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A COMPARISON OF THE ORDER COMPONENTS IN FROBENIUS AND 2-FROBENIUS GROUPS WITH FINITE SIMPLE GROUPS

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Abstract. Let G be a finite group. Based on the Gruenberg-Kegel graph $\mathrm{GK}(G)$, the order of G can be divided into a product of coprime positive integers. These integers are called the order components of G and the set of order components is denoted by $\mathrm{OC}(G)$. In this article we prove that, if S is a non-Abelian finite simple group with a disconnected graph $\mathrm{GK}(S)$, with an exception of $U_4(2)$ and $U_5(2)$, and G is a finite group with $\mathrm{OC}(G) = \mathrm{OC}(S)$, then G is neither Frobenius nor 2-Frobenius. For a group S isomorphic to $U_4(2)$ or $U_5(2)$, we construct examples of 2-Frobenius groups G such that $\mathrm{OC}(S) = \mathrm{OC}(G)$. In particular, the simple groups $U_4(2)$ and $U_5(2)$ are not recognizable by their order components.

1. Introduction

Throughout this article, all groups are assumed to be finite and all simple groups are non-Abelian. The *spectrum* $\omega(G)$ of a group G is the set of element orders of G. The set $\omega(G)$ determines the *Gruenberg-Kegel graph* $\mathrm{GK}(G)$, or the *prime graph* of G, whose vertices are all prime divisors of the order of G, and two vertices p and q are adjacent if $pq \in \omega(G)$. We denote by $s(G)$ the number of connected components of $\mathrm{GK}(G)$ and by $\pi_i(G)$, $i = 1, 2, \ldots, s(G)$, the ith connected component of $\mathrm{GK}(G)$. For a group G of even order, we set $2 \in \pi_1(G)$.

For a natural number n, let $\pi(n)$ be the set of prime divisors of n. Now, we can write

$$|G| = \prod_{i=1}^{s(G)} n_i(G),$$

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where $n_i(G)$'s are positive integers with $\pi(n_i(G)) = \pi_i(G)$. These $n_i(G)$'s are called the *order components* of G. The set of order components of G will be denoted by $\mathrm{OC}(G)$.

Gruenberg and Kegel gave the following description for finite groups with disconnected prime graph.

Gruenberg-Kegel Theorem (See [12]). *If G is a finite group with a disconnected graph $\mathrm{GK}(G)$, then one of the following assertions holds:*

(a) *$s(G) = 2$ and G is a Frobenius group;*

(b) *$s(G) = 2$ and G is a 2-Frobenius group, i.e., $G = ABC$, where A and AB are the normal subgroups of G and AB and BC are Frobenius groups with kernels A and B and complements B and C, respectively.*

(c) *There exists a non-Abelian simple group P such that $P \leq \overline{G} = G/K \leq \mathrm{Aut}(P)$, where K is the maximal normal soluble subgroup of G; furthermore, K and $\overline{G}/P$ are $\pi_1(G)$-groups, the Gruenberg-Kegel graph $\mathrm{GK}(P)$ is disconnected and $s(P) > s(G)$.*

The recognition of finite groups through their order components was first introduced by G. Y. Chen in [3]. A group G is said to be *recognizable by its order components* if $H \cong G$ for every group H such that $\mathrm{OC}(H) = \mathrm{OC}(G)$. There are scattered results in the literature showing that certain groups are recognizable by their order components. For example, the following simple groups are recognizable by their order components: All sporadic simple groups [3], Suzuki-Ree groups [4], $\mathrm{PSL}(2, q)$ [5]. Evidently, a simple group S with one connected component is not recognizable by its order component, because $\mathrm{OC}(S) = \mathrm{OC}(\mathbb{Z}_{|S|}) = \{|S|\}$ but $S \ncong \mathbb{Z}_{|S|}$.

Let S be a simple group with a disconnected Gruenberg-Kegel graph $\mathrm{GK}(S)$, which is recognizable by its order components. Usually, the proof of this statement

$$\text{“ } \mathrm{OC}(G) = \mathrm{OC}(S) \Longrightarrow G \cong S \text{ ”}$$

can be divided into the following steps:

Step 1. G is neither Frobenius nor 2-Frobenius,

Step 2. The condition (c) of Gruenberg-Kegel Theorem holds for G, and

Step 3. $\overline{G} = P \cong S$, which implies that $G \cong S$.

According to three steps, one may pose a question: Let S be a simple group with $s(S) > 1$. Is there any Frobenius or 2-Frobenius group F such that $\mathrm{OC}(F) =$

$\mathrm{OC}(S)$? In this article we demonstrate that for all simple groups S with $s(S) > 1$, except $U_4(2)$ and $U_5(2)$, the answer to this question is negative. In fact, by using the classification of the finite simple groups with a disconnected Gruenberg-Kegel graph, we prove the following theorem.

Main Theorem. *Let S be a simple group with a disconnected Gruenberg-Kegel graph $\mathrm{GK}(S)$, except $U_4(2)$ and $U_5(2)$. If G is a finite group with $\mathrm{OC}(G) = \mathrm{OC}(S)$, then G is neither Frobenius nor 2-Frobenius.*

Concerning $U_4(2)$ and $U_5(2)$ we will show that there exist no Frobenius group G such that $\mathrm{OC}(G) = \mathrm{OC}(U_4(2))$ or $\mathrm{OC}(G) = \mathrm{OC}(U_5(2))$. But, we will construct 2-Frobenius groups F_1 and F_2 such that $\mathrm{OC}(F_1) = \mathrm{OC}(U_4(2))$ and $\mathrm{OC}(F_2) = \mathrm{OC}(U_5(2))$. In this way we can conclude that:

Corollary. *The simple groups $U_4(2)$ and $U_5(2)$ are not recognizable by their order components.*

Of course this corollary can specify a mistake made in [8]. In fact the simple group $U_5(2)$ is not recognizable by its order component and should be omitted from these simple groups.

Here, it should be mentioned that a similar research concerning finite simple groups with the same spectrum as a Frobenius group or a 2-Frobenius group has been done by M.R. Aleeva in [1].

We finally introduce some notations. Given a group G, we put $\pi(G) := \pi(|G|)$ the set of all prime divisors of the order of G. Also, $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of G for each $p \in \pi(G)$. We denote by $N \rtimes H$ a semidirect product of N by H. If m and n are natural numbers and p is a prime, the notation $p^m \| n$ means that $p^m | n$ and $p^{m+1} \nmid n$. By $\lfloor x \rfloor$ we denote the integer part of x, i.e., the greatest integer that is less than or equal to x. We denote by $r_{[x]}$ the largest prime not exceeding x. All further unexplained notations are standard and can be found in [6] and [7], for instance.

2. Preliminary Results

In this section we collect all the results that we need to prove our main results. We start with some definitions. A *Frobenius group* with kernel K and complement C is a semidirect product $F = K \rtimes C$ such that $C_K(x) = 1$ for every non-identity element x of C. Also, G is a 2-*Frobenius group* if $G = ABC$, where A and AB are the normal subgroups of G and AB and BC are Frobenius groups with kernels A and B and complements B and C, respectively.

Lemma 1. (See [7, 11, 13, 14]). *If $F = K \rtimes C$ is a Frobenius group with the kernel K and complement C, then the following assertions hold:*

(1) *K is nilpotent and the prime graph* $\mathrm{GK}(K)$ *is a complete graph;*

(2) *If H is a subgroup of order rs in C, where $r, s \in \pi(C)$ (not necessary distinct), then H is a cyclic group; in particular, the Sylow r-subgroup of C is cyclic for any odd prime $r \in \pi(C)$;*

(3) *If $2 \in \pi(C)$ then C has a unique element z of order 2, in particular, the Sylow 2-subgroup of C is either cyclic or a (generalized) quaternion group and the subgroup K is Abelian.*

(4) *Either the group C is soluble and the prime graph $\mathrm{GK}(C)$ is complete or C contains a normal subgroup $L \cong \mathrm{SL}(2,5)$ such that $(|L|, |C : L|) \leq 2$ and the prime graph $\mathrm{GK}(C)$ can be obtained from the complete graph on $\pi(C)$ by deleting the edge $\{3, 5\}$.*

(5) $s(F) = 2$ *and* $\mathrm{OC}(F) = \{|K|, |C|\}$.

(6) *Every non-identity element of C induces by conjugation an automorphism of K which is fixed-point-free.*

(7) $|K| \equiv 1 \pmod{|C|}$.

The following lemma deals with the structure of 2-Frobenius groups and their Gruenberg-Kegel graphs. One may find its proof in [10].

Lemma 2. *In case* (b) *of the Gruenberg-Kegel Theorem:*

(1) *C and B are cyclic groups, and $|B|$ is odd;*

(2) *G is soluble, and*

(3) $\mathrm{GK}(B)$ *and* $\mathrm{GK}(AC)$ *are connected components of the prime graph* $\mathrm{GK}(G)$*, and both of them are complete graphs. In particular,* $s(G) = 2$, $\pi_1(G) = \pi(AC)$, $\pi_2(G) = \pi(B)$, *and* $\mathrm{OC}(G) = \{|AC|, |B|\}$.

The following lemma will be used so as to analyze the finite simple groups of Lie type.

Lemma 3. (Zsigmondy [15]). *Let q and f be integers greater than 1. There exists a prime divisor r of $q^f - 1$ such that r does not divide $q^e - 1$ for all $0 < e < f$, except in the following cases:*

(a) $f = 6$ *and* $q = 2$;

(b) $f = 2$ *and* $q = 2^l - 1$ *for some natural number l.*

Such a prime r is called a *primitive prime divisor* of $q^f - 1$. If $q > 1$ is fixed, we denote by q_f any primitive prime divisors of $q^f - 1$. Of course, there may be more than one primitive prime divisor of $q^f - 1$, however the symbol q_f denotes any one of these primes. For example, the primitive prime divisors of $53^5 - 1$ are

11, 131, 5581 and thus 53_5 denotes any one of these primes. Evidently, f divides $q_f - 1$, thus $q_f \geq f + 1$.

The following lemma is an immediate consequence of Lemma 3.

Lemma 4. *Let p and q be two primes and m, n be natural numbers such that $p^m - q^n = 1$. Then one of the following holds:*

(a) $(p, n) = (2, 1)$*, and* $q = 2^m - 1$ *is a Mersenne prime;*

(b) $(q, m) = (2, 1)$*, and* $p = 2^n + 1$ *is a Fermat prime;*

(c) $(p, n) = (3, 3)$ *and* $(q, m) = (2, 2)$.

Lemma 5. *Let q be a power of a prime and $n \geq 2$ be an odd natural number. Then the primitive prime divisor q_{n-1} does not divide the numbers $q^{n-2i} + 1$ for all natural numbers i, $1 \leq i \leq (n-1)/2$.*

Proof. Suppose that $s := q_{n-1}$. Then s does not divide $q^{(n-1)/2} - 1$, therefore s divides $q^{(n-1)/2} + 1$. Assume that our claim is false. Then s divides $q^{n-2i} + 1$ for some i with $1 \leq i \leq (n-1)/2$, and so

$$s \mid (q^{n-2i} + 1) - (q^{(n-1)/2} + 1) = q^{n-2i} - q^{(n-1)/2}.$$

Furthermore, we have

$$q^{n-2i} - q^{(n-1)/2} = \begin{cases} q^{(n-1)/2}(q^{(n-4i+1)/2} - 1) & \text{if } 1 \leq i \leq (n+1)/4; \\ q^{n-2i}(1 - q^{(4i-n-1)/2}) & \text{if } (n+1)/4 < i \leq (n-1)/2, \end{cases}$$

and since $(s, q) = 1$, we deduce that s divides $q^{(n-4i+1)/2} - 1$ or $q^{(4i-n-1)/2} - 1$. On the other hand, since we have $(n-4i+1)/2 < n-1$ and $(4i-n-1)/2 < n-1$, this contradicts with the primitivity of s. ■

Lemma 6. *Let $q, n \geq 2$ be integers. Then for all i, $0 < i < n$, we have $(q_n, q^{n+i} - 1) = 1$.*

Proof. It is easy to notice that $(q^n - 1, q^{n+i} - 1) = q^{(n,n+i)} - 1 = q^{(n,i)} - 1$. Now, considering $(n, i) < n$, we obtain $(q_n, q^{(n,i)} - 1) = 1$, therefore we can see that $(q_n, q^{n+i} - 1) = 1$. ■

The next two lemmas reduce the problem to a study of simple groups of Lie type or alternating groups with two connected components in their prime graphs.

Lemma 7. (See [2]). *Let G be a finite group with more than or equal 3 prime graph components. Then G is neither Frobenius nor 2-Frobenius.*

Lemma 8. (See [3]). *Let S be a sporadic simple group. If G is a group with* $\mathrm{OC}(G) = \mathrm{OC}(S)$*, then G is neither Frobenius nor 2-Frobenius.*

The following lemma presents the order components for simple groups of Lie type or alternating groups with two connected components in their prime graphs.

Lemma 9. (See [9, Lemma 4]). *Let S be a finite simple group of Lie type or an alternating group with $s(S) = 2$. Then $n_1(S)$ and $n_2(S)$ are as shown in Table* 1.

3. Main Results

As mentioned before, the problem under study can be reduced to investigating the finite simple groups of Lie type or alternating groups with two connected components. We prefer to examine the cases, Frobenius and 2-Frobenius, separately.

Theorem 1. *Let S be a simple group of Lie type or an alternating group with $s(S) = 2$. If G is a finite group with* $\mathrm{OC}(G) = \mathrm{OC}(S)$*, then G is not a Frobenius group.*

Proof. Let S be a finite simple group with $s(S) = 2$ and let G be a finite group with $\mathrm{OC}(G) = \mathrm{OC}(S) = \{n_1(S), n_2(S)\}$. Clearly $|G| = |S| = n_1(S).n_2(S)$. We must show that G is not a Frobenius group. Assume the contrary that G is a Frobenius group with the kernel K and complement C. Then by Lemma 1, parts (5) and (7), $s(G) = 2$, $\mathrm{OC}(G) = \{|K|, |C|\}$, and $|C|$ divides $|K|-1$. From $|C| < |K|$ and Table 1 we can easily conclude that $|K| = n_1(S)$ and $|C| = n_2(S)$. Notice that, if $r \in \pi(K)$ and $s \in \pi(C)$, then $\mathrm{Syl}_r(G) = \mathrm{Syl}_r(K)$ and $\mathrm{Syl}_s(G) = \mathrm{Syl}_s(C)$.

Now, suppose that $R \in \mathrm{Syl}_r(K)$. Since K is nilpotent, $R \trianglelefteq G$. Hence, C acts on R by conjugation. By Lemma 1(6), this action is fixed-point-free on R, and so $R \rtimes C$ is a Frobenius group. Therefore

$$|R| \equiv 1 \pmod{|C|}. \tag{1}$$

On the basis of the information provided above, the proof is made through a case by case analysis.

(1) $S \cong A_n$, where $6 < n = p, p+1$, or $p+2$ and n or $n-2$ is not a prime. In this case $|K| = (n!)/(2p)$ and $|C| = p$. Assume $r := r_{[p]}$ and $R \in \mathrm{Syl}_r(G)$. Clearly $r > \lfloor (p+1)/2 \rfloor$ and it follows that $|R| = r$. Now by (1), we must have $p | r-1$, which is a contradiction.

(2) $S \cong A_{p-1}(q)$, $(p, q) \notin \{(3, 2), (3, 4)\}$. In this case we have

$$|K| = q^{p(p-1)/2} \prod_{i=1}^{p-1} (q^i - 1) \qquad \text{and} \qquad |C| = \frac{q^p - 1}{(q-1)(p, q-1)}.$$

Table 1. The order components of alternating groups and simple groups of Lie type with two connected components.

S	Conditions	$n_1(S)$	$n_2(S)$
A_n	$6 < n = p, p+1$ or $p+2$ n or $n-2$ is not a prime number	$(n!)/(2p)$	p
$A_{p-1}(q)$	$(p,q) \neq (3,2),(3,4)$	$q^{p(p-1)/2}\prod_{i=1}^{p-1}(q^i-1)$	$\frac{q^p-1}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1)\vert(p+1)$	$q^{p(p+1)/2}(q^{p+1}-1)\prod_{i=1}^{p-1}(q^i-1)$	$\frac{q^p-1}{q-1}$
${}^2A_{p-1}(q)$		$q^{p(p-1)/2}\prod_{i=1}^{p-1}(q^i-(-1)^i)$	$\frac{q^p+1}{(q+1)(p,q+1)}$
${}^2A_p(q)$	$(q+1)\vert(p+1)$ $(p,q)\neq(3,3),(5,2)$	$q^{p(p+1)/2}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-(-1)^i)$	$\frac{q^p+1}{q+1}$
${}^2A_3(2)$		$2^6.3^4$	5
$B_n(q)$	$n=2^m\geq 4,\ 2\nmid q$	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{2}$
$B_p(3)$		$3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p-1}{2}$
$C_n(q)$	$n=2^m\geq 2$	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$	$q=2,3$	$q^{p^2}(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$	$p\geq 5,\ q=2,3,5$	$q^{p(p-1)}\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$	$q=2,3$	$q^{p(p+1)}(q^p+1)(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{(2,q-1)}$
${}^2D_n(q)$	$n=2^m\geq 4$	$q^{n(n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$	$n=2^m+1,\ m\geq 2$	$2^{n(n-1)}(2^n+1)(2^{n-1}-1)\prod_{i=1}^{n-2}(2^{2i}-1)$	$2^{n-1}+1$
${}^2D_p(3)$	$5\leq p\neq 2^m+1$	$3^{p(p-1)}\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p+1}{4}$
${}^2D_n(3)$	$9\leq n=2^m+1\neq p,$	$3^{n(n-1)}(3^n+1)(3^{n-1}-1)\prod_{i=1}^{n-2}(3^{2i}-1)$	$\frac{3^{n-1}+1}{2}$
$G_2(q)$	$2<q\equiv\epsilon \pmod 3,\ \epsilon=\pm$	$q^6(q^2-1)^2(q^2+\epsilon q+1)$	$q^2+\epsilon q+1$
${}^3D_4(q)$		$q^{12}(q^4+q^2+1)(q^6-1)(q^2-1)$	q^4-q^2+1
$F_4(q)$	q is odd	$q^{24}(q^6-1)(q^8-1)(q^4-1)(q^2-1)$ (q^4+q^2+1)	q^4-q^2+1
${}^2F_4(2)'$		$2^{11}.3^3.5^2$	13
$E_6(q)$		$q^{36}(q^{12}-1)(q^3-1)(q^8-1)(q^6-1)$ $(q^5-1)(q^2-1)$	$\frac{q^6+q^3+1}{(3,q-1)}$
${}^2E_6(q)$	$q>2$	$q^{36}(q^{12}-1)(q^3+1)(q^8-1)(q^6-1)$ $(q^5+1)(q^2-1)$	$\frac{q^6-q^3+1}{(3,q+1)}$

First, assume that $p=3$ and $S\cong A_2(q)\cong \mathrm{PSL}(3,q)$. Here, we have

$$|K| = q^3(q-1)(q^2-1) \qquad \text{and} \qquad |C| = (q^2+q+1)/(3,q-1).$$

First, we assume that q is not a Mersenne prime. Then, by Lemma 3, we can consider the primitive prime divisor $r := q_2 \in \pi(q+1)$. Now, if $R\in \mathrm{Syl}_r(G)$, then $|R|$ divides $q+1$, and by (1) we deduce that $(q^2+q+1)/(3,q-1)\leq |R|-1\leq q+1$, which is a contradiction. Next, we suppose $3\neq q$ is a Mersenne prime. In this case if $R\in \mathrm{Syl}_2(G)$, then $|R| = 4(q+1)$. Moreover, by (1) we must have

$(q^2+q+1)/(3,q-1) \leq |R|-1 \leq 4(q+1)$, which is a contradiction. For $q=3$, we have $|K| = 3^3.2^4$ and $|C| = 13$. Again, by (1) we obtain $13 \mid 16-1$, which is a contradiction.

Next, we assume that $p > 3$. We consider two cases separately.

(a) Let $(p, q-1) = 1$. Assume first that $q \neq 2$. Now, we consider the primitive prime divisor $r := q_{p-1}$. Clearly $r|q^{(p-1)/2}+1$. Assume that $r^m \parallel q^{(p-1)/2}+1$ and $R \in \mathrm{Syl}_r(G)$. Then $|R| = r^m$ and by (1) we must have $(q^p-1)/(q-1)$ divides $r^m - 1$. Thus, we deduce that

$$q^{p-1} + q^{p-2} + \cdots + q + 1 \leq r^m - 1 \leq q^{(p-1)/2} + 1,$$

which is a contradiction.

Next, suppose $q = 2$. In this case, if $p \neq 7$, we may similarly derive a contradiction as the previous case. Now, we assume that $p = 7$. In this case we have $S \cong A_6(2) \cong \mathrm{PSL}(7,2)$, $|K| = 2^{7(7-1)/2}\prod_{i=2}^{6}(2^i-1)$ and $|C| = 2^7 - 1$. Let $R \in \mathrm{Syl}_7(G)$. Then $|R| = 7^2$ and by (1) we must have $2^7 - 1|31 - 1$, which is a contradiction.

(b) $(p, q-1) = p$. In this case $q \neq 2$ and $p < q$. Now, we consider the primitive prime divisor $r := q_{p-1}$. Evidently $r|q^{(p-1)/2} + 1$. Let $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $q^{(p-1)/2}+1$. By (1), we obtain that $(q^p-1)/p(q-1) \leq |R|-1 \leq q^{(p-1)/2} + 1$, and through an easy calculation, we get

$$q^{p-1} + q^{p-2} + \cdots + q + 1 \leq p(q^{(p-1)/2} + 1) \leq q^{(p+1)/2} + q,$$

which is impossible.

(3) $S \cong A_p(q)$, $(q-1)|(p+1)$. Here, we have

$$|K| = q^{p(p+1)/2}(q^{p+1}-1)\prod_{i=1}^{p-1}(q^i-1) \qquad \text{and} \qquad |C| = \frac{q^p-1}{q-1}.$$

First assume that $(p, q) \neq (5, 2)$. Then, by Lemma 3, the number $q^{p+1} - 1$ has a primitive prime divisor $r := q_{p+1}$; in particular, r divides $q^{\frac{p+1}{2}} + 1$. Let $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $q^{\frac{p+1}{2}} + 1$, and by (1) we must have

$$q^{p-1} + q^{p-2} + \cdots + q + 1 \leq |R| - 1 \leq q^{(p+1)/2} + 1,$$

which is a contradiction.

For the case $(p, q) = (5, 2)$, we have $|K| = 2^{15}.3^4.5.7^2$ and $|C| = 31$. In this case, we take $R = \mathrm{Syl}_7(G)$ and we similarly obtain $31|7^2-1$, a contradiction.

(4) $S \cong {}^2A_{p-1}(q)$. Here, the orders of K and C are

$$|K| = q^{p(p-1)/2}\prod_{i=1}^{p-1}(q^i - (-1)^i) \qquad \text{and} \qquad |C| = \frac{q^p+1}{(q+1)(p,q+1)}.$$

First we assume that $p = 3$. Then

$$|K| = q^3(q+1)^2(q-1) \qquad \text{and} \qquad |C| = \frac{q^2-q+1}{(3,q+1)}.$$

Since ${}^2A_2(2) \cong \mathrm{PSL}(3,2)$ is a solvable group, we may assume that $q > 2$. Now, by Lemma 3, the number q^6-1 has a primitive prime divisor $r := q_6$; in particular, $5 < r \in \pi(q^2-q+1)$. Let $R \in \mathrm{Syl}_{p'}(G)$, where $p' \in \pi(q)$. Then $|R| = q^3$, and by (1), $(q^2-q+1)/(3,q+1)$ must divides q^3-1. But this is a contradiction, since $(r, q^3-1) = 1$.

Henceforth, we may assume that $p \geq 5$.

First assume that $q \neq 2$. Then, we consider the primitive prime divisor $r := q_{p-1}$. Obviously r divides $q^{(p-1)/2}+1$. Note that, by Lemma 5 we have $(r, q^{p-2i}+1) = 1$, where $i = 1, 2, \ldots, (p-1)/2$. Now, if $R \in \mathrm{Syl}_r(G)$, then $|R|$ divides $q^{(p-1)/2}+1$. Moreover, by (1) we must have $|C|$ divides $|R|-1$, which implies the following

$$\frac{q^p+1}{(q+1)(p,q+1)} \leq |R| - 1 \leq q^{(p-1)/2}+1,$$

and it is a contradiction.

Next, suppose $q = 2$. In this case, if $p \neq 7$, the proof is similar as in the previous paragraph. For $p = 7$ we have $|K| = 2^{21}.3^8.5.7.11$ and $|C| = 43$. Now, if $R \in \mathrm{Syl}_{11}(G)$ then it follows that $43|11-1$, which is a contradiction.

(5) $S \cong {}^2A_p(q), (q+1)|(p+1), (p,q) \notin \{(3,3),(5,2)\}$. In this case

$$|K| = q^{p(p+1)/2}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-(-1)^i) \qquad \text{and} \qquad |C| = (q^p+1)/(q+1).$$

Here, we consider the primitive prime divisor $r := q_{p+1}$. An argument similar to that in case (4) shows that the case under consideration is impossible.

(6) $S \cong {}^2A_3(2)$. Here, $|K| = 2^6.3^4$ and $|C| = 5$. Let $R \in \mathrm{Syl}_2(G)$. Then by (1), we get $5|2^6-1$, which is impossible.

(7) $S \cong B_n(q)$, $n = 2^m \geq 4$, q is odd. Here, we have

$$|K| = q^{n^2}(q^n - 1)\prod_{i=1}^{n-1}(q^{2i} - 1) \qquad \text{and} \qquad |C| = \frac{q^n + 1}{2}.$$

By Lemma 3, the number $q^{2(n-1)} - 1$ has a primitive prime divisor $r := q_{2(n-1)}$; in particular, r divides $q^{n-1} + 1$. If $R \in \mathrm{Syl}_r(G)$ then $|R|$ divides $q^{n-1} + 1$, and by (1) we must have

$$\frac{q^n + 1}{2} \leq |R| - 1 \leq q^{n-1} + 1,$$

which is false.

(8) $S \cong C_n(q)$, $n = 2^m \geq 2$. In this case we have

$$|K| = q^{n^2}(q^n - 1)\prod_{i=1}^{n-1}(q^{2i} - 1) \qquad \text{and} \qquad |C| = (q^n + 1)/(2, q - 1).$$

If $(q, n) \notin \{(2, 4), (\text{a Mersenne prime}, 2)\}$, then we consider the primitive prime divisor $r := q_{2(n-1)}$. Clearly r divides $q^{n-1} + 1$ and $(r, q^n - 1) = 1$. Now, if $R \in \mathrm{Syl}_r(G)$, then $|R|$ divides $q^{n-1} + 1$, and by (1) we obtain that

$$(q^n + 1)/(2, q - 1) \leq |R| - 1 \leq q^{n-1} + 1,$$

which is a contradiction.

In the case that $(q, n) = (2, 4)$, we have $|K| = 2^{16}.3^5.5^2.7$ and $|C| = 17$. If $R = \mathrm{Syl}_7(G)$ then by (1) we must have $17 \mid 7-1$, which is clearly impossible.

Finally, suppose q is a Mersenne prime and $n = 2$. In this case, we get $|K| = q^4(q^2 - 1)^2$ and $|C| = (q^2 + 1)/2$. Now, if there exists a prime $2 \neq r \in \pi(q - 1)$, then $r|(q - 1)/2$. Let $r^m \| (q - 1)/2$ and $R \in \mathrm{Syl}_r(G)$. Clearly, the order of R is r^{2m}. Therefore, by (1) we obtain $(q^2+1)/2|r^{2m}-1$, which implies $(q^2 + 1)/2 \leq r^{2m} - 1 \leq (q - 1)^2/4$, a contradiction. Thus, we must have $\pi(q - 1) = \{2\}$. Now it is easy to see that $q - 1 = 2$, because q is a Mresenne prime. Hence $q = 3$, $|K| = 2^6.3^4$ and $|C| = 5$. In this case we consider $R = \mathrm{Syl}_2(G)$. Clearly, $|R| = 2^6$ and by (1) we arrive at the contradiction of $5 \mid 2^6 - 1$.

For the cases $B_p(3)$; ${}^2D_p(3)$, $5 \leq p \neq 2^n + 1$ we consider the primitive prime divisor $r := 3_{2(p-1)}$ and for the cases $C_p(q)$, $q = 2, 3$; $D_p(q)$, $p \geq 5$, $q = 2, 3, 5$; $D_{p+1}(q)$, $q = 2, 3$, we assume $r := q_{2(p-2)}$, and similarly we get a contradiction.

(9) $S \cong {}^2D_n(q)$, $n = 2^m \geq 4$. Here, we have

$$|K| = q^{n(n-1)} \prod_{i=1}^{n-1} (q^{2i} - 1) \quad \text{and} \quad |C| = \frac{q^n + 1}{(2, q+1)}.$$

First, we assume that $(n, q) \neq (4, 2)$. Then, by Lemma 3, the number $q^{2(n-1)} - 1$ has a primitive prime divisor $r := q_{2(n-1)}$; in particular, r divides $(q^{n-1}+1)/(q+1)$. Now, if $R \in \mathrm{Syl}_r(G)$, then $|R|$ divides $(q^{n-1}+1)/(q+1)$, and by (1) we must have

$$\frac{q^n + 1}{(2, q-1)} \leq |R| - 1 \leq \frac{q^{n-1} + 1}{q + 1},$$

which is a contradiction. Next, we suppose $(n, q) = (4, 2)$ and $S \cong {}^2D_4(2)$. In this case we have $|K| = 2^{12}.3^4.5.7$ and $|C| = 17$; and we take $R \in \mathrm{Syl}_7(G)$. Therefore, by (1) we must have $17 \mid 7 - 1$, which is a contradiction again.

(10) $S \cong {}^2D_n(2)$, $5 \leq n = 2^m + 1$. In this case we have

$$|K| = 2^{n(n-1)}(2^n + 1)(2^{n-1} - 1) \prod_{i=1}^{n-2} (2^{2i} - 1) \qquad \text{and} \qquad |C| = 2^{n-1} + 1.$$

If $n \neq 5$, we consider the primitive prime divisor $r := 2_{2(n-2)}$. Evidently $r | 2^{n-2} + 1$. Let $r^m \| 2^{n-2} + 1$ and $R \in \mathrm{Syl}_r(G)$. Clearly $|R| = r^m$. On the other hand, since $|C|$ divides $|R| - 1$, we must have $2^{n-1} + 1 \leq r^m - 1 \leq 2^{n-2} + 1$, but this is a contradiction. Now, we assume $n = 5$, then $|K| = 2^{20}.3^6.5^2.7.11$ and $|C| = 17$. Clearly, K contains a normal subgroup of order 11, and we must have $17 \mid 11 - 1$, which is a contradiction.

(11) $S \cong {}^2D_n(3)$, $5 \leq n = 2^m + 1 \neq p$. Then

$$|K| = 3^{n(n-1)}(3^n + 1)(3^{n-1} - 1) \prod_{i=1}^{n-2} (3^{2i} - 1) \qquad \text{and} \qquad |C| = (3^{n-1} + 1)/2.$$

Now, we consider the primitive prime divisor $r := 3_{2(n-2)}$. Therefore, similar to the previous case, we get a contradiction.

(12) $S \cong G_2(q)$, where $2 < q \equiv 1 \pmod 3$. In this case we have

$$|K| = q^6(q^2 - 1)^2(q^2 + q + 1) \quad \text{and} \quad |C| = q^2 - q + 1.$$

Obviously $(q - 1, q^2 + q + 1) = 1$ or 3 and $(q - 1, q + 1) = 1$ or 2. First assume that there exists a prime r such that $r \in \pi(q - 1) \backslash \{2, 3\}$ and

$r^m \| q-1$. Now, we consider the Sylow r-subgroup R of K. Therefore, we may easily see that $|R| = r^{2m}$ and $R \in \mathrm{Syl}_r(G)$. Therefore, from (1) we get $q^2 - q + 1 \leq r^m - 1 \leq (q-1)^2$, which is a contradiction.

In what follows, we suppose that $\pi(q-1) \subseteq \{2, 3\}$ and we consider two cases separately.

(a) q is an odd number. Since $3 \in \pi(q-1)$, we consider the Sylow 3-subgroup R of G. Let $3^m \| q-1$. Since $(q, 3) = (q+1, 3) = 1$ and $(q^2+q+1, q-1) = 3$, we obtain $|R| = 3^{2m+1}$. According to (1) and $2|q-1$, we conclude

$$q^2 - q + 1 \leq 3^{2m+1} - 1 < 4.3^{2m} \leq (q-1)^2,$$

which is a contradiction.

(b) $q = 2^n$. By our assumptions, we have $\pi(2^n - 1) = \{3\}$, and hence $2^n - 1 = 3^m$ for some $m \in \mathbb{N}$. Now, by Lemma 4, the only solution is $(m, n) = (1, 2)$, and we get $S \cong G_2(4)$. In this case, we have $|K| = 2^{12}.3^3.5^2.7$ and $|C| = 13$. Let $R \in \mathrm{Syl}_7(G)$. Now, by (1) it follows that $13|7-1$, which is impossible.

(13) $S \cong G_2(q)$, where $2 < q \equiv -1 \pmod 3$. In this case we have

$$|K| = q^6(q^2-1)^2(q^2-q+1) \qquad \text{and} \qquad |C| = q^2+q+1.$$

First, assume that there exists a prime $3 \neq r \in \pi(q^2-q+1)$. Since $(q^2 - q + 1, q^2 - 1) = 1$ or 3, we get $(r, q^2-1) = 1$. Therefore, if $R \in \mathrm{Syl}_r(G)$ then $|R|$ divides $q^2 - q + 1$. But as $|C|$ divides $|R| - 1$ we must have $q^2 + q + 1 \leq r^m - 1 \leq q^2 - q + 1$, which is a contradiction.

Next, suppose $\pi(q^2 - q + 1) = \{3\}$. Now we claim that $\pi(q-1) \neq \{2\}$. In fact, if this is false, then $q - 1 = 2^k$ for some $k \in \mathbb{N}$. This yields that $2^{2k} + 2^k + 1 = 3^s$, for some $s \in \mathbb{N}$. But this is impossible as the equation $x^2+x+1-3^s = 0$ where x is a power of 2, has no solution in natural numbers. Now, we consider a prime $2 \neq r \in \pi(q-1)$. Let $r^m \| q-1$ and $R \in \mathrm{Syl}_r(G)$. Then $|R| = r^{2m}$, and by (1) we must have $q^2+q+1 \leq r^{2m}-1 \leq (q-1)^2$, which is again a contradiction.

(14) $S \cong {}^3D_4(q)$. In this case we have

$$|K| = q^{12}(q^2-1)^2(q^4+q^2+1)^2 \ \text{ and } \ |C| = q^4 - q^2 + 1.$$

Obviously, $(q^2-1, q^4+q^2+1) = 1$ or 3. First, we assume that there exists a prime $3 \neq r \in \pi(q^2-1)$. Let R be a Sylow r-subgroup of G. Then by (1) we have $q^4 - q^2 + 1 \leq |R| - 1 \leq (q^2-1)^2$, which is a contradiction. Next, suppose $\pi(q^2-1) \subseteq \{3\}$. Now, by Lemma 4, the only solution is $q = 2$. Then $S \cong {}^3D_4(2)$, $|K| = 2^{12}.3^4.7^2$ and $|C| = 13$. Let $R \in \mathrm{Syl}_7(G)$. From (1) we must have $13|7^2 - 1$, which is a contradiction.

(15) $S \cong F_4(q)$, q is an odd number. In this case we have

$$|K| = q^{24}(q^6-1)^2(q^8-1)(q^4-1) \quad \text{and} \quad |C| = q^4-q^2+1.$$

Here, we consider the primitive prime divisor $r := q_8$. We easily see that r divides $(q^4+1)/2$, because q is odd. Let $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $(q^4+1)/2$ and from (1) we can conclude that $q^4-q^2+1 \leq |R|-1 \leq (q^4+1)/2$, which is a contradiction.

(16) $S \cong {}^2F_4(2)'$. In this case we have $|K| = 2^{11}.3^3.5^2$ and $|C| = 13$. Now, by considering $R \in \mathrm{Syl}_2(G)$, we must have $13 \mid 2^{11}-1$, which is impossible.

(17) $S \cong E_6(q)$. In this case we have

$$|K| = q^{36}(q^{12}-1)(q^3-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1) \text{ and } |C| = (q^6+q^3+1)/(3,q-1).$$

Now we consider the primitive prime divisor $r := q_{12}$. Also, from

$$q^{12}-1 = (q^6-1)(q^2+1)(q^4-q^2+1),$$

we notice that r divides q^4-q^2+1. Now, if $R \in \mathrm{Syl}_r(G)$, it is easy to see that $|R|$ divides q^4-q^2+1 and by (1) we deduce that $(q^6+q^3+1)/(3,q-1) \leq |R|-1 \leq q^4-q^2+1$, which is a contradiction.

(18) $S \cong {}^2E_6(q)$. In this case we have

$$|K| = q^{36}(q^{12}-1)(q^3+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1) \quad \text{and} \quad |C| = (q^6-q^3+1)/(3,q+1).$$

Here, we consider the primitive prime divisor $r := q_{10}$. Certainly, r divides $(q^5+1)/(q+1)$. By Lemma 6, $(r, q^{12}-1) = 1$ and so $(r, |K|/(q^5+1)) = 1$. Now if $R \in \mathrm{Syl}_r(G)$, then $|R|$ divides $(q^5+1)/(q+1)$, and by (1) we must have

$$\frac{q^6-q^3+1}{(3,q+1)} \leq |R|-1 \leq \frac{q^5+1}{q+1} = q^4-q^3+q^2-q+1,$$

which is a contradiction.

In this way, the case by case analysis has been done and the proof is finished. ■

Theorem 2. *Let S be a simple group of Lie type or an alternating group over n letters ($n \geq 5$), with $s(S) = 2$, with the exception of $U_4(2)$ and $U_5(2)$. If G is a finite group with $\mathrm{OC}(G) = \mathrm{OC}(S)$, then G is not a 2-Frobenius group.*

Proof. Assume that S is a finite simple group of Lie type or an alternating group with $s(S) = 2$, except $U_4(2)$ and $U_5(2)$. Also, we suppose G is a finite group with $\mathrm{OC}(G) = \mathrm{OC}(S) = \{n_1(S), n_2(S)\}$. Now, we must show that G is not a 2-Frobenius group.

On the contrary, suppose that G is a 2-Frobenius group. Then $G = ABC$, where A and AB are the normal subgroups of G and AB and BC are Frobenius groups with the kernels A and B and complements B and C, respectively. By Lemma 2(3), we know that the connected components of $\mathrm{GK}(G)$ are:

$$\pi_1(G) = \pi(AC) \qquad \text{and} \qquad \pi_2(G) = \pi(B).$$

Put $a := |A|$, $b := |B|$, $c := |C|$, and $c' := |C : A \cap C|$. Then by the definition and Lemma 1(7), c divides $b-1$ and b divides $a-1$. Moreover, we have $a = |G|/(bc')$. Notice that for all $r \in \pi(A)\backslash\pi(C)$, $\mathrm{Syl}_r(A) = \mathrm{Syl}_r(G)$. Since AB is a Frobenius group with the kernel A and complement B, by Lemma 1(6), B acts fixed-point-freely on A and this is true for every Sylow r-subgroup of A, say $R \in \mathrm{Syl}_r(A)$. Hence, the semidirect product $R \rtimes B$ is also a Frobenius group and by Lemma 1(7) we have

$$(2) \qquad |R| \equiv 1 \pmod{b}.$$

On the basis of the information provided above, the proof is made by a case by case study.

(1) Suppose $S \cong A_n$, where $6 < n = p, p+1$ or $p+2$ and one of n, $n-2$ is not a prime number. In this case we have $|G| = (n!)/2$, $|B| = p$, and $|A| = (n!)/(2pc')$, where $c'|p-1$. Now we consider $r := r_{[p]}$. Since $\pi(c) \subseteq \pi(\frac{p-1}{2}) \cup \{2\}$ and $r > \frac{p-1}{2}$, we deduce that $r \in \pi(A)\backslash\pi(C)$. Now, if $R \in \mathrm{Syl}_r(G)$, then $|R| = r$, and by (2) we must have $p|r-1$, which is a contradiction.

(2) Suppose $S \cong A_{p-1}(q)$, $(p,q) \neq (3,2), (3,4)$. In this case we have

$$a = \frac{1}{c'}q^{p(p-1)/2}\prod_{i=1}^{p-1}(q^i-1) \qquad \text{and} \qquad b = (q^p-1)/(q-1)(p,q-1),$$

where $c'|b-1$. First, we assume that $p = 3$, and $S \cong A_2(q) \cong \mathrm{PSL3}(q)$. Here, we have

$$a = \frac{1}{c'}q^3(q-1)^2(q+1) \qquad \text{and} \qquad b = (q^2+q+1)/(3,q-1),$$

where $c' \mid q(q+1)$ if $(3,q-1) = 1$, and $c' \mid (q+2)(q-1)/3$ if $(3,q-1) = 3$. We consider two cases separately.

Case 1. $(3, q-1) = 1$. If there exists an odd prime number $r \in \pi(q-1)$, then $(r, c') = 1$, i.e., $r \in \pi(A)\backslash\pi(C)$. Let $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $(q-1)^2$ and by (2) we must have $q^2+q+1 \leq |R|-1 \leq (q-1)^2$, which is a contradiction. Hence, we may assume that $\pi(q-1) = \{2\}$. Now, by Lemma 4(b), it follows that q is a Fermat prime. Let $R \in \mathrm{Syl}_2(A)$. If $(2, c') = 1$, then $|R| = 2(q-1)^2$. Moreover, by (2) we must have q^2+q+1 divides $2(q-1)^2 - 1$, i.e., $2(q-1)^2 - 1 = k(q^2+q+1)$ for some $k \in \mathbb{N}$. Now by easy calculations we obtain that

$$(k-2)q^2 + (k+4)q + k - 1 = 0.$$

Thus $k < 2$ and q divides $k-1$, and so we deduce that $k = 1$ and $q = 5$. In this case, if $G = ABC$ is a 2-Frobenius group of order $2^5{\cdot}5^3{\cdot}31{\cdot}3$, then it is easy to see that $a = 2^5 \cdot 5^3$, $b = 31$ and $c = 3$. In particular, this means that $\mathrm{GL}(5, 2)$ contains a Frobenius subgroup of order 31.3, which is a contradiction. Now we assume that $(2, c') = 2$. Then $|R| = (q-1)^2$ and by (2) we deduce that q^2+q+1 divides $|R|-1$, and so $q^2+q+1 \leq |R|-1 \leq (q-1)^2 - 1$, which is a contradiction.

Case 2. $(3, q-1) = 3$. If there exists an odd prime number $r \in \pi(q+1)$, then $(r, c') = 1$ since $(q+1, q-1) = 1$ and $(q+1, q+2) = 1$. Let $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $q+1$ and by (2) we must have $(q^2+q+1)/3 \leq |R|-1 \leq q+1$, which is a contradiction. Hence, we may assume that $\pi(q+1) = \{2\}$. Now, by Lemma 4(a), it follows that q is a Mersenne prime, say $q = 2^s - 1$ where s is a prime number. Evidently $q^2+q+1 = 2^{2s} - 2^s + 1$. Now we consider the primitive prime divisor 2_{6s}. Since $2^{6s} - 1 = (2^{3s}-1)(2^s+1)(2^{2s} - 2^s + 1)$, it is easy to see that $2_{6s} \in \pi(2^{2s} - 2^s + 1) = \pi(q^2+q+1)$. Now we assume that $R \in \mathrm{Syl}_2(A)$. Then $|R|$ divides $4(q+1) = 2^{s+2}$ and by (2) we deduce that $(q^2+q+1)/3$ divides $|R|-1$. But this is a contradiction, since $2_{6s} \notin \pi(|R|-1)$.

Now, we may assume that $p \geq 5$. We study the two following cases separately:

- (a) $(p, q-1) = 1$. In this case we consider the primitive prime divisor $r := q_{p-2}$. Since $b-1 = q(q^{p-1}-1)/(q-1)$, it follows that $(r, b-1) = (r, c') = 1$, hence $r \in \pi(A)\backslash\pi(C)$. Let $R \in \mathrm{Syl}_r(G)$. Therefore from (2) we have $|R| \equiv 1 \pmod{b}$, thus $(q^p-1)/(q-1) \leq |R|-1 < q^{p-2}+1$, which is a contradiction.
- (b) $(p, q-1) = p$. In this case we have

$$b - 1 = \frac{q^p - 1 - p(q-1)}{p(q-1)}.$$

If $d = (q^{p-1} - 1, q^p - 1 - p(q-1))$, it follows that

$$d \mid q(q^{p-1} - 1) - [q^p - 1 - p(q-1)] = (q-1)(p-1).$$

Now we consider the primitive prime divisor $r := q_{p-1}$. Evidently r divides $q^{(p-1)/2} + 1$. Since $q_{p-1} \geq p$, $(r, p-1) = 1$. Thus, $(r, b-1) = (r, c') = 1$. Let $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $q^{(p-1)/2} + 1$, and by (2) we must have $(q^p - 1)/p(q-1) \leq |R| - 1 \leq q^{(p-1)/2} - 1$, which is impossible.

(3) $S \cong A_p(q)$, $(q-1)|(p+1)$. Here, we have

$$a = \frac{1}{c'} q^{p(p+1)/2}(q^{p+1} - 1) \prod_{i=1}^{p-1}(q^i - 1) \quad \text{and} \quad b = \frac{q^p - 1}{q - 1},$$

where c' divides $b - 1 = q(q^{p-1} - 1)/(q-1)$. First, we assume that $(p, q) \neq (5, 2)$. By Lemma 3, the number $q^{p+1} - 1$ has a primitive prime divisor $r := q_{p+1}$; in particular, r divides $q^{(p+1)/2} - 1$. Clearly $(r, b-1) = (r, c') = 1$. Let $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $q^{(p+1)/2} + 1$, and by (2) we must have $(q^p - 1)/(q-1) \leq |R| - 1 \leq q^{(p+1)/2} - 1$, which is impossible. Next, we assume that $(p, q) = (5, 2)$, i.e., $S \cong A_5(2) \cong \mathrm{PSL}(6, 2)$. In this case we have $a = \frac{1}{c'} 2^{15}.3^4.7^2$ and $b = 31$, where $c' \mid 30$. Now, if $R \in \mathrm{Syl}_7(G)$, then $|R| = 7^2$ and by (2) we must have $31 \mid 7^2 - 1$, which is a contradiction.

(4) $S \cong {}^2A_{p-1}(q)$. In this case, we have

$$a = \frac{1}{c'} q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i - (-1)^i) \quad \text{and} \quad b = \frac{q^p + 1}{(q+1)(p, q+1)},$$

where c' divides $b - 1$. Here, we consider two cases separately.

(a) $(p, q+1) = 1$. In this case, c' divides $b - 1 = q(q^{p-1} - 1)/(q+1)$. Since we assumed before $S \not\cong U_5(2)$, we have $(p, q) \neq (5, 2)$, and by Lemma 3, the number $q^{2(p-2)} - 1$ has a primitive prime divisor $r := q_{2(p-2)}$; in particular, r divides $q^{p-2} + 1$. Moreover, it is easy to see that $(r, b-1) = (r, c') = 1$. Now, assume that $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $q^{p-2} + 1$, and by (2) we must have

$$\frac{q^p + 1}{q + 1} \leq |R| - 1 \leq q^{p-2} + 1,$$

which is a contradiction.

(b) The case when $(p, q+1) = p$ is the same.

(5) $S \cong {}^2A_p(q), (q+1)|(p+1), (p,q) \notin \{(3,3),(5,2)\}$. In this case

$$a = \frac{1}{c'} q^{p(p+1)/2}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-(-1)^i) \quad \text{and} \quad |C| = (q^p+1)/(q+1).$$

Here, we consider the primitive prime divisor $r := q_{p+1}$. An argument similar to that in case (4) shows that the case under consideration is impossible.

(6) $S \cong B_n(q)$, $n = 2^m \geq 4$ and q is an odd number. Here, we have

$$a = \frac{1}{c'} q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1) \quad \text{and} \quad b = \frac{q^n+1}{2},$$

where c' divides $b-1 = (q^n-1)/2$. Now, we consider the primitive prime divisor $r := q_{2(n-1)}$. Evidently r divides $q^{n-1}+1$ and $(r, b-1) = (r, c') = 1$. Let $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $q^{n-1}+1$ and by (2) we deduce that

$$\frac{q^n+1}{2} \leq |R|-1 \leq q^{n-1}+1,$$

which of course is impossible, since $n \geq 4$.

(7) $S \cong B_p(3)$. Since by the assumption $S \ncong U_4(2) \cong B_2(3)$, we may assume that $p \neq 2$. In this case we have

$$a = \frac{1}{c'} 3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1) \quad \text{and} \quad b = \frac{3^p-1}{2},$$

where c' divides $b-1 = 3(3^{p-1}-1)/2$. Since $p \neq 2$, we may consider the primitive prime divisor $r := 3_{2(p-1)} > 2$. Hence $r \in \pi((3^{p-1}+1)/2)$ and $(r, b-1) = (r, c') = 1$. Assume that $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $(3^{p-1}+1)/2$ and by (2) we must have

$$\frac{3^p-1}{2} \leq |R|-1 \leq \frac{3^{p-1}+1}{2},$$

which is a contradiction.

(8) $S \cong C_n(q)$, $n = 2^m \geq 2$. In this case we have

$$a = \frac{1}{c'} q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1) \quad \text{and} \quad b = (q^n+1)/(2, q-1),$$

where c' divides $b-1$. First, we assume that $(q,n) \neq (2,4)$. Now, by Lemma 3, we may consider the primitive prime divisor $r := q_{2(n-1)}$; in particular, r divides $q^{n-1}+1$. Since $b-1=q^n$ if q is an odd number and $b-1=(q^n-1)/2$ if q is an even number, it follows that $(r,b-1)=(r,c')=1$. Let $R\in \mathrm{Syl}_r(G)$. Then $|R|$ divides $q^{n-1}+1$, and by (2) we deduce that $b\leq |R|-1\leq q^{n-1}+1$, which is a contradiction. Next, we suppose $(q,n)=(2,4)$ and $S\cong C_4(2)$. In this case, we have $a=\frac{1}{c'}2^{16}.3^5.5^2.7$ and $b=17$, where c' divides 16. Now, if $R\in \mathrm{Syl}_7(G)$, then by (2) we must have $17\mid 7-1$, which is a contradiction.

(9) $S\cong C_p(q)$, $q=2,3$. For $S\cong C_p(3)$ the proof is similar to the case (6). Let $S\cong C_p(2)$, $p>2$. In this case we have

$$a=\frac{1}{c'}2^{p^2}(2^p+1)\prod_{i=1}^{p-1}(2^{2i}-1) \qquad \text{and} \qquad b=2^p-1,$$

where c' divides $b-1=2(2^{p-1}-1)$. Now by Lemma 3, we may consider the primitive prime divisor $r:=2_{2(p-1)}\in\pi(2^{p-1}+1)$. Evidently $(r,b-1)=(r,c')=1$. Hence, if $R\in \mathrm{Syl}_r(G)$, then $|R|$ divides $2^{p-1}+1$, and by (2) we get $2^p-1\leq|R|-1\leq 2^{p-1}+1$, which is a contradiction.

(10) $S\cong D_p(q)$, $p\geq 5$, $q=2,3,5$.

$$a=\frac{1}{c'}q^{p(p-1)}\prod_{i=1}^{p-1}(q^{2i}-1) \quad \text{and} \quad b=\frac{q^p-1}{q-1},$$

where c' divides $b-1$. For these simple groups of Lie type we may consider the primitive prime divisor $r:=q_{2(p-1)}$, and the proof is similar to the previous cases.

(11) $S\cong D_{p+1}(q)$, $q=2,3$. In this case we have

$$a=\frac{1}{c'}q^{p(p+1)}(q^p+1)(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1) \quad \text{and} \quad b=\frac{q^p-1}{(2,q-1)},$$

where c' divides $b-1$. First, we assume that $q=2$ and $S\cong D_{p+1}(2)$. Now, by Lemma 3 we may consider the primitive prime divisor $r:=2_{2(p-1)}\in\pi(2^{p-1}+1)$. Since $b-1=2(2^{p-1}-1)$, it follows that $(r,b-1)=(r,c')=1$. Let $R\in \mathrm{Syl}_r(G)$. Then $|R|$ divides $2^{p-1}+1$ and by (2) we obtain that $2^p-1\leq|R|-1\leq 2^{p-1}+1$, which is a contradiction. The case when $q=3$ is the same therefore we can omit its proof.

(12) $S\cong {}^2D_n(q)$, $n=2^m\geq 4$. In this case we have

$$a=\frac{1}{c'}q^{n(n-1)}\prod_{i=1}^{n-1}(q^{2i}-1) \quad \text{and} \quad b=\frac{q^n+1}{(2,q+1)},$$

where c' divides q^n or $(q^n-1)/2$ according to $(2,q-1)=1$ or 2 respectively. First we assume that $(n,q)\neq(4,2)$. Now, by Lemma 3, we consider the primitive prime divisor $r:=q_{2(n-1)}\in\pi(q^{n-1}+1)$. Evidently $(r,b-1)=(r,c')=1$, and if $R\in \mathrm{Syl}_r(G)$, then $|R|$ divides $q^{n-1}+1$. Now, by (2) we deduce that $(q^n+1)/(2,q+1)\leq |R|-1\leq q^{n-1}+1$, which is a contradiction. Next, we suppose that $(n,q)=(4,2)$ and $S\cong {}^2D_4(2)$. In this case we have $a=\frac{1}{c'}2^{12}.3^4.5.7$ and $b=17$, and also $c'\mid 16$. Now, if $R\in\mathrm{Syl}_5(G)$, then by (2) we must have $17\mid 5-1$, a contradiction.

(13) $S\cong {}^2D_n(2)$, $5\leq n=2^m+1$. In this case we have

$$a=\frac{1}{c'}2^{n(n-1)}(2^n+1)(2^{n-1}-1)\prod_{i=1}^{n-2}(2^{2i}-1)\quad\text{and}\quad b=2^{n-1}+1,$$

where $c'\mid 2^{n-1}$. By Lemma 3, the number $2^{2n}-1$ has a primitive prime divisor $r:=2_{2n}$; in particular, $r\in\pi((2^n+1)/3)$ and $(r,c')=1$. Let $R\in\mathrm{Syl}_r(G)$. Then $|R|$ divides $(2^n+1)/3$, and by (2) we must have $2^{n-1}+1\leq|R|-1\leq(2^n+1)/3$, which is a contradiction.

(14) $S\cong {}^2D_p(3)$, $5\leq p\neq 2^m+1$. Then

$$a=\frac{1}{c'}3^{p(p-1)}\prod_{i=1}^{p-1}(3^{2i}-1)\quad\text{and}\quad b=\frac{3^p+1}{4},$$

where c' divides $3(3^{p-1}-1)/4$. By Lemma 3, the number $3^{2(p-1)}-1$ has a primitive prime divisor $r:=3_{2(p-1)}$; in particular, $r\in\pi((3^{p-1}+1)/2)$ and $(r,c')=1$. Assume that $R\in\mathrm{Syl}_r(G)$. Then $|R|$ divides $3^{p-1}+1$, and from (2) it follows that $(3^p+1)/4\leq|R|-1\leq(3^{p-1}+1)/2$, which is a contradiction.

(15) $S\cong {}^2D_n(3)$, $5\leq n=2^m+1\neq p$. Then

$$a=\frac{1}{c'}3^{n(n-1)}(3^n+1)(3^{n-1}-1)\prod_{i=1}^{n-2}(3^{2i}-1)\quad\text{and}\quad b=(3^{n-1}+1)/2,$$

where c' divides $(3^{n-1}-1)/2$. By Lemma 3, the number $3^{2(n-2)}-1$ has a primitive prime divisor $r:=3_{2(n-2)}$; in particular, $r\in\pi((3^{n-2}+1)/4)$ and $(r,c')=1$. Let $R\in\mathrm{Syl}_r(G)$. Then $|R|$ divides $(3^{n-2}+1)/4$, because $(3^{2(n-2)}-1,3^{2n}-1)=3^{(2(n-2),2n)}-1=3^4-1$, and by (2) we must have $(3^{n-1}+1)/2\leq|R|-1\leq(3^{n-2}+1)/4$, which is a contradiction.

(16) $S\cong G_2(q)$, where $2<q\equiv 1 \pmod 3$. In this case we have

$$a=\frac{1}{c'}q^6(q^2-1)^2(q^2+q+1)\ \text{ and }\ b=q^2-q+1,$$

where c' divides $q(q-1)$. Since $q > 2$, by Lemma 3, the number q^6-1 has a primitive prime divisor $r := q_6$; in particular, $r \in \pi((q^2+q+1)/3)$ and $(r, c') = 1$. Let $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides $(q^2+q+1)/3$, because 3 divides q^2+q+1, and by (2) we must have $q^2-q+1 \leq |R|-1 \leq (q^2+q+1)/3$, which is a contradiction.

(17) $S \cong G_2(q)$, where $2 < q \equiv -1 \pmod 3$. Here we have

$$a = \frac{1}{c'} q^6(q^2-1)^2(q^2-q+1) \quad \text{and} \quad b = q^2+q+1,$$

where c' divides $b-1 = q(q+1)$. Evidently $(q^2-q+1, q(q^2-1)) = 1$ or 3. First, assume that there exists a prime $3 \neq r \in \pi(q^2-q+1)$. Then $(r, q(q^2-1)) = (r, b-1) = (r, c') = 1$. Now, we assume $R \in \mathrm{Syl}_r(G)$. From (2) we must have $|R| \equiv 1 \pmod{q^2+q+1}$, which implies that $q^2+q+1 < |R|-1 < q^2-q+1$, which is a contradiction. Next, suppose q^2-q+1 is a power of 3. In this case we claim that $\pi(q-1) \neq \{2\}$. Otherwise, in a similar way as in the proof of Theorem 1(7), we get a contradiction . Now, we consider an odd prime $r \in \pi(q-1)$. Clearly $(r, c') = 1$. Let $R \in \mathrm{Syl}_r(G)$. Then, since $(r, q(q+1)(q^2-q+1)) = 1$, it follows that $|R|$ divides $(q-1)^2$. Now, by (2) we must have $q^2+q+1 \leq |R|-1 \leq (q-1)^2$, which is a contradiction.

(18) $S \cong {}^3D_4(q)$. In this case we have

$$a = \frac{1}{c'} q^{12}(q-1)^2(q^2+q+1)^2(q+1)^2(q^2-q+1)^2 \quad \text{and} \quad b = q^4-q^2+1,$$

where c' divides $b-1 = q^2(q^2-1)$. Note that $(q^2-q+1, q+1) = 1$ or 3, $(q^2-q+1, q-1) = 1$ and $(q^2-q+1, q^2+q+1) = 1$. First, assume that there exists a prime $3 \neq r \in \pi(q^2-q+1)$. Now, we can easily see that $(r, c') = 1$ and if $R \in \mathrm{Syl}_r(G)$, then $|R|$ divides $(q^2-q+1)^2$. From (2) we get $q^4-q^2+1 \leq |R|-1 \leq (q^2-q+1)^2$, which is a contradiction. Next, suppose that $\pi(q^2-q+1) \subseteq \{3\}$. If $q \neq 2$, then we consider the primitive prime divisor $r := q_6$. Clearly r divides q^2-q+1 and $r \neq 3$. Since, it is impossible, we must have $q = 2$. In this case, $S \cong {}^3D_4(2)$ and we have $a = \frac{1}{c'} 2^{12}.3^4.7^2$ and $b = 13$, where $c' \mid 12$. Now, we consider a Sylow 7-subgroup of G, and by (2) it follows that $13 | 7^2-1$, which is a contradiction.

(19) $S \cong F_4(q)$, q is an odd number. In this case we have

$$a = \frac{1}{c'} q^{24}(q^6-1)^2(q^8-1)(q^4-1) \quad \text{and} \quad b = q^4-q^2+1,$$

where c' divides $q^2(q^2-1)$. Now, we consider the primitive prime divisor $r := q_8$. It is not difficult to see that $r \geq 11$, $(r, b-1) = (r, c') = 1$, and for

$R \in \mathrm{Syl}_r(G)$, $|R|$ divides $(q^4+1)/2$. Also, from (2) we get $q^4-q^2+1 \leq |R|-1 \leq (q^4+1)/2$, which is a contradiction.

(20) $S \cong {}^2F_4(2)'$. In this case we have $a = \frac{1}{c'}2^{11}.3^3.5^2$ and $|C| = 13$. Now, by considering $R \in \mathrm{Syl}_2(G)$, we must have $13 \mid 2^{11}-1$, which is impossible.

(21) $S \cong E_6(q)$. In this case we have

$$\begin{aligned} a &= \frac{1}{c'}q^{36}(q^{12}-1)(q^3-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1) \text{ and} \\ b &= (q^6+q^3+1)/(3,q-1), \end{aligned}$$

where c' divides $b-1$. By Lemma 3, the number $q^{12}-1$ has a primitive prime divisor $r := q_{12}$; in particular, r divides q^4-q^2+1 and $r \geq 13$. Since

$$b-1 = \begin{cases} q^3(q^3+1) & \text{if } (3,q-1)=1; \\ \dfrac{(q^3-1)(q^3+2)}{3} & \text{if } (3,q-1)=3, \end{cases}$$

and $(q^6+1, q^3+2) = 1$ or 5, we may easily see that $(r, b-1) = (r, c') = 1$. Now suppose that $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides q^4-q^2+1, hence by (2) we obtain the contradiction of

$$\frac{q^6+q^3+1}{(3,q-1)} \leq |R|-1 \leq q^4-q^2+1.$$

(22) $S \cong {}^2E_6(q)$, $q > 2$. In this case we have

$$\begin{aligned} a &= \frac{1}{c'}q^{36}(q^{12}-1)(q^3+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1) \quad \text{and} \\ b &= (q^6-q^3+1)/(3,q+1), \end{aligned}$$

where c' divides $b-1$. By Lemma 3, the number $q^{12}-1$ has a primitive prime divisor $r := q_{12}$; in particular, r divides q^4-q^2+1 and $r \geq 13$. Since

$$b-1 = \begin{cases} q^3(q^3-1) & \text{if } (3,q+1)=1; \\ \dfrac{(q^3+1)(q^3-2)}{3} & \text{if } (3,q+1)=3, \end{cases}$$

and $(q^6+1, q^3-2) = 1$ or 5, we may easily see that $(r, b-1) = (r, c') = 1$. Now suppose that $R \in \mathrm{Syl}_r(G)$. Then $|R|$ divides q^4-q^2+1, hence by (2) we must have

$$\frac{q^6-q^3+1}{(3,q-1)} \leq |R|-1 \leq q^4-q^2+1,$$

which is a contradiction.

Now the proof of theorem is complete. ∎

Through Theorems 1 and 2, the Main Theorem can be proved.

At the end, we should focus on the simple groups $U_4(2)$ and $U_5(2)$. As mentioned before, we construct the 2-Frobenius groups having the same order as $U_4(2)$ and $U_5(2)$. The existence of these 2-Frobenius groups shows that the simple groups $U_4(2)$ and $U_5(2)$ are not recognizable by their order components.

Some Examples. (V. D. Mazurov).

(a) *There exists a* 2*-Frobenius group* F_1 *with* $\mathrm{OC}(F_1) = \mathrm{OC}(U_4(2))$. Indeed, we consider the general linear groups $\mathrm{GL}(4,2)$ and $\mathrm{GL}(4,3)$. In the general linear group $\mathrm{GL}(4,2)$ and also in $\mathrm{GL}(4,3)$ there exists a Frobenius group $E := K \rtimes C$ of order 20 such that K acts fixed-point-freely on corresponding natural modules V_1 and V_2. Now, we take $(V_1 \times V_2) \cdot E$ with the natural action of E on direct factors. Then we obtain a required group $(2^4 \times 3^4) : 5 : 4$.

(b) *There exists a* 2*-Frobenius group* F_2 *with* $\mathrm{OC}(F_2) = \mathrm{OC}(U_5(2))$. Similarly, in the general linear group $\mathrm{GL}(10,2)$ and also in $\mathrm{GL}(5,3)$ there exists a Frobenius group $E := K \rtimes C$ of order 55 such that K acts fixed-point-freely on corresponding natural modules V_1 and V_2. Again, we take $(V_1 \times V_2) \cdot E$ with the natural action of E on direct factors. Now we obtain a required group $(2^{10} \times 3^5) : 11 : 5$.

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