsilon}.$$

Then from (3), (4) and (5), we have

$$M_1=\frac{J(q)}{2}\sum_{\substack{n=1\\ (n,2q)=1}}^{\infty}\frac{\tau_k^2(n)}{n^2}+O(q^{\epsilon}). \tag{6}$$

(ii) From Lemma 4 of [12], we have the estimate

$$\sum_{\chi\ne\chi_0}|A(y,\chi)|^2\ll y^{2-4/2^k+\epsilon}\phi^2(q),$$

where χ_0 denotes the principal character modulo q. Then from the Cauchy inequality we can easily get

$$\sum_{\chi(-1)=-1}|A(y,\chi)|\ll\sum_{\chi\ne\chi_0}|A(y,\chi)|\ll y^{1-2/2^k+\epsilon}q^{\frac{3}{2}}.$$

Using this estimate we have

$$M_2=\sum_{\chi(-1)=-1}{}^{*}\left(\sum_{1\le n_1\le N}\frac{\bar{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1}\right)\left(\int_N^{\infty}\frac{A(y,\chi\chi_4)}{y^2}dy\right)$$

$$\begin{aligned} &\ll \sum_{1\le n_1\le N} n_1^{\epsilon-1}\int_N^\infty \frac{1}{y^2}\left(\sum_{\chi\chi_4(-1)=-1}|A(y,\chi\chi_4)|\right)dy \\ &\ll N^\epsilon \int_N^\infty \frac{q^{\frac{3}{2}}y^{1-2/2^k+\epsilon_1}}{y^2}dy \ll \frac{q^{\frac{3}{2}}}{N^{\frac{2}{2^k}-\epsilon}}. \end{aligned} \tag{7}$$

(iii) Similar to (ii), we can also get

$$M_3 \ll \frac{q^{\frac{3}{2}}}{N^{\frac{2}{2^k}-\epsilon}}. \tag{8}$$

(iv) By the same argument in (ii), and noting that the absolute convergent properties of the integral we can write

$$\begin{aligned} &M_4 \\ &= \sideset{}{^*}\sum_{\chi(-1)=-1}\left(\int_N^\infty \frac{A(y,\bar\chi\chi_4)}{y^2}dy\right)\left(\int_N^\infty \frac{A(y,\chi\chi_4)}{y^2}dy\right) \\ &\le \int_N^\infty\int_N^\infty \frac{1}{y^2z^2}\sideset{}{^*}\sum_{\chi(-1)=-1}|A(y,\bar\chi\chi_4)||A(z,\chi\chi_4)|dydz \\ &\ll \int_N^\infty \frac{1}{y^2}\int_N^\infty\frac{1}{z^2}\left(\sum_{\chi\chi_4\ne\chi_0}|A(y,\bar\chi\chi_4)|^2\right)^{\frac{1}{2}}\left(\sum_{\chi\chi_4\ne\chi_0}|A(z,\chi\chi_4)|^2\right)^{\frac{1}{2}}dydz \\ &\ll \left(\int_N^\infty \frac{1}{y^2}\left(\sum_{\chi\chi_4\ne\chi_0}|A(y,\chi\chi_4)|^2\right)^{\frac{1}{2}}dy\right)^2 \\ &\ll \left(\int_N^\infty \frac{\phi(q)}{y^{1+2/2^k-\epsilon}}dy\right)^2 \ll \frac{\phi^2(q)}{N^{4/2^k-\epsilon}}. \end{aligned} \tag{9}$$

Now taking $N = q^{2^k}$ and $\epsilon < \frac{2}{2^k}$, combining (1) and (6)-(9) we obtain the asymptotic formula

$$\sideset{}{^*}\sum_{\chi(-1)=1}|L(1,\bar\chi\chi_4)|^{2k} = \frac{J(q)}{2}\sum_{\substack{n=1\\(n,2q)=1}}^\infty \frac{\tau_k^2(n)}{n^2} + O(q^\epsilon).$$

This proves Lemma 4.

Lemma 5. *Let q be any odd integer with $q > 2$, χ be the Dirichlet character modulo q and χ_8 be the real primitive character modulo 8 with $\chi(-1) = -1$. Then* *we have the asymptotic formula:*

$$\sum_{\chi(-1)=1}^{*} |L(1, \bar{\chi}\chi_8)|^{2k} = J(q) \sum_{\substack{n=1 \\ (n,2q)=1}}^{\infty} \frac{\tau_k^2(n)}{n^2} + O(q^{\epsilon}).$$

Proof. By using the same method as proving Lemma 4, we can also get this Lemma. ■

Lemma 6. *Let q be any odd integer with $q > 2$, χ be the Dirichlet character modulo q, χ_4 be the primitive character modulo 4, χ_8 be the real primitive character modulo 8 with $\chi(-1) = -1$ and $k \geq 2$ be any positive integer. Then for any fixed nonnegative integer $a, b \leq k$ such that if $a = 0$ then $b \neq 0$ and if $a = k$, then $b \neq k$, we have*

$$\sum_{\chi(-1)=1}^{*} \bar{\chi}^a(2)\chi^b(2)L^a(1, \bar{\chi}\chi_4)L^b(1, \chi\chi_4)L^{k-a}(1, \bar{\chi}\chi_8)L^{k-b}(1, \chi\chi_8)$$
$$= \begin{cases} \frac{1}{2}C(a)J(q) + O(q^{\epsilon}), & \text{if } a = b; \\ O(q^{\epsilon}), & \text{if } a \neq b, \end{cases}$$

where $C(a) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{r_a^2(n)}{n^2}$ *is a constant and* $r_a(n) = \sum_{t|n} \tau_a(t)\chi_4(t)\tau_{k-a}\left(\frac{n}{t}\right)\chi_8\left(\frac{n}{t}\right)$.

Proof. During the procedure of proving this lemma, we put

$$A(y, \chi) = \sum_{N<n\leq y} \chi(n)r_a(n) \quad \text{and} \quad B(y, \chi) = \sum_{N<n\leq y} \chi(n)r_b(n),$$

where N is a parameter with $q \leq N < q^{2^k+1}$. Then from Abel's identity we have

$$L^a(1, \bar{\chi}\chi_4)L^{k-a}(1, \bar{\chi}\chi_8)$$
$$= \sum_{n_1=1}^{\infty} \frac{\bar{\chi}\chi_4(n_1)\tau_a(n_1)}{n_1} \sum_{n_2=1}^{\infty} \frac{\bar{\chi}\chi_8(n_2)\tau_{k-a}(n_2)}{n_2}$$
$$= \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)r_a(n)}{n} = \sum_{1\leq n\leq N} \frac{\bar{\chi}(n)r_a(n)}{n} + \int_N^{\infty} \frac{A(y, \bar{\chi})}{y^2} dy.$$

Similarly, we can write

$$L^b(1, \chi\chi_4)L^{k-b}(1, \chi\chi_8) = \sum_{1\leq n\leq N} \frac{\chi(n)r_b(n)}{n} + \int_N^{\infty} \frac{B(y, \chi)}{y^2} dy.$$

Hence, we have

$$
\begin{aligned}
&\sum_{\chi(-1)=1}^{*} \bar{\chi}^a(2)\chi^b(2)L^a(1,\bar{\chi}\chi_4)L^b(1,\chi\chi_4)L^{k-a}(1,\bar{\chi}\chi_8)L^{k-b}(1,\chi\chi_8)\\
&= \sum_{\chi(-1)=1}^{*} \bar{\chi}^a(2)\chi^b(2)\left(\sum_{1\le n_1\le N}\frac{\bar{\chi}(n_1)r_a(n_1)}{n_1}+\int_N^{\infty}\frac{A(y,\bar{\chi})}{y^2}dy\right)\\
&\quad\times\left(\sum_{1\le n_2\le N}\frac{\chi(n_2)r_b(n_2)}{n_2}+\int_N^{\infty}\frac{B(y,\chi)}{y^2}dy\right).
\end{aligned}\tag{10}
$$

From the proof of Lemma 4, we know that only the term which does not contain the infinite integral will make contribution to the main term of (10). That is,

$$
\begin{aligned}
&\sum_{\chi(-1)=1}^{*} \bar{\chi}^a(2)\chi^b(2)L^a(1,\bar{\chi}\chi_4)L^b(1,\chi\chi_4)L^{k-a}(1,\bar{\chi}\chi_8)L^{k-b}(1,\chi\chi_8)\\
&= \sum_{\chi(-1)=1}^{*} \bar{\chi}^a(2)\chi^b(2)\sum_{1\le n_1\le N}\sum_{1\le n_2\le N}\frac{\chi(\bar{n_1}n_2)r_a(n_1)r_b(n_2)}{n_1n_2}+O(q^{\epsilon}).
\end{aligned}\tag{11}
$$

Then from Lemma 3, we can write

$$
\begin{aligned}
&\sum_{\chi(-1)=1}^{*} \bar{\chi}^a(2)\chi^b(2)\sum_{1\le n_1\le N}\sum_{1\le n_2\le N}\frac{\chi(\bar{n_1}n_2)r_a(n_1)r_b(n_2)}{n_1n_2}\\
&= \frac{1}{2}\sum_{1\le n_1\le N}{}^{'}\sum_{1\le n_2\le N}{}^{'}\frac{r_a(n_1)r_b(n_2)}{n_1n_2}\sum_{d\mid(q,\bar{2}^a2^b\bar{n_1}n_2-1)}\mu(d)\phi\left(\frac{q}{d}\right)\\
&\quad+\tfrac{1}{2}\sum{}^{'}_{1\le n_1\le N}\sum{}^{'}_{1\le n_2\le N}\tfrac{r_a(n_1)r_b(n_2)}{n_1n_2}\sum_{d\mid(q,\bar{2}^a2^b\bar{n_1}n_2+1)}\mu(d)\phi\left(\tfrac{q}{d}\right)\\
&= \frac{1}{2}\sum_{d\mid q}\mu(d)\phi\left(\frac{q}{d}\right)\sum_{\substack{1\le n_1\le N\\ \bar{2}^a2^b\bar{n_1}n_2\equiv 1(\bmod d)}}{}^{'}\sum_{1\le n_2\le N}{}^{'}\frac{r_a(n_1)r_b(n_2)}{n_1n_2}\\
&\quad+\frac{1}{2}\sum_{d\mid q}\mu(d)\phi\left(\frac{q}{d}\right)\sum_{\substack{1\le n_1\le N\\ \bar{2}^a2^b\bar{n_1}n_2\equiv -1(\bmod d)}}{}^{'}\sum_{1\le n_2\le N}{}^{'}\frac{r_a(n_1)r_b(n_2)}{n_1n_2},
\end{aligned}\tag{12}
$$

where $\sum_{1\le n\le N}{}^{'}$ denotes the summation over all integers n with $(n,q)=1$ and $1\le n\le N$. Now we study the first summation in (12).

(I) If $b\ge a$, we split the sum over n_1 or n_2 into following four cases:

(i) $d\le n_1\le N$ and $\frac{d}{2^{b-a}}\le n_2\le N$;

(ii) $d \le n_1 \le N$ and $1 \le n_2 \le \frac{d}{2^{b-a}} - 1$;
(iii) $1 \le n_1 \le d-1$ and $\frac{d}{2^{b-a}} \le n_2 \le N$;
(iv) $1 \le n_1 \le d-1$ and $1 \le n_2 \le \frac{d}{2^{b-a}} - 1$.

(II) If $b < a$, we split the sum over n_1 or n_2 into following cases:
(i) $d \le n_2 \le N$ and $\frac{d}{2^{a-b}} \le n_1 \le N$;
(ii) $d \le n_2 \le N$ and $1 \le n_1 \le \frac{d}{2^{a-b}} - 1$;
(iii) $1 \le n_2 \le d-1$ and $\frac{d}{2^{a-b}} \le n_1 \le N$;
(iv) $1 \le n_2 \le d-1$ and $1 \le n_1 \le \frac{d}{2^{a-b}} - 1$.

By using the same method as proving Lemma 4, we know that the main term of (12) will be

$$\begin{aligned}
&\frac{1}{2}\sum_{d|q}\mu(d)\phi\left(\frac{q}{d}\right)\sideset{}{'}\sum_{\substack{1\le n_1\le d-1\\ \bar{2}^a 2^b \bar{n}_1 n_2\equiv 1(\bmod d)}}\sideset{}{'}\sum_{1\le n_2\le \frac{d}{2^{b-a}}-1}\frac{r_a(n_1)r_b(n_2)}{n_1 n_2}\\
&=\frac{1}{2}\sum_{d|q}\mu(d)\phi\left(\frac{q}{d}\right)\sum_{\substack{n_2=1\\(n_2,q)=1}}^{\infty}\frac{r_a(2^{b-a}n_2)r_b(n_2)}{2^{b-a}n_2^2}+O(q^{\epsilon}),\quad \text{if}\quad b\ge a
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{2}\sum_{d|q}\mu(d)\phi\left(\frac{q}{d}\right)\sideset{}{'}\sum_{\substack{1\le n_1\le \frac{d}{2^{a-b}}-1\\ \bar{2}^a 2^b \bar{n}_1 n_2\equiv 1(\bmod d)}}\sideset{}{'}\sum_{1\le n_2\le d-1}\frac{r_a(n_1)r_b(n_2)}{n_1 n_2}\\
&=\frac{1}{2}\sum_{d|q}\mu(d)\phi\left(\frac{q}{d}\right)\sum_{\substack{n_1=1\\(n_1,q)=1}}^{\infty}\frac{r_a(n_1)r_b(2^{a-b}n_2)}{2^{a-b}n_1^2}+O(q^{\epsilon}),\quad \text{if}\quad b< a.
\end{aligned}\tag{13}$$

In fact, noting that $r_a(n_1) \ll n_1^{\epsilon}$ we have the estimates, for case I) $b \ge a$,

$$\begin{aligned}
&\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{d\le n_1\le N\\ 2^{b-a}n_2\equiv n_1(\bmod d)}}\sideset{}{'}\sum_{\frac{d}{2^{b-a}}\le n_2\le N}\frac{r_a(n_1)r_b(n_2)}{n_1 n_2}\\
&\ll\sum_{d|q}\phi(d)\sum_{1\le r_1\le \frac{N}{d}}\sum_{1\le r_2\le \frac{2^{b-a}N}{d}}\sideset{}{'}\sum_{\substack{l_1=1\\ l_2\equiv l_1(\bmod d)}}^{d-1}\sideset{}{'}\sum_{l_2=1}^{d-1}\frac{r_a(r_1d+l_1)r_b(r_2d+l_2)}{(r_1d+l_1)(r_2d+l_2)}\\
&\ll\sum_{d|q}\phi(d)\sum_{1\le r_1\le \frac{N}{d}}\sum_{1\le r_2\le \frac{2^{b-a}N}{d}}\sideset{}{'}\sum_{l_1=1}^{d-1}\frac{[(r_1d+l_1)(r_2d+l_1)]^{\epsilon}}{(r_1d+l_1)(r_2d+l_1)}
\end{aligned}$$

$$\ll \sum_{d|q} \frac{\phi(d)}{d} \sum_{1\le r_1\le \frac{N}{d}} \sum_{1\le r_2\le \frac{2^{b-a}N}{d}} \frac{[(r_1d+1)(r_2d+1)]^\epsilon}{r_1r_2}$$
$$\ll q^\epsilon,$$

$$\sum_{d|q} \mu\left(\frac{q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{d\le n_1\le N \\ 2^{b-a}n_2\equiv n_1(\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le \frac{d}{2^{b-a}}-1} \frac{r_a(n_1)r_b(n_2)}{n_1n_2}$$
$$\ll \sum_{d|q} \phi(d) \sum_{1\le r_1\le \frac{N}{d}} \sum_{1\le n_2\le \frac{d}{2^{b-a}}-1} (r_1n_2d)^{\epsilon-1}$$
$$\ll q^\epsilon,$$

$$\sum_{d|q} \mu\left(\frac{q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le d-1 \\ 2^{b-a}n_2\equiv n_1(\bmod d)}} \sideset{}{'}\sum_{\frac{d}{2^{b-a}}\le n_2\le N} \frac{r_a(n_1)r_b(n_2)}{n_1n_2}$$
$$\ll \sum_{d|q} \phi(d) \sum_{1\le n_1\le d-1} \sum_{1\le r_2\le \frac{2^{b-a}N}{d}} (n_1r_2d)^{\epsilon-1} \ll q^\epsilon$$

and
$$\frac{1}{2}\sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right) \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ \bar{2}^a2^b\bar{n}_1n_2\equiv -1(\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{r_a(n_1)r_b(n_2)}{n_1n_2} \ll q^\epsilon,$$

which can be deduced by same argument in (5). For case II) $b < a$, the same estimations can be obtained similarly.

Noting that $r_a(n) = r_b(n) = 0$ if $2|n$, we can conclude that

$$\frac{1}{2}\sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right) \sideset{}{'}\sum_{\substack{1\le n_1\le d-1 \\ \bar{2}^a2^b\bar{n}_1n_2\equiv 1(\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le \frac{d}{2^{b-a}}-1} \frac{r_a(n_1)r_b(n_2)}{n_1n_2} = 0, \quad \text{if} \quad b > a$$

or

$$\frac{1}{2}\sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right) \sideset{}{'}\sum_{\substack{1\le n_1\le \frac{d}{2^{a-b}}-1 \\ \bar{2}^a2^b\bar{n}_1n_2\equiv 1(\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le d-1} \frac{r_a(n_1)r_b(n_2)}{n_1n_2} = 0, \quad \text{if} \quad b < a. \tag{14}$$

Hence, from (10), (11), (12), (13) and (14) we have

$$\sideset{}{^*}\sum_{\chi(-1)=1} \bar{\chi}^a(2)\chi^b(2)L^a(1,\bar{\chi}\chi_4)L^b(1,\chi\chi_4)L^{k-a}(1,\bar{\chi}\chi_8)L^{k-b}(1,\chi\chi_8)$$
$$= \begin{cases} \frac{1}{2}J(q) \sum\limits_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{r_a^2(n)}{n^2} + O(q^\epsilon), & \text{if } a = b; \\ O(q^\epsilon), & \text{if } a \ne b. \end{cases}$$

Since $r_a(n) \ll n^{\epsilon}$, $\sum\limits_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{r_a^2(n)}{n^2}$ is convergent. This proves Lemma 6.

Lemma 7. *Let* $r(n) = \sum\limits_{t|n} \chi_4(t)\chi_8\left(\frac{n}{t}\right)$*, then we have*

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{r^2(n)}{n^2} = \frac{\pi^4}{96}\prod_{p|q}\left(1-\frac{1}{p^4}\right) \prod_{\substack{p\nmid q\\ p\equiv 1,7(\mod 8)}} \left(1+\frac{2}{p^2-1}\right)^2.$$

Proof. Noting that $\chi_8(n) = \chi_4(n)\chi_8'(n)$, where $\chi_8'(n)$ is another real primitive character modulo 8 such that $\chi_8'(-1) = 1$, we can write

$$r(n) = \sum_{t|n} \chi_4(t)\chi_4\left(\frac{n}{t}\right)\chi_8'\left(\frac{n}{t}\right) = \chi_4(n)\sum_{t|n}\chi_8'(t).$$

Since

$$\chi_8'(n) = \begin{cases} 1 & \text{if } n \equiv 1,7(\mod 8); \\ -1 & \text{if } n \equiv 3,5(\mod 8), \end{cases}$$

from Euler product formula we can write

$$\begin{aligned}
&\sum_{\substack{n=1\\(n,2q)=1}}^{\infty} \frac{r^2(n)}{n^2} \\
&= \prod_{p\nmid 2q}\left(1+\frac{r^2(p)}{p^2}+\frac{r^2(p^2)}{p^4}+\frac{r^2(p^3)}{p^6}+\cdots\right) \\
&= \prod_{\substack{p\nmid q\\ p\equiv 3,5(\mod 8)}} \left(1+\frac{1}{p^4}+\frac{1}{p^8}+\cdots+\frac{1}{p^{4i}}+\cdots\right) \\
&\times \prod_{\substack{p\nmid q\\ p\equiv 1,7(\mod 8)}} \left(1+\frac{4}{p^2}+\frac{9}{p^4}+\cdots+\frac{(i+1)^2}{p^{2i}}+\cdots\right).
\end{aligned} \tag{15}$$

Let

$$S = 1+\frac{4}{p^2}+\frac{9}{p^4}+\cdots+\frac{(i+1)^2}{p^{2i}}+\cdots.$$

It is clear that

$$S\left(1-\frac{1}{p^2}\right)^2 = 1+\frac{2}{p^2}\left(\frac{1}{1-\frac{1}{p^2}}\right).$$

That is,

$$S = \frac{1+\frac{1}{p^2}}{\left(1-\frac{1}{p^2}\right)^3}.$$

So from (15), we have

$$\begin{aligned}
\sum_{n=1\,(n,q)=1}^{\infty} \frac{r^2(n)}{n^2} &= \prod_{\substack{p\nmid q\\ p\equiv 3,5(\bmod 8)}} \frac{1}{1-\frac{1}{p^4}} \prod_{\substack{p\nmid q\\ p\equiv 1,7(\bmod 8)}} \frac{1+\frac{1}{p^2}}{(1-\frac{1}{p^2})^3} \\
&= \prod_{p\nmid 2q} \frac{1}{1-\frac{1}{p^4}} \prod_{\substack{p\nmid q\\ p\equiv 1,7(\bmod 8)}} \frac{\left(1+\frac{1}{p^2}\right)^2}{(1-\frac{1}{p^2})^2} \\
&= \zeta(4)\prod_{p|2q}\left(1-\frac{1}{p^4}\right) \prod_{\substack{p\nmid q\\ p\equiv 1,7(\bmod 8)}} \left(1+\frac{2}{p^2-1}\right)^2 \\
&= \frac{\pi^4}{96}\prod_{p|q}\left(1-\frac{1}{p^4}\right) \prod_{\substack{p\nmid q\\ p\equiv 1,7(\bmod 8)}} \left(1+\frac{2}{p^2-1}\right)^2,
\end{aligned}$$

where we have used the fact $\zeta(4)=\frac{\pi^4}{90}$. This proves Lemma 7. ■

3. Proofs of the Theorems

In this section, we will complete the proofs of the theorems.

Proof of Theorem 1. Note that $|\tau(\chi)| = \sqrt{q}$ if χ is a primitive character modulo q. So from Lemma 2, we can write

$$\begin{aligned}
&\sideset{}{^*}\sum_{\chi(-1)=1} \left|\sum_{a<\frac{q}{8}} \chi(a)\right|^{2k} \\
&= \frac{q^k}{4^k\pi^{2k}} \sideset{}{^*}\sum_{\chi(-1)=1} \left|\bar{\chi}(2)L(1,\bar{\chi}\chi_4)+\sqrt{2}L(1,\bar{\chi}\chi_8)\right|^{2k} \\
&= \frac{q^k}{4^k\pi^{2k}} \sideset{}{^*}\sum_{\chi(-1)=1} \left|\sum_{i=1}^{k}\binom{k}{i}\bar{\chi}^i(2)L^i(1,\bar{\chi}\chi_4)\sqrt{2}^{k-i}L^{k-i}(1,\bar{\chi}\chi_8)\right|^2 \\
&= \frac{q^k}{4^k\pi^{2k}} \sum_{i=0}^{k}\sum_{j=0}^{k} \sqrt{2}^{2k-i-j}\binom{k}{i}\binom{k}{j}
\end{aligned}$$

$$\times \sideset{}{^*}\sum_{\chi(-1)=1} \bar{\chi}^i(2)\chi^j(2)L^i(1,\bar{\chi}\chi_4)L^j(1,\chi\chi_4)L^{k-i}(1,\bar{\chi}\chi_8)L^{k-j}(1,\chi\chi_8)$$

$$\begin{aligned}
&= \frac{q^k}{4^k\pi^{2k}}\left(\sideset{}{^*}\sum_{\chi(-1)=1} |L(1,\bar{\chi}\chi_4)|^{2k} + 2^k \sideset{}{^*}\sum_{\chi(-1)=1} |L(1,\bar{\chi}\chi_8)|^{2k}\right.\\
&\quad + \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \sqrt{2}^{2k-i-j}\binom{k}{i}\binom{k}{i}\\
&\quad \times \sideset{}{^*}\sum_{\chi(-1)=1} \bar{\chi}^i(2)\chi^j(2)L^i(1,\bar{\chi}\chi_4)L^j(1,\chi\chi_4)L^{k-i}(1,\bar{\chi}\chi_8)L^{k-j}(1,\chi\chi_8)\\
&\quad + \sum_{j=1}^{k}\sqrt{2}^{2k-j}\binom{k}{j}\sideset{}{^*}\sum_{\chi(-1)=1} \chi^j(2)L^j(1,\chi\chi_4)L^k(1,\bar{\chi}\chi_8)L^{k-j}(1,\chi\chi_8)\\
&\quad + \sum_{j=0}^{k-1}\sqrt{2}^{k-j}\binom{k}{j}\sideset{}{^*}\sum_{\chi(-1)=1} \bar{\chi}^k(2)\chi^j(2)L^k(1,\bar{\chi}\chi_4)L^j(1,\chi\chi_4)L^{k-j}(1,\chi\chi_8)\\
&\quad + \sum_{i=1}^{k}\sqrt{2}^{2k-i}\binom{k}{i}\sideset{}{^*}\sum_{\chi(-1)=1} \bar{\chi}^i(2)L^i(1,\bar{\chi}\chi_4)L^k(1,\chi\chi_8)L^{k-i}(1,\bar{\chi}\chi_8)\\
&\quad \left. + \sum_{i=0}^{k-1}\sqrt{2}^{k-i}\binom{k}{i}\sideset{}{^*}\sum_{\chi(-1)=1} \bar{\chi}^i(2)\chi^k(2)L^i(1,\bar{\chi}\chi_4)L^k(1,\chi\chi_4)L^{k-i}(1,\bar{\chi}\chi_8)\right).
\end{aligned} \tag{16}$$

From Lemma 6 and (16), we can easily get

$$\sideset{}{^*}\sum_{\chi(-1)=1}\left|\sum_{a<\frac{q}{8}}\chi(a)\right|^{2k} = \frac{J(q)q^k}{2^{2k+1}\pi^{2k}}\sum_{i=0}^{k}2^{k-i}\binom{k}{i}^2 C_i + O\left(q^{k+\epsilon}\right),$$

where $C_i = \sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{r_i^2(n)}{n^2}$ is a constant related to i,

$$r_i(n) = \sum_{t|n}\tau_i(t)\chi_4(t)\tau_{k-i}\left(\frac{n}{t}\right)\chi_8\left(\frac{n}{t}\right)$$

for $1 < i < k$ and $r_0(n) = \sqrt{2}\tau_k(n), r_k(n) = \tau_k(n)$. This proves Theorem 1.

Proof of Theorem 2. Putting $k = 2$, combining Lemma 4, Lemma 5, Lemma 6 and Lemma 7, after some simple calculation we have

$$\begin{aligned}
&\sum_{\chi(-1)=1}^{*}\left|\sum_{a<\frac{q}{8}}\chi(a)\right|^4\\
&=\frac{q^2}{16\pi^4}\left(\sum_{\chi(-1)=1}^{*}|L(1,\bar{\chi}\chi_4)|^4+4\sum_{\chi(-1)=1}^{*}|L(1,\bar{\chi}\chi_8)|^4\right.\\
&\quad+8\sum_{\chi(-1)=1}^{*}L(1,\bar{\chi}\chi_4)L(1,\chi\chi_4)L(1,\bar{\chi}\chi_8)L(1,\chi\chi_8)\\
&\quad+2\sum_{\chi(-1)=1}^{*}\chi^2(2)L^2(1,\chi\chi_4)L^2(1,\bar{\chi}\chi_8)\\
&\quad+4\sqrt{2}\sum_{\chi(-1)=1}^{*}\chi(2)L(1,\chi\chi_4)L^2(1,\bar{\chi}\chi_8)L(1,\chi\chi_8)\\
&\quad+2\sum_{\chi(-1)=1}^{*}\chi^2(2)L^2(1,\chi\chi_4)L^2(1,\chi\chi_8)\\
&\quad+2\sqrt{2}\sum_{\chi(-1)=1}^{*}\bar{\chi}(2)L^2(1,\bar{\chi}\chi_4)L(1,\bar{\chi}\chi_4)L(1,\chi\chi_8)\\
&\quad+4\sqrt{2}\sum_{\chi(-1)=1}^{*}\bar{\chi}(2)L(1,\bar{\chi}\chi_4)L(1,\bar{\chi}\chi_8)L^2(1,\chi\chi_8)\\
&\quad+2\sqrt{2}\sum_{\chi(-1)=1}^{*}\chi(2)L(1,\bar{\chi}\chi_4)L^2(1,\chi\chi_4)L(1,\bar{\chi}\chi_8)\\
&\quad\left.+4\sum_{\chi(-1)=1}^{*}\bar{\chi}^2(2)L^2(1,\bar{\chi}\chi_4)L^2(1,\chi\chi_8)\right)\\
&=\frac{q^2J(q)}{384}\left[\prod_{p|q}\left(1-\frac{1}{p^4}\right)\prod_{\substack{p\nmid q\\ p\equiv 1,7(\bmod 8)}}\left(1+\frac{2}{p^2-1}\right)^2\right.\\
&\quad\left.+\frac{27}{32}\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)}\right]+O(q^{2+\epsilon}).
\end{aligned}$$

This proves Theorem 2.

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References

1. G. Pólya, *Über die Verteilung der quadratische Reste und Nichtreste*, Göttingen Nachrichten, 1918, pp. 21-29.
2. I. M. Vinogradov, On the diswtribution of residues and non-residues of powers, *Journal of the Physico-Mathematical society of Perm.*, **1** (1918), 94-96.
3. A. V. Sokolovskiĭ, On a theorem of Sárkozy, *Acta Arithmetica*, **41** (1982), 27-31.
4. D. A. Burgess, On a conjecture of Norton, *Acta Arithmetica*, **27** (1975), 265-267.
5. K. K. Norton, On character sums and power residues, *Trans. Amer. Math. Soc.*, **167** (1972), 203-226.
6. D. A. Burgess, Mean value of character sums, *Mathematika*, **33** (1986), 1-5.
7. D. A. Burgess, Mean value of character sums II, *Mathematika*, **33** (1987), 1-7.
8. H. L. Montgomery and R. C. Vaughan, Mean value of character sums, *Canadian Journal of Mathematics*, **31(3)** (1979), 476-487.
9. Xu zhefeng and Zhang Wenpeng, On the $2k$th power mean of the character sums over short intervals, *Acta Arithmetica*, **121**, (2006), 149-160.
10. Juan C. Peral, Character sums and explicit estimates for L-functions, *Contemporary Mathematics*, **189** (1995), 449-459.
11. Wenpeng Zhang, On a Cochrane sum and its hybrid mean value formula (II), *Journal of Mathematical Analysis and Applications*, **276** (2002), 446-457.
12. Zhang Wenpeng, Yi Yuan and He Xiali, On the $2k$-th power mean of Dirichlet L-functions with the weight of general Kloosterman sums, *Journal of Number Theory*, **84** (2000), 199-213.

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