TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 1, pp. 137-155, February 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON EXISTENCE AND APPROXIMATION OF SOLUTIONS OF ABSTRACT CAUCHY PROBLEM

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Abstract. Let A be the generator of a nondegenerate α-times integrated C-semigroup $T(\cdot)$ on a complex Banach space X for some $\alpha \geq 0$, $x \in X$ and $f \in L^1_{loc}([0,\infty),X) \cap C((0,\infty),X)$. We first show that the abstract Cauchy problem $ACP(A,Cf,Cx)$: $u'(t) = Au(t) + Cf(t)$ for $t > 0$ and $u(0) = Cx$ has a strong solution is equivalent to the function $v(\cdot) = T(\cdot)x + T * f(\cdot) \in C^\alpha([0,\infty),X)$ and $D^\alpha v(\cdot) \in C^1((0,\infty),X)$, and then use it to prove some new existence and approximation theorems concerning strong solutions of $ACP(A, Cy + j_{\alpha-1} * Cg, Cx)$ in $C^1([0,\infty),X)$ and mild solutions of $ACP(A, Cy + j_{\alpha-2} * Cg, Cx)$ (for $\alpha \geq 1$) in $C([0,\infty),X)$ when vectors x and y both satisfy some suitable regularity assumptions and $T(\cdot)$ is locally Lipschitz continuous.

1. Introduction

Let X be a complex Banach space with norm $\|\cdot\|$, and let $B(X)$ denote the family of all bounded linear operators from X into itself. For $\alpha > 0$ and $C \in B(X)$, a family $T(\cdot)(=\{T(t)|t \geq 0\})$ in $B(X)$ is called an α-times integrated C-semigroup on X if

$$\text{(1.1)} \quad \begin{array}{l} T(\cdot) \text{ is strongly continuous, that is,} \\ \text{for each } x \in X, T(\cdot)x : [0,\infty) \to X \text{ is continuous,} \end{array}$$

$$\text{(1.2)} \quad \begin{array}{l} T(\cdot)C = CT(\cdot), \text{ that is, } T(t)C = CT(t) \\ \text{on } X \text{ for each } t \geq 0 \text{ and} \end{array}$$

Received January 18, 2007, accepted July 4, 2007.
Communicated by Sen-Yen Shaw.
2000 *Mathematics Subject Classification*: 47D60, 47D62.
Key words and phrases: α-Times integrated C-semigroup, Generator, Abstract Cauchy problem.
Research supported in part by the National Science Council of R.O.C.

(1.3) $$T(t)T(s)x = \frac{1}{\Gamma(\alpha)}[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^{\alpha-1}T(r)Cxdr$$
for each $x \in X$ and $t, s \geq 0$(see [9]);

or called a (0-times integrated) C-semigroup on X if it satisfies (1.1), (1.2) and (1.4)

$$T(t)T(s)x = T(t+s)Cx \quad \text{for each } x \in X \text{ and } t, s \geq 0 \text{ (see [4,12,18,20])}.$$

Here $\Gamma(\cdot)$ denotes the Gamma function. Moreover, we say that $T(\cdot)$ is

(i) nondegenerate, if $x = 0$ whenever $T(t)x = 0$ for all $t \geq 0$. In this case, its (integrated) generator $A : D(A) \subset X \to X$ is a closed linear operator in X defined by $D(A) = \{x \in X|$ there exists a $y_x \in X$ such that $T(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx = \int_0^t T(r)y_x dr$ for $t \geq 0\}$ and $Ax = y_x$ for each $x \in D(A)$;

(ii) locally Lipschitz continuous if for each $t_0 > 0$ there exists a $K_{t_0} > 0$ such that

(1.5) $$\|T(t+h) - T(t)\| \leq K_{t_0}h \quad \text{for all } 0 \leq t, h \leq t+h \leq t_0;$$

(iii) exponentially bounded if there exist $M, \omega \geq 0$ such that

(1.6) $$\|T(t)\| \leq Me^{\omega t} \quad \text{for all } t \geq 0.$$

In this case, we write $T(\cdot) \in g(M, \omega)$;

(iv) exponentially Lipschitz continuous, if there exist $M, \omega \geq 0$ such that

(1.7) $$\|T(t+h) - T(t)\| \leq Me^{\omega(t+h)}h \quad \text{for all } t, h \geq 0.$$

In this case, we write $T(\cdot) \in \epsilon(M, \omega)$.

In general, a (0-times integrated) I_X-semigroup on X is also called a semigroup on X(see [1,5-6]) and an α-times integrated I_X-semigroup on X is also called an α-times integrated semigroup on X (see[1-3,7-8,13-19]). Here I_X denotes the identity operator on X.

In this paper we consider the following abstract Cauchy problem :

$$ACP(A, f, x) \quad \begin{cases} u'(t) = Au(t) + f(t) & \text{for } t > 0, \\ u(0) = x, \end{cases}$$

where $x \in X$ is given, $A : D(A) \subset X \to X$ is a closed linear operator in X with domain $D(A)$ and range $R(A)$ and f is an X-valued function defined on a subset of $\mathbb{R}$. The concept of α-times integrated C-semigroups has been extensively applied to discuss the existence of (strong, mild or weak) solutions of $ACP(A, f, x)$ when $C = I_X$ or $\alpha \in \mathbb{N} \cup \{0\}$(see [1,4-6,9-10,15-16,18-19]). Some equivalence conditions between the existence of an α-times integrated C-semigroup and the unique existence of (strong or weak) solutions of $ACP(A, f, x)$ are also discussed in [9,10]. As an application of Arendt [2,Proposition 5.1 and Theorem 5.2], Hilber [8] first presented some meaningful sufficient conditions for the existence of strong solutions of $ACP(A, Cy + j_{\alpha-1} * Cg, Cx)$ when $g \in L^1_{loc}([0, \infty), X)$ the set of all locally Bochner integrable functions from $[0, \infty)$ into X, vectors x and y both satisfy some suitable regularity assumptions and A generates an exponentially Lipschitz continuous $(\alpha + 1)$-times integrated semigroup on X. Here $j_\beta(t) = t^\beta/\Gamma(\beta + 1)$ for $t > 0$ and $\beta > -1$, and j_{-1} denotes the Dirac measure at 0. That is, $j_{-1}(t) = \{\begin{matrix} 1 \text{ for } t = 0 \\ 0 \text{ for } t \neq 0. \end{matrix}$ In 2000, together with solving Nicaise's problem ([17]) in essence, Xiao and Liang [19] extended considerably previous Laplace transform versions of the Trotter-Kato theorem and established some significant existence and approximation theorems of mild solutions of $ACP(A, Cy + j_{\alpha-2} * Cg, Cx)$ (for $\alpha \geq 1$). The purpose of this paper is to explore if the aforementional results in [8, Theorem 4.6] and in [19, Lemma 4.1] are still true when the exponential Lipschitz continuity of an α-times integrated semigroup is replaced by the local Lipschitz continuity, and also to investigate if some approximation theorems in [19] concerning mild solutions of $ACP(A, Cy+j_{\alpha-2}*Cg, Cx)$ (for $\alpha \geq 1$) are still true when the considered integrated semigroups are replaced by integrated C-semigroups (see Theorems 3.8 and 3.9 below).

In section 2, we first prove that $ACP(A, Cf, Cx)$ has a strong solution in $C([0, \infty), X)$ is equivalent to $v(\cdot) = T(\cdot)x + T * f(\cdot) \in C^\alpha([0, \infty), X)$ and $D^\alpha v \in C^1((0, \infty), X)$ when $x \in X$, $f \in L^1_{loc}([0, \infty), X) \cap C((0, \infty), X)$ and A generates an α-times integrated C-semigroup $T(\cdot)$ on X for some $\alpha \geq 0$. In this case, $u = D^\alpha v$ the αth order derivative of v on $[0, \infty)$(see Theorem 2.5 and Corollary 2.6 below). Then, assuming A generates a locally Lipschitz continuous α-times integrated C-semigroup on X, $g \in L^1_{loc}([0, \infty), X)$ and $x, y \in X$,we show $ACP(A, Cy+j_{\alpha-1}*Cg, Cx)$ has a unique strong solution in $C^1([0, \infty), X)$ when either $0 \leq \alpha < 1$; or $\alpha \geq 1$, $x \in D(A)$ with $Ax + y \in D(A^{k-1})$ and

$$A^{k-1}(Ax + y) \in \{ \begin{matrix} \overline{D(A)} \text{ for } \alpha \in \mathbb{N} \\ D(A) \text{ for } \alpha \notin \mathbb{N} \end{matrix}$$

(see Theorem 2.10 below), and $ACP(A, Cy + j_{\alpha-2} * Cg, Cx)$ has a unique mild solution in $C([0, \infty), X)$ when $x \in D(A)$ and either $1 \leq \alpha < 2$; or $\alpha \geq 2$ with

$Ax + y \in D(A^{k-2})$ and

$$A^{k-2}(Ax+y) \in \begin{cases} \overline{D(A)} \text{ for } \alpha \in \mathbb{N} \\ D(A) \text{ for } \alpha \notin \mathbb{N} \end{cases}$$

(see Theorem 2.11 below). Here $k = [\alpha]$ the Gauss integer of α. Applying these results we can also deduce some new approximation theorems(see Theorems 3.6 and 3.7 below) concerning strong solutions of $ACP(A, Cy + j_{\alpha-1} * Cg, Cx)$ and mild solutions of $ACP(A, Cy + j_{\alpha-2} * Cg, Cx)$(for $\alpha \geq 1$).

2. Existence Theorems

From now on we always write $[\alpha]$ to denote the largest integer that is less than or equal to the real number α and set $f * g(\cdot) = \int_0^{\cdot} f(\cdot - s)g(s)ds$ on $[0, t_0]$ for each $t_0 > 0$, $f \in L^1([0, t_0])$ the set of all complex Lebesgue integrable functions on $[0, t_0]$ and $g \in L^1([0, t_0], X)$ the set of all complex Bochner integrable functions from $[0, t_0]$ into X.

Definition 2.1. Let $\alpha > 0$, $k = [\alpha] + 1$ and $v : I \to X$ for some subinterval of $[0, \infty)$ containing $\{0\}$. We write $v \in C^{\alpha}(I, X)$ if $v = v(0) + j_{\alpha-k} * u$ on I for some $u \in C^{k-1}(I, X)$. In this case, we say that v is α-times continuously differentiable on I, the $(k-1)$th order derivative $u^{(k-1)}$ of u on I is called the αth order derivative of v on I and denoted by $D^{\alpha}v$(on I) or $D^{\alpha}v : I \to X$. Here $C^k(I, X)$ denotes the set of all k-times continuously differentiable functions from I into X and $C^0(I, X) = C(I, X)$ the set of all continuous functions from I into X.

Next we state some basic properties concerning nondegenerate α-times integrated C-semigroups, which have been obtained in [9] and are frequently applied in the following.

Proposition 2.2. *Let $\alpha \geq 0$, and A be the generator of a nondegenerate α-times integrated C-semigroup $T(\cdot)$ on X. Then*

(2.1) $$C \text{ is injective and } C^{-1}AC = A;$$

(2.2) $$T(t)x \in D(A) \text{ and } AT(t)x = T(t)Ax \quad \text{for all } x \in D(A) \text{ and } t \geq 0;$$

(2.3) $$\int_0^t T(r)xdr \in D(A) \text{ and } A\int_0^t T(r)xdr = T(t)x - j_{\alpha}(t)Cx$$

for all $x \in X$ and $t \geq 0$;

(2.4) $T(0) = C$ on X if $\alpha = 0$, and $T(0) = 0$ the zero operator on X if $\alpha > 0$.

Definition 2.3. For a closed linear operator $A : D(A) \subset X \to X$, $f : [0, \infty) \to X$ and $x \in X$, a function $u : [0, \infty) \to X$ is called a (strong) solution of $ACP(A, f, x)$, if $u \in C^1((0, \infty), X) \cap C([0, \infty), X) \cap C((0, \infty), [D(A)])$ and satisfies $ACP(A, f, x)$, where $[D(A)]$ denotes the Banach space $D(A)$ with the graph norm $|x|_A = \|x\| + \|Ax\|$.

The next lemma is a direct consequence of Definition 2.1, and so its proof is omitted.

Lemma 2.4. *Let $\alpha \geq 0$, $v \in C^\alpha(I, X)$ for some subinterval I of $[0, \infty)$ containing $\{0\}$ with $v(0) = 0$ and $k = [\alpha] + 1$. Then $j_{k-\alpha-1} * v \in C^k(I, X)$, $v \in C^{\alpha-i}(I, X)$ and $D^{\alpha-i}v = (j_{k-\alpha-1} * v)^{(k-i)}$ on I for each integer $0 \leq i \leq k-1$. In particular, for each $x \in X$, we have $j_\alpha(\cdot)x \in C^\alpha([0, \infty), X)$ and $D^{\alpha-i}j_\alpha(\cdot)x = D^{k-i}j_k(\cdot)x = j_i(\cdot)x$ on $[0, \infty)$ for each integer $0 \leq i \leq k-1$.*

Combining Proposition 2.2 with Lemma 2.4, we can deduce the following result which is a generalization of Hieber [8, Proposition 4.5] and Arendt [2, Proposition 5.1 and Theorem 5.2].

Theorem 2.5. *Let A be the generator of a nondegenerate α-times integrated C-semigroup $T(\cdot)$ on X for some $\alpha \geq 0$, $x \in X$ and $f \in L^1_{loc}([0, \infty), X) \cap C((0, \infty), X)$. Assume that $v(\cdot) = T(\cdot) + T * f(\cdot)$ on $[0, \infty)$. Then $ACP(A, f, x)$ has a strong solution u if and only if $v(t) \in R(C)$ for each $t \geq 0$, $C^{-1}v(\cdot) \in C^\alpha([0, \infty), X)$ and $D^\alpha C^{-1}v(\cdot) \in C^1((0, \infty), X)$. Here $T * f(t) = \int_0^t T(t-s)f(s)ds$ for $t \geq 0$. In this case, we have $u = D^\alpha C^{-1}v$. Moreover, $C^{-1}v \in C^{\alpha+1}([0, \infty), X)$ (resp., $C^{-1}v \in C^\alpha([0, \infty), [D(A)])$) if and only if $u \in C^1([0, \infty), X)$ (resp., $u \in C([0, \infty), [D(A)])$).*

Proof. We consider only the case $\alpha > 0$, for the case $\alpha = 0$ can be treated similarly. Now let u be a strong solution of $ACP(A, f, x)$. For each $0 < t < \infty$, we set $w(\cdot) = T(t - \cdot)u(\cdot)$ on $[0, t]$. Since $u \in C^1(0, \infty), X) \cap C((0, \infty), [D(A)])$ we have

$$\begin{aligned}
&\frac{d}{ds}T(t-s)u(s)|_{s=s_0} \\
&= -j_{\alpha-1}(t-s_0)Cu(s_0) - T(t-s_0)Au(s_0) + T(t-s_0)u'(s_0) \\
&= -j_{\alpha-1}(t-s_0)Cu(s_0) + T(t-s_0)f(s_0)
\end{aligned}$$

for each $0 < s_0 \leq t$, which together with the continuity of u on $[0,\infty)$ implies that

$$\begin{aligned} T(t)x &= w(0) - w(t) \\ &= -\lim_{s_0\to 0^+}\int_{s_0}^{t} w'(r)dr \\ &= \lim_{s_0\to 0^+}[\int_{s_0}^{t} j_{\alpha-1}(t-r)Cu(r)dr - \int_{s_0}^{t} T(t-r)f(r)dr] \\ &= j_{\alpha-1} * Cu(t) - T * f(t), \end{aligned}$$

so that $v(t) = Cj_{\alpha-1} * u(t) \in R(C)$. Hence $C^{-1}v(\cdot) = j_{\alpha-1} * u(\cdot) = j_{\alpha-k} * j_{k-2} * u \in C^{\alpha}([0,\infty), X)$ and $D^{\alpha}C^{-1}v = u \in C^1((0,\infty), X)$. Conversely, if $v(t) \in R(C)$ for all $t \geq 0$, $C^{-1}v(\cdot) \in C^{\alpha}([0,\infty), X)$ and $D^{\alpha}C^{-1}v(\cdot) \in C^1((0,\infty), X)$. Then from (2.3) and (2.4) with $\alpha > 0$, we have $v(0) = 0$, $j_0 * v(t) \in D(A)$ and

$$\begin{aligned} Aj_0 * v(t) &= T(t)x - j_{\alpha}(t)Cx + T * f(t) - j_{\alpha} * Cf(t) \\ &= v(t) - C[j_{\alpha}(t)x + j_{\alpha} * f(t)] \end{aligned}$$

for all $t \geq 0$, so that AC$j_0 * C^{-1}v(t) = Aj_0 * v(t) \in R(C)$ and

$$\begin{aligned} Aj_0 * C^{-1}v(t) &= C^{-1}ACj_0 * C^{-1}v(t) \\ &= C^{-1}v(t) - [j_{\alpha}(t)x + j_{\alpha} * f(t)] \end{aligned}$$

for all $t \geq 0$. Now if we set $k = [\alpha] + 1$, then from Lemma 2.4, we have $D^{\alpha-i}j_{\alpha}(t)x = j_i(t)x$ and $D^{\alpha+1}j_{\alpha}(t)x = 0$ for all integer $0 \leq i \leq k-1$ and all $t \geq 0$. Combining this, and the closedness of A with the fact that $j_{k-\alpha-1}*C^{-1}v(\cdot) \in C^{k+1}((0,\infty), X) \cap C^k([0,\infty), X)$, we have

$$Aj_0 * j_{k-\alpha-1} * C^{-1}v(t) = j_{k-\alpha-1} * C^{-1}v(t) - [j_k(t)x + j_k * f(t)] \text{ for all } t \geq 0,$$

$$AD^i(j_{k-\alpha-1}*C^{-1}v)(\cdot) = D^{i+1}(j_{k-\alpha-1} * C^{-1}v)(\cdot) - [j_{k-(i+1)}(\cdot)x + j_{k-(i+1)}*f(\cdot)]$$

on $[0,\infty)$ for each integer $0 \leq i \leq k-1$ and $AD^k j_{k-\alpha-1}*C^{-1}v(t) = D^{k+1}j_{k-\alpha-1}*C^{-1}v(t) - f(t)$ for each $t > 0$. By induction, we also have $D^i j_{k-\alpha-1}*C^{-1}v(0) = 0$ for each integer $0 \leq i \leq k-1$ and $D^k j_{k-\alpha-1} * C^{-1}v(0) - x = 0$. Consequently, $D^k j_{k-\alpha-1} * C^{-1}v(\cdot)$ is a strong solution of $ACP(A, f, x)$.

By slightly modifying the proof of Theorem 2.5 the next corollary is also attained. ∎

Corollary 2.6. *Under assumptions of Theorem 2.5, the following statements are equivalent:*

(i) *$ACP(A, Cf, Cx)$ has a strong solution u;*

(*ii*) $v \in C^{\alpha}([0,\infty),X)$ *and* $D^{\alpha}v \in C^1((0,\infty),X)$.

In this case, we have $u = D^{\alpha}v$. *Moreover,* $v \in C^{\alpha+1}([0,\infty),X)$ (*resp.,* $v \in C^{\alpha}([0,\infty),[D(A)])$) *if and only if* $u \in C^1([0,\infty),X)$ (*resp.,* $u \in C([0,\infty),[D(A)])$).

Proposition 2.7. *Let* $\alpha \geq 1$, *and* $T(\cdot)$ *be a locally Lipschitz continuous* α-*times integrated* C-*semigroup on* X *with generator* A. *Then* A_1 *the part of* A *in* $X_1(=\overline{D(A)})$ *generates an* $(\alpha-1)$-*times integrated* C_1-*semigroup* $T_1(\cdot)$ *on* X_1. *Here* C_1 *denotes the part of* C *in* X_1 *and* $T_1(t)x = \frac{d}{dt}T(t)x$ *for each* $x \in X_1$ *and* $t \geq 0$.

Proof. We first show that $A_1 = C_1^{-1}A_1C_1$. Indeed, if $x \in D(C_1^{-1}A_1C_1)$ is given, then $x \in D(C^{-1}AC) = D(A)$ and $Ax = C^{-1}ACx = C_1^{-1}A_1C_1x \in \overline{D(A)}$, so that $x \in D(A_1)$ and $A_1x = Ax = C_1^{-1}A_1C_1x \in \overline{D(A)}$. Hence $C_1^{-1}A_1C_1 \subset A_1$, which together with the inclusion $A_1 \subset C_1^{-1}A_1C_1$ implies that $C_1^{-1}A_1C_1 = A_1$. Similarly, we can show that $A_1 : D(A_1) \subset X_1 \to X_1$ is a closed linear operator in X. Since $\{x \in X | T(\cdot)$ is coninuously differentiable on $[0,\infty)\}$ is a closed subspace of X containing $D(A)$, we also have, for $x \in \overline{D(A)}$, $T(\cdot)x$ is continuously differentiable on $[0,\infty)$ and $\frac{d}{dt}T(t)x \in \overline{D(A)}$ for each $t \geq 0$. It follows from the closedness of A and (2.3) that we have

$$
\begin{aligned}
T_1(t)x - j_{\alpha-1}(t)C_1x &= \frac{d}{dt}T(t)x - j_{\alpha-1}(t)Cx \\
&= A_1\int_0^t T_1(r)xdr
\end{aligned}
$$

for each $x \in \overline{D(A)}$ and $t \geq 0$. The uniqueness of solutions of $ACP(A, j_{\alpha-1}Cx, 0)$ implies that $T(\cdot)x = \int_0^t T_1(r)xdr$ is the unique strong solution of $ACP(A_1, j_{\alpha-1}C_1x, 0)$ in $C^1([0,\infty),X_1) \cap C([0,\infty),[D(A_1)])$. We conclude from [9, Theorem 2.3] that $T_1(\cdot)$ is an $(\alpha-1)$-times integrated C_1-semigroup on X_1 with generator A_1. ■

Proposition 2.8. *Let* $\alpha \geq 1$, *and* $T(\cdot)$ *be a locally Lipschitz continuous* α-*times integrated* C-*semigroup on* X *with generator* A. *Then for each* $0 < \theta < 1$ *there exists an* $(\alpha-1+\theta)$-*times integrated* C-*semigroup* $\widetilde{T}(\cdot)$ *on* X *with generator* A *such that for each* $t_0 > 0$, *we have*

$$
\|\widetilde{T}(t+h) - \widetilde{T}(t)\| \leq K_{t_0}h^{\theta} \quad \textit{for all } 0 \leq t, h \leq t+h \leq t_0, \tag{2.5}
$$

where K_{t_0} *is given as in (1.5).*

Proof. Clearly, $-1 < \theta - 1 < 0$. It follows that $T * j_{\theta-1}(\cdot)x \in C^1([0,\infty),X)$ for each $x \in X$ and $T * j_{\theta-1}(\cdot)$ is an $(\alpha+\theta)$-times integrated C-semigroup on X

with generator A. Now let $\widetilde{T}(t) : X \to X$ be defined by

$$\widetilde{T}(t)x = \frac{d}{dt} T * j_{\theta-1}(t)x \tag{2.6}$$

for each $x \in X$. As in the proof of Proposition 2.7, we can show that $\widetilde{T}(\cdot)$ is an $(\alpha - 1 + \theta)$-times integrated C-semigroup on X with generator A and (2.5) is satisfied. ∎

Remark 2.9. If A generates a locally Lipschitz continuous α-times integrated C-semigroup $T(\cdot)$ on X for some $0 \le \alpha < 1$, then A also generates a C-semigroup $\widetilde{T}(\cdot)$ on X which is defined by

$$\widetilde{T}(t)x = \frac{d}{dt} T * j_{-\alpha}(t)x \quad \text{for each } x \in X \text{ and } t \ge 0. \tag{2.7}$$

The next result is an extension of [8, Theorem 4.6] in which $\alpha \ge 1$ and exponentially Lipschitz continuous integrated semigroups are replaced , respectively, by $\alpha \ge 0$ and locally Lipschitz continuous integrated C-semigroups here.

Theorem 2.10. *Let $\alpha \ge 0$, $k = [\alpha]$, and $T(\cdot)$ be a locally Lipschitz continuous α-times integrated C-semigroup on X. Assume that $x \in D(A)$, $y \in X$ and $g \in L^1_{loc}([0,\infty), X)$. Then $ACP(A, Cy + j_{\alpha-1} * Cg, Cx)$ has a unique strong solution u in $C^1([0,\infty), X)$ when $0 \le \alpha < 1$; or $\alpha \ge 1$, $z(= Ax + y) \in D(A^{k-1})$ and*

$$A^{k-1}z \in \begin{cases} \overline{D(A)} & \text{if } \alpha \in \mathbb{N} \\ D(A) & \text{if } \alpha \notin \mathbb{N}. \end{cases}$$

In this case, we have

$$u(\cdot) = T * g(\cdot) + Cx + \begin{cases} j_0 * \widetilde{T}(\cdot)z & \text{if } k = 0 \\ \widetilde{T}(\cdot)z & \text{if } k = 1 \\ \widetilde{T}(\cdot)A^{k-1}z + \sum_{i=0}^{k-2} j_{i+1}(\cdot)CA^i z & \text{if } k \ge 2 \end{cases} \tag{2.8}$$

on $[0,\infty)$ and

$$u'(t) = \frac{d}{dt} T * g(t) + \begin{cases} \widetilde{T}(t)z & \text{if } k = 0 \\ \frac{d}{dt}\widetilde{T}(t)z & \text{if } k = 1 \\ \frac{d}{dt}\widetilde{T}(t)A^{k-1}z + \sum_{i=0}^{k-2} j_i(t)CA^i z & \text{if } k \ge 2 \end{cases} \tag{2.9}$$

for each $t \geq 0$*, where* $\widetilde{T}(\cdot)$ *denotes the* k*-times integrated* C*-semigroup on* X *with generator* A *which is given as in Remark 2.9 when* $0 \leq \alpha < 1$ *or in Proposition 2.8 with* $\theta = k - (\alpha - 1)$ *when* $\alpha \geq 1$.

Proof. From (2.3) and Corollary 2.6, we need only to show that $\widetilde{T}(\cdot)x + \widetilde{T} * f(\cdot) = j_k(\cdot)Cx + j_0 * \widetilde{T}(\cdot)z + j_{\alpha-1} * \widetilde{T} * g(\cdot) \in C^{k+1}([0,\infty), X)$ if we set $f = y + j_{\alpha-1} * g$ on $[0,\infty)$. Indeed, if $0 \leq \alpha < 1$, then $k = 0$, so that $j_{\alpha-1} * \widetilde{T} * g = \widetilde{T} * g \in C^1([0,\infty), X)$. Hence $v \in C^{k+1}([0,\infty), X)$. Next if $\alpha \geq 1$, then $k \geq 1$, $\frac{d^k}{dt^k} j_{\alpha-1} * \widetilde{T} * g(t) = j_{\alpha-k-1} * \widetilde{T} * g(t) = T * g(t)$ and

$$\frac{d^k}{dt^k} j_0 * \widetilde{T}(t)z = \frac{d^{k-1}}{dt^{k-1}} \widetilde{T}(t)z = \begin{cases} \widetilde{T}(t)z & \text{if } k = 1 \\ \widetilde{T}(t)A^{k-1}z + \sum\limits_{i=0}^{k-2} j_{i+1}(t)CA^i z & \text{if } k \geq 2 \end{cases}$$

for all $t \geq 0$. Clearly, $\sum\limits_{i=0}^{k-2} j_{i+1}(\cdot)CA^i z$, $T*g(\cdot) \in C^1([0,\infty), X)$, and $\widetilde{T}(\cdot)A^{k-1}z \in C^1([0,\infty), X)$ when $z \in D(A^k)$. Now if $\alpha \in \mathbb{N}$, then $k = \alpha$ and $\widetilde{T} = T$ is locally Lipschitz continuous, so that $\widetilde{T}(\cdot)A^{k-1}z$ is continuously differentiable on $[0,\infty)$ when $A^{k-1}z \in \overline{D(A)}$. Hence $v \in C^{k+1}([0,\infty), X)$. We conclude from Corollary 2.6 that $v^{(k)} = u$ is the unique strong solution of $ACP(A, Cy + j_{\alpha-1} * Cg, Cx)$ in $C^1([0,\infty), X)$. ∎

Similarly we can obtain the next result which has been established by Xiao and Liang [19] when $\alpha \geq 1$, $C = I_X$ and $T(\cdot)$ is exponentially Lipschitz continuous.

Theorem 2.11. *Let* $\alpha \geq 0$, $k = [\alpha]$*, and* $T(\cdot)$ *be a locally Lipschitz continuous* α*-times integrated* C*-semigroup on* X *with generator* A*. Assume that* $x, y \in X$ *and* $g \in L^1_{loc}([0,\infty), X)$*. Then* $ACP(A, Cx + j_1Cy + j_{\alpha-1} * Cg, 0)$ *has a unique strong solution* u *in* $C^1([0,\infty), X)$ *when* $0 \leq \alpha < 1$*; or* $1 \leq \alpha < 2$, $x \in D(A)$ *with* $z = Ax + y$*; or* $\alpha \geq 2$, $x \in D(A)$, $z(= Ax + y) \in D(A^{k-2})$ *and*

$$A^{k-2}z \in \begin{cases} \overline{D(A)} & \textit{if } \alpha \in \mathbb{N} \\ D(A) & \textit{if } \alpha \notin \mathbb{N}. \end{cases}$$

In this case, we have

$$u(\cdot) = T * g(\cdot) + j_1(\cdot)Cx + \begin{cases} j_0 * \widetilde{T}(\cdot)z & \textit{if } k = 1 \\ \widetilde{T}(\cdot)z & \textit{if } k = 2 \\ \widetilde{T}(\cdot)A^{k-2}z + \sum\limits_{i=0}^{k-3} j_{i+1}(\cdot)CA^i z & \textit{if } k \geq 3 \end{cases} \tag{2.10}$$

on $[0,\infty)$ *and*

$$(2.11)\quad u'(t)=\frac{d}{dt}T*g(t)+Cx+\begin{cases}\widetilde{T}(t)z & \text{if } k=1\\ \frac{d}{dt}\widetilde{T}(t)z & \text{if } k=2\\ \frac{d}{dt}\widetilde{T}(t)A^{k-2}z+\sum_{i=0}^{k-3}j_i(t)CA^iz & \text{if } k\geq 3\end{cases}$$

for each $t\geq 0$.

Remark 2.12. If $\alpha\geq 1$, and u is the unique strong solution of $ACP(A,Cx+j_1Cy+j_{\alpha-1}*Cg,0)$ in $C^1([0,\infty),X)$ given as in Theorem 2.11. Then u' is the unique mild solution of $ACP(A,Cy+j_{\alpha-2}*Cg,Cx)$, that is, u' is the unique function $w\in C([0,\infty),X)$ satisfying the integral equation $w=A(j_0*w)+Cx+j_0*[Cy+j_{\alpha-2}*Cg]$ on $[0,\infty)$.

Corollary 2.13. *Let* $\alpha\geq 0$, $k=[\alpha]$, *and* $T(\cdot)$ *be a locally Lipschitz continuous* α*-times integrated* C*-semigroup on* X *with generator* A. *Assume that* $x\in D(A)$, $y\in X$ *and* $g\in L^1_{loc}([0,\infty),X)$. *Then* $ACP(A,Cy+j_{\alpha-1}*Cg,Cx)$ *has a unique strong solution* u *in* $C^1([0,\infty),X)$ *when* $0\leq\alpha<1$*; or* $\alpha\geq 1$ *and* $z(=Ax+y)\in D(A^k)$.

Corollary 2.14. *Let* $\alpha\geq 0$, $k=[\alpha]$, *and* $T(\cdot)$ *be a locally Lipschitz continuous* α*-times integrated* C*-semigroup on* X *with generator* A. *Assume that* $x,y\in X$ *and* $g\in L^1_{loc}([0,\infty),X)$. *Then* $ACP(A,Cx+j_1Cy+j_{\alpha-1}*Cg,0)$ *has a unique strong solution* u *in* $C^1([0,\infty),X)$ *when* $0\leq\alpha<1$*; or* $\alpha\geq 1$, $x\in D(A)$ *and* $z(=Ax+y)\in D(A^{k-1})$.

Corollary 2.15. *Let* $T(\cdot)$ *be an* α*-times integrated* C*-semigroup on* X *with the densely defined generator* A *for some* $\alpha\in\mathbb{N}\cup\{0\}$ *and* $k=\alpha+1$. *Assume that* $x\in D(A)$, $y\in X$ *and* $g\in L^1_{loc}([0,\infty),X)$. *Then* $ACP(A,Cy+j_\alpha*Cg,Cx)$ *has a unique strong solution* u *in* $C^1([0,\infty),X)$ *when* $z(=Ax+y)\in D(A^{k-1})$. *In this case, we have*

$$(2.12)\quad u(\cdot)=j_0*T*g(\cdot)+Cx+\begin{cases}j_0*T(\cdot)z & \text{if } k=1\\ j_0*T(\cdot)A^{k-1}z+\sum_{i=0}^{k-2}j_{i+1}(\cdot)CA^iz & \text{if } k\geq 2\end{cases}$$

and

$$(2.13)\qquad u'(\cdot)=T*g(\cdot)+\begin{cases}T(\cdot)z & \text{if } k=1\\ T(\cdot)A^{k-1}z+\sum_{i=0}^{k-2}j_i(\cdot)CA^iz & \text{if } k\geq 2\end{cases}$$

on $[0, \infty)$.

Corollary 2.16. *Let* $T(\cdot)$ *be an* α*-times integrated* C*-semigroup on* X *with the densely defined generator* A *for some* $\alpha \in \mathbb{N} \cup \{0\}$ *and* $k = \alpha + 1$. *Assume that* $x \in D(A)$, $y \in X$ *and* $g \in L^1_{loc}([0, \infty), X)$. *Then* $ACP(A, Cx + j_1 Cy + j_\alpha * Cg, 0)$ *has a unique strong solution* u *in* $C^1([0, \infty), X)$ *when* $\alpha = 0$; *or* $\alpha \geq 1$ *and* $z(= Ax + y) \in D(A^{k-2})$. *In this case, we have*

$$(2.14)\quad u(\cdot) = j_0 * T * g(\cdot) + j_1(\cdot)Cx + \begin{cases} j_1 * T(\cdot)z & \text{if } k = 1 \\ j_0 * T(\cdot)z & \text{if } k = 2 \\ j_0 * T(\cdot)A^{k-2}z + \sum_{i=0}^{k-3} j_{i+1}(\cdot)CA^i z & \text{if } k \geq 3 \end{cases}$$

and

$$(2.15)\qquad u'(\cdot) = T * g(\cdot) + \begin{cases} j_0 * T(\cdot)z & \text{if } k = 1 \\ T(\cdot)z & \text{if } k = 2 \\ T(\cdot)A^{k-2}z + \sum_{i=0}^{k-3} j_i(\cdot)CA^i z & \text{if } k \geq 3 \end{cases}$$

on $[0, \infty)$.

3. Approximation Theorems

In this section we first extend the well known properties on convergence and approximation of integrated semigroups and resolvent sets due to Lizama [13], Xiao and Liang [19] to the context of integrated C-semigroups.

Proposition 3.1. *Let* $C \in B(X)$ *be an injection, and let* A *and* A_m *for* $m \in \mathbb{N}$, *be closed linear operators in* X. *Assume that* $(\lambda - A_m)^{-1}Cy \to (\lambda - A)^{-1}Cy$ *in* X *for each* $y \in X$ *and for some fixed* $\lambda \in \bigcap_{m \in \mathbb{N}} \rho_C(A_m) \cap \rho_C(A)$. *Then for each* $w \in (\lambda - A)^{-1}C(X)$ *there exists a* $w_m \in (\lambda - A_m)^{-1}C(X)$ *such that* $w_m \to w$ *and* $A_m w_m \to Aw$ *in* X. *Here* $\rho_C(A)$ *denotes the* C*-resolvent set of* A. *That is,* $\rho_C(A) = \{\lambda \in \mathbb{C} | \lambda - A$ *is injective and* $R(C) \subset R(\lambda - A)\}$.

Proof. Indeed, if $w \in (\lambda - A)^{-1}C(X)$ is given, then we set $w_m = (\lambda - A_m)^{-1}C(C^{-1}(\lambda - A)w)$ for $m \in \mathbb{N}$, so that $w_m \to (\lambda - A)^{-1}C(C^{-1}(\lambda - A)w) = w$ and $A_m w_m = -(\lambda - A)w + \lambda(\lambda - A_m)^{-1}C\,(C^{-1}(\lambda - A)w) \to -(\lambda - A)w + \lambda w = Aw$ in X. ∎

Combining [11, Theorem 2.4] with [19, Theorem 2.2], we can obtain the following Trotter-Kato type approximation theorem concerning integrated C-semigroups

Proposition 3.2. *Let $\alpha \geq 0$, $T(\cdot)$ and $T_m(\cdot)$ for $m \in \mathbb{N}$, be α-times integrated C-semigroups on X generated by A and A_m, respectively. Assume that $T(\cdot)$, $T_m(\cdot) \in g(M,\omega)$ for $m \in \mathbb{N}$. Then $T_m(\cdot)v \to T(\cdot)v$ uniformly on compact subsets of $[0,\infty)$ for each $v \in X$ if and only if $\{T_m(\cdot)w\}_{m\in\mathbb{N}}$ is equicontinuous at t and $(\lambda - A_m)^{-1}Cw \to (\lambda - A)^{-1}Cw$ in X for each $w \in X$, $t \geq 0$ and $\lambda > \omega$.*

Proposition 3.3. *Let $C \in B(X)$ be an injection, and let A and A_m for $m \in \mathbb{N}$, be closed linear operators in X. Assume that $\{(\lambda - A_m)^{-1}C | m \in \mathbb{N}\}$ is bounded in $B(X)$ for some $\lambda \in \bigcap_{m\in\mathbb{N}} \rho_C(A_m) \cap \rho_C(A)$ and D is a core of A such that for each $w \in D_\lambda (= D \cap (\lambda - A)^{-1}C(X))$, we have $w_m \to w$ and $A_m w_m \to Aw$ in X for some $w_m \in D(A_m)$. Then $(\lambda - A_m)^{-1}Cy \to (\lambda - A)^{-1}Cy$ in X for each $y \in \overline{R(C)}$.*

Proof. Indeed, if $w \in D_\lambda$ is given, then we set $y = (\lambda - A)w$ and $y_m = (\lambda - A_m)w_m$ for $m \in \mathbb{N}$, so that $y_m \to y$ and $(\lambda - A_m)^{-1}Cy_m (= Cw_m) \to (\lambda - A)^{-1}Cy (= Cw)$ in X. Hence

$$
\begin{aligned}
&\|(\lambda - A_m)^{-1}Cy - (\lambda - A)^{-1}Cy\| \\
&\leq \|(\lambda - A_m)^{-1}C(y - y_m)\| + \|(\lambda - A_m)^{-1}Cy_m - (\lambda - A)^{-1}Cy\| \\
&\leq \|(\lambda - A_m)^{-1}C\|\|y - y_m\| + \|(\lambda - A_m)^{-1}Cy_m - (\lambda - A)^{-1}Cy\| \\
&\to 0 \text{ as } m \to \infty.
\end{aligned}
\tag{3.1}
$$

Now if $y \in R(C)$ is given, then $y = Cx$ for some $x \in X$. By hypotheses, we have $z_n \to (\lambda - A)^{-1}Cx$ and $Az_n \to A(\lambda - A)^{-1}Cx$ in X for some sequence $\{z_n\}_{n=1}^{\infty}$ in D_λ, so that $(\lambda - A)z_n \to Cx$ in X. Hence $(\lambda - A)(D_\lambda)$ is dense in $R(C)$, which together with (3.1) and the boundedness of $\{(\lambda - A_m)^{-1}C | m \in \mathbb{N}\}$ in $B(X)$ implies that $(\lambda - A_m)^{-1}Cy \to (\lambda - A)^{-1}Cy$ in X for each $y \in \overline{R(C)}$. ∎

Proposition 3.4. *Let $\alpha \geq 0$, $T(\cdot)$ and $T_m(\cdot)$ for $m \in \mathbb{N}$, be α-times integrated C-semigroups on X generated by A and A_m, respectively. Assume that $T(\cdot)$, $T_m(\cdot) \in g(M,\omega)$ for $m \in \mathbb{N}$, and $(\lambda - A_m)^{-1}Cw \to (\lambda - A)^{-1}Cw$ in X for each $w \in X$ and $\lambda > \omega$. Then $T_m(\cdot)v \to T(\cdot)v$ uniformly on compact subsets of $[0,\infty)$ for each $\lambda > \omega$ and $v \in \overline{(\lambda - A)^{-1}C(X)}$.*

Proof. Clearly, $j_0 * T(\cdot)$, $j_0 * T_m(\cdot) \in \epsilon(M,\omega)$ for $m \in \mathbb{N}$. It follows from Proposition 3.2 that we have $j_0 * T_m(\cdot)x \to j_0 * T(\cdot)x$ uniformly on compact subsets of $[0, \infty)$ for each $x \in X$. Now if $\lambda > \omega$ is fixed, then for each $w \in X$ we set $z = (\lambda - A)^{-1}Cw$ and $z_m = (\lambda - A_m)^{-1}Cw$ for $m \in \mathbb{N}$, so that

$$\begin{aligned}
&T_m(t)z - T(t)z = T_m(t)(z - z_m) + T_m(t)z_m - T(t)z, \\
&T_m(t)z_m - T(t)z = j_\alpha(t)(Cz_m - Cz) + j_0 * T_m(t)A_m z_m - j_0 * T(t)Az, \\
&j_0 * T_m(t)A_m z_m - j_0 * T(t)Az \\
= &\lambda(j_0 * T_m(t)z_m - j_0 * T(t)z) - (j_0 * T_m(t)Cw - j_0 * T(t)Cw)
\end{aligned}$$

and

$$\begin{aligned}
&j_0 * T_m(t)z_m - j_0 * T(t)z \\
= &j_0 * T_m(t)(z_m - z) + (j_0 * T_m(t)z - j_0 * T(t)z)
\end{aligned}$$

for each $t \geq 0$ and $m \in \mathbb{N}$. Hence $T_m(\cdot)z \to T(\cdot)z$ uniformly on compact subsets of $[0, \infty)$, which together with the uniform boundedness of $\{T_m(\cdot)|m \in \mathbb{N}\}$ on compact subsets of $[0, \infty)$ implies that $T_m(\cdot)v \to T(\cdot)v$ uniformly on compact subsets of $[0, \infty)$ for each $v \in \overline{(\lambda - A)^{-1}C(X)}$. ∎

Definition 3.5. A sequence of α-times integrated C-semigroups $\{T_m(\cdot)\}_{m=1}^{\infty}$ on X is said to be uniformly locally Lipschitz continuous, if for each $t_0 > 0$ there exists a $K_{t_0} > 0$ such that

$$\|T_m(t + h) - T_m(t)\| \leq K_{t_0} h \tag{3.2}$$

for each $m \in \mathbb{N}$ and $0 \leq t, h \leq t + h \leq t_0$.

Theorem 3.6. *Let the hypotheses of Corollary 2.13 hold for $T(\cdot)$, A, g, x, y and $z(= Ax + y)$, and also for $T_m(\cdot)$, A_m, g_m, x_m, y_m and $z_m(= A_m x_m + y_m)$ in place of $T(\cdot)$, A, g, x, y and z, respectively. Assume that*

- (i) *$\{T_m(\cdot)\}_{m=1}^{\infty}$ is uniformly locally Lipschitz continuous and $\lim_{m\to\infty} T_m(\cdot)v = T(\cdot)v$ uniformly on compact subsets of $[0, \infty)$ for each $v \in X$;*
- (ii) *$x_m \to x$ in X and $A_m^i z_m \to A^i z$ in X for each integer $0 \leq i \leq k$;*
- (iii) *$g_m \to g$ in $L^1_{loc}([0, \infty), X)$. That is, $\|g_m - g\|_{L^1([0,t_0],X)} (= \int_0^{t_0} \|g_m(s) - g(s)\|ds) \to 0$ in $\mathbb{R}$ for each $t_0 > 0$.*

Then the strong solution u_m of $ACP(A_m, Cy_m + j_{\alpha-1} * Cg_m, Cx_m)$ converges to the strong solution u of $ACP(A, Cy + j_{\alpha-1} * Cg, Cx)$ in $C^1([0, \infty), X)$, that is, $u_m \to u$ and $u'_m \to u'$ uniformly on compact subsets of $[0, \infty)$.

Proof. Indeed, if $\widetilde{T}_m(\cdot)$ denotes the k-times integrated C-semigroup on X generated by A_m which is given as in either (2.6) or (2.7), then from (2.8) and

(2.9), we have

$$(3.3)\quad u_m(t)=T_m*g_m(t)+Cx_m+\begin{cases} j_0*\widetilde{T}_m(t)z_m & \text{if } k=0\\ \widetilde{T}_m(t)z_m & \text{if } k=1\\ \widetilde{T}_m(t)A_m^{k-1}z_m+\sum_{i=0}^{k-2} j_{i+1}(t)CA_m^i z_m & \text{if } k\geq 2\end{cases}$$

and

$$(3.4)\quad u_m'(t)=\frac{d}{dt}T_m*g_m(t)+\begin{cases}\widetilde{T}_m(t)z_m & \text{if } k=0\\ \widetilde{T}_m(t)A_m^k z_m+\sum_{i=0}^{k-1} j_i(t)CA_m^i z_m & \text{if } k\geq 1\end{cases}$$

for each $t\geq 0$ and $m\in\mathbb{N}$. We observe from (3.3), (3.4) and (i)-(iii) that we need only to be shown that $T_m*g_m\to T*g$, $(T_m*g_m)'\to(T*g)'$ and $\widetilde{T}_m(\cdot)A_m^i z_m\to\widetilde{T}(\cdot)A^i z$ uniformly on compact subsets of $[0,\infty)$ for each integer $0\leq i\leq k$, and shall first show that

$$(3.5)\qquad T_m*\phi\to T*\phi$$

uniformly on compact subsets of $[0,\infty)$ for each $\phi\in L^1_{loc}([0,\infty),X)$. Here $\widetilde{T}(\cdot)$ denotes the k-times integrated C-semigroup on X generated by A. Indeed, if $t_0>0$ is fixed, then for each $\phi\in C([0,t_0],X)$ we deduce from the uniform continuity of ϕ on $[0,t_0]$, the uniform boundedness of $\{\|T_m(\cdot)\|\}_{m=1}^\infty$ on $[0,t_0]$ and (i) that $T_m(t-\cdot)\phi(\cdot)\to T(t-\cdot)\phi(\cdot)$ uniformly on $[0,t]$ for each $0<t<t_0$, so that $T_m*\phi(t)\to T*\phi(t)$ in X for each $0\leq t\leq t_0$. The uniform Lipschitz continuity of $\{T_m(\cdot)\}_{m=1}^\infty$ on $[0,t_0]$ implies that $\{T_m*\phi(\cdot)\}_{m=1}^\infty$ is uniformly bounded and equicontinuous on $[0,t_0]$. It follows from the pointwise convergence of $\{T_m*\phi(\cdot)\}_{m=1}^\infty$ to $T*\phi(\cdot)$ on $[0,t_0]$ and Arzela-Ascoli's theorem that each subsequence of $\{T_m*\phi\}_{m=1}^\infty$ contains a subsequence which converges to $T*\phi$ uniformly on $[0,t_0]$. Hence $T_m*\phi\to T*\phi$ uniformly on $[0,t_0]$ for each $\phi\in C([0,t_0],X)$. Combining this, and the uniform boundedness of $\{\|T_m(\cdot)\|\}_{m=1}^\infty$ on $[0,t_0]$ with the denseness of $C([0,t_0],X)$ in $L^1([0,t_0],X)$, we have $T_m*\phi\to T*\phi$ uniformly on $[0,t_0]$ for each $\phi\in L^1([0,t_0],X)$. Consequently, $T_m*\phi\to T*\phi$ uniformly on compact subsets of $[0,\infty)$ for each $\phi\in L^1_{loc}([0,\infty),X)$. In particular,

$$T_m*g_m=T_m*(g_m-g)+T_m*g\to 0+T*g=T*g$$

uniformly on compact subsets of $[0,\infty)$. Next, we shall show that $(T_m*\phi)'(\cdot)\to(T*\phi)'(\cdot)$ uniformly on compact subsets of $[0,\infty)$ for each $\phi\in L^1_{loc}([0,\infty),X)$. Indeed, if $t_0>0$ is fixed, then from (3.5) and (i), we have

$$(T_m*\phi)'(\cdot)=T_m(\cdot)*\phi'(\cdot)+T_m(\cdot)\phi(0)\to T*\phi'(\cdot)+T(\cdot)\phi(0)=(T*\phi)'(\cdot)$$

uniformly on $[0,t_0]$ for each $\phi\in C^1([0,t_0],X)$. Combining this, and the denseness of $C^1([0,t_0],X)$ in $L^1([0,t_0],X)$ with the fact that

$$\|(T_m*\phi)'(t)\|\le K_{t_0}\int_0^t\|\phi(s)\|ds \tag{3.6}$$

for each $\phi\in L^1([0,t_0],X)$, $m\in\mathbb{N}$ and $0\le t\le t_0$, we have $(T_m*\phi)'\to(T*\phi)'$ uniformly on $[0,t_0]$ for each $\phi\in L^1([0,t_0],X)$, where K_{t_0} is given as in (3.2). Consequently, $(T_m*\phi)'\to(T*\phi)'$ uniformly on compact subsets of $[0,\infty)$ for each $\phi\in L^1_{loc}([0,\infty),X)$. In particular,

$$\widetilde{T}_m(\cdot)A^iz=(T_m*j_{k-\alpha}(\cdot)A^iz)'\to(T*j_{k-\alpha}(\cdot)A^iz)'=\widetilde{T}(\cdot)A^iz$$

uniformly on compact subsets of $[0,\infty)$ for each integer $0\le i\le k$. Applying (i)-(iii) and (3.6) again, we have

$$(T_m*g_m)'=(T_m*(g_m-g))'+(T_m*g)'\to 0+(T*g)'=(T*g)'$$

and

$$\begin{aligned}\widetilde{T}_m(\cdot)A^i_mz_m&=\widetilde{T}_m(\cdot)(A^i_mz_m-A^iz)+\widetilde{T}_m(\cdot)A^iz\\&=(T_m*j_{k-\alpha}(\cdot)(A^i_mz_m-A^iz))'+\widetilde{T}_m(\cdot)A^iz\to\widetilde{T}(\cdot)A^iz\end{aligned}$$

uniformly on compact subsets of $[0,\infty)$ for each integer $0\le i\le k$. Hence the proof of this theorem is complete.

By slightly modifying the proof of Theorem 3.6 the next result is also attained. ∎

Theorem 3.7. *Let the hypotheses of Corollary 2.14 hold for $T(\cdot)$, A, g, x,y and $z(=Ax+y)$, and also for $T_m(\cdot)$, A_m, g_m,x_m,y_m and $z_m(=A_mx_m+y_m)$ in place of $T(\cdot)$, A,g,x,y and z, respectively. Assume that*

- *(i) $\{T_m(\cdot)\}_{m=1}^\infty$ is uniformly locally Lipschitz continuous and $\lim_{m\to\infty}T_m(\cdot)v=T(\cdot)v$ uniformly on compact subsets of $[0,\infty)$ for each $v\in X$;*
- *(ii) $x_m\to x$ in X and either $y_m\to y$ in X if $0\le\alpha<1$; or $A^i_mz_m\to A^iz$ in X for each integer $0\le i\le k-1$ if $\alpha\ge1$;*
- *(iii) $g_m\to g$ in $L^1_{loc}([0,\infty),X)$.*

Then the strong solution u_m of $ACP(A_m,Cx_m+j_1Cy_m+j_{\alpha-1}*Cg_m,0)$ converges to the strong solution u of $ACP(A,Cx+j_1Cy+j_{\alpha-1}*Cg,0)$ in $C^1([0,\infty),X)$.

Theorem 3.8. *Let the hypotheses of Theorem 2.10 hold for $T(\cdot)$, A,g,x,y and $z(=Ax+y)$, and also for $T_m(\cdot)$, A_m,g_m,x_m,y_m and $z_m(=A_mx_m+y_m)$ in place of $T(\cdot)$, A,g,x,y and z, respectively. Assume that $\alpha\in\mathbb{N}$ and*

(i) $T_m(\cdot) \in \epsilon(M,\omega)$ *for all* $m \in \mathbb{N}$ *and* $\lim_{m\to\infty} T_m(\cdot)v = T(\cdot)v$ *uniformly on compact subsets of* $[0,\infty)$ *for each* $v \in X$*;*

(ii) $x_m \to x$ *and* $A_m^i z_m \to A^i z$ *in* X *for each integer* $0 \leq i \leq k-1$*;*

(iii) $A^{k-1}z \in \overline{(\lambda - A)^{-1}C(X)}$ *and* $(\lambda - A)^{-1}C(X) \subset \overline{D(A_m)}$ *for all* $m \in \mathbb{N}$ *and for some* $\lambda > \omega$*;*

(iv) $g_m \to g$ *in* $L^1_{loc}([0,\infty), X)$.

Then the conclusion of Theorem 3.6 holds.

Proof. Indeed, we observe from (2.8) and (2.9) that u_m is given as in (3.3) and

$$u'_m(t) = \frac{d}{dt}T_m * g_m(t) + \begin{cases} \frac{d}{dt}\widetilde{T}_m(t)z & \text{if } k=1 \\ \frac{d}{dt}\widetilde{T}_m(t)A_m^{k-1}z_m + \sum_{i=0}^{k-2} j_i(t)CA_m^i z_m & \text{if } k \geq 2 \end{cases}$$

for each $t \geq 0$ and $m \in \mathbb{N}$. Just like in the proof of Theorem 3.6, we need only to show that $(\widetilde{T}_m(\cdot)A_m^{k-1}z_m)' \to (\widetilde{T}(\cdot)A^{k-1}z)'$ uniformly on compact subsets of $[0,\infty)$. Since $k = \alpha$ we have $\widetilde{T}(\cdot) = T(\cdot)$ and $\widetilde{T}_m(\cdot) = T_m(\cdot)$ for $m \in \mathbb{N}$. Now if $w \in (\lambda - A)^{-1}C(X)$ is given, then from Proposition 3.1, we have $w_m \to w$ and $A_m w_m \to Aw$ in X for some $w_m \in D(A_m)$, which together with the uniform boundedness of $\{\|T_m(\cdot)\|\}_{m=1}^{\infty}$ on compact subsets of $[0,\infty)$ implies that $(T_m(\cdot)w_m)' = j_{\alpha-1}(\cdot)Cw_m + T_m(\cdot)A_m w_m \to j_{\alpha-1}(\cdot)Cw + T(\cdot)Aw = (T(\cdot)w)'$ uniformly on compact subsets of $[0,\infty)$. Combining this with the fact that

$$\|(T_m(\cdot)v)'\| \leq K_{t_0}\|v\| \text{ on } [0,t_0] \tag{3.7}$$

for each $m \in \mathbb{N}$, $v \in \overline{D(A_m)}$ and $t_0 > 0$, we have

$$(T_m(\cdot)w)' = (T_m(\cdot)(w - w_m))' + (T_m(\cdot)w_m)' \to (T(\cdot)w)'$$

uniformly on compact subsets of $[0,\infty)$, which together with (3.7) and the denseness of $(\lambda - A)^{-1}C(X)$ in $\overline{(\lambda - A)^{-1}C(X)}$ implies that $(T_m(\cdot)w)' \to (T(\cdot)w)'$ uniformly on compact subsets of $[0,\infty)$ for each $w \in \overline{(\lambda - A)^{-1}C(X)}$. Combining this, and (3.7) with the assumption that $A^{k-1}z \in \overline{(\lambda - A)^{-1}C(X)}$, we have

$$(T_m(\cdot)A_m^{k-1}z_m)' = (T_m(\cdot)(A_m^{k-1}z_m - A^{k-1}z))' + (T_m(\cdot)A^{k-1}z)' \to (T(\cdot)A^{k-1}z)'$$

uniformly on compact subsets of $[0,\infty)$. ■

Similarly the next theorem is also attained.

Theorem 3.9. *Let the hypotheses of Theorem 2.11 hold for* $T(\cdot)$*,* A*,* g, x, y *and* $z(= Ax + y)$*, and also for* $T_m(\cdot)$*,* A_m*,* g_m, x_m, y_m *and* $z_m(= A_m x_m + y_m)$ *in place of* $T(\cdot)$*,* A*,* g, x, y *and* z*, respectively. Assume that* $\alpha \in \mathbb{N} \setminus \{1\}$ *and* (*i*) $T_m(\cdot) \in \epsilon(M, \omega)$ *for* $m \in \mathbb{N}$ *and* $\lim_{m\to\infty} T_m(\cdot)v = T(\cdot)v$ *uniformly on compact subsets of* $[0, \infty)$ *for each* $v \in X$*;*

(*ii*) $x_m \to x$ *and* $A_m^i z_m \to A^i z$ *in* X *for each integer* $0 \leq i \leq k-2$*;*

(*iii*) $A^{k-2}z \in \overline{(\lambda - A)^{-1}C(X)}$ *and* $(\lambda - A)^{-1}C(X) \subset \overline{D(A_m)}$ *for all* $m \in \mathbb{N}$ *and for some* $\lambda > \omega$*;* $g_m \to g$ *in* $L^1_{loc}([0, \infty), X)$.

Then the conclusion of Theorem 3.7 holds.

Remark 3.10. The conclusion of Theorem 3.9 has been deduced by Xiao and Liang in [19] when $C = I_X$.

Corollary 3.11. *Let the hypotheses of Corollary 2.15 hold for* $T(\cdot)$*,* A*,* g, x, y *and* $z(= Ax + y)$*, and also for* $T_m(\cdot)$*,* A_m*,* g_m, x_m, y_m *and* $z_m(= A_m x_m + y_m)$ *in place of* $T(\cdot)$*,* A*,* g, x, y *and* z*, respectively. Assume that*

(*i*) $T(\cdot)$*,* $T_m(\cdot) \in g(M, \omega)$ *for* $m \in \mathbb{N}$*,* $\overline{R(C)} = X$ *and for each* $\lambda > \omega$ *there exists a core* D *of* A *such that for each* $w \in D_\lambda(= D \cap (\lambda - A)^{-1}C(X))$*, we have* $w_m \to w$ *and* $A_m w_m \to Aw$ *in* X *for some* $w_m \in D(A_m)$*;*

(*ii*) $x_m \to x$ *and* $A_m^i z_m \to A^i z$ *in* X *for each integer* $0 \leq i \leq k-1$*;*

(*iii*) $g_m \to g$ *in* $L^1_{loc}([0, \infty), X)$.

Then the strong solution u_m *of* $ACP(A, Cy_m + j_\alpha * Cg_m, Cx_m)$ *converges to the strong solution* u *of* $ACP(A, Cy + j_\alpha * Cg, Cx)$ *in* $C^1([0, \infty), X)$.

Proof. From the denseness of $D(A)$ in X, we have $\overline{R(C)} \subset \overline{(\lambda - A)^{-1}C(X)}$ for each $\lambda > \omega$. Combining this, and Proposition 3.3 with the assumption that $\overline{R(C)} = X$, we also have $(\lambda - A_m)^{-1}Cw \to (\lambda - A)^{-1}Cw$ in X for each $w \in X$ and $\lambda > \omega$. Applying Proposition 3.4, we have $T_m(\cdot)v \to T(\cdot)v$ uniformly on compact subsets of $[0, \infty)$ for each $v \in X$, which together with (ii)-(iii) and Theorem 3.8 implies that the conclusion of Theorem 3.6 holds. ∎

Similarly the next corollary is also attained.

Corollary 3.12. *Let the hypotheses of Corollary 2.16 hold for* $T(\cdot)$*,* A*,* g, x, y *and* $z(= Ax + y)$*, and also for* $T_m(\cdot)$ A_m*,* g_m, x_m, y_m *and* $z_m(= A_m x_m + y_m)$ *in place of* $T(\cdot)$*,* A*,* g, x, y *and* z*, respectively. Assume that* $\alpha \geq 1$ *and*

(*i*) $T(\cdot)$*,* $T_m(\cdot) \in g(M, \omega)$ *for* $m \in \mathbb{N}$*,* $\overline{R(C)} = X$ *and for each* $\lambda > \omega$ *there exists a core* D *of* A *such that for each* $w \in D_\lambda(= D \cap (\lambda - A)^{-1}C(X))$*, we have* $w_m \to w$ *and* $A_m w_m \to Aw$ *in* X *for some* $w_m \in D(A_m)$*;*

(*ii*) $x_m \to x$ *and* $A_m^i z_m \to A^i z$ *in* X *for each integer* $0 \leq i \leq k-2$*;*

(*iii*) $g_m \to g$ *in* $L^1_{loc}([0, \infty), X)$.

Then the strong solution u_m of $ACP(A_m, Cx_m + j_1Cy_m + j_\alpha * Cg_m, 0)$ converges to the strong solution u of $ACP(A, Cx + j_1Cy + j_\alpha * Cg, 0)$ in $C^1([0,\infty), X)$.

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