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NONLINEAR KLEIN-GORDON EQUATIONS AND LORENTZIAN MINIMAL SURFACES IN LORENTZIAN COMPLEX SPACE FORMS

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Abstract. We investigate Lorentzian minimal surfaces in Lorentzian complex space forms. First, we prove that for such surfaces the equation of Ricci is a consequence of the equations of Gauss and Codazzi. Next, we classify Lorentzian minimal surfaces in the Lorentzian complex plane $\mathbf{C}^2_1$. Finally, we classify minimal slant surfaces in the Lorentzian complex projective plane $CP^2_1(4)$ and in the Lorentzian complex hyperbolic plane $CH^2_1(-4)$. In particular, our latter results show that if a minimal slant surface in $CP^2_1(4)$ or in $CH^2_1(-4)$ contains no open subset of constant curvature, then it is of Klein-Gordon type which arises from the solutions of certain nonlinear Klein-Gordon equations.

1. Introduction

Let $\tilde{M}^n_i(4c)$ be a complete simply-connected indefinite complex space form of complex dimension n and *complex index* i. Here, the complex index is defined as the complex dimension of the largest complex negative definite subspace of the tangent space. If $i = 1$, we say that $\tilde{M}^n_i(4c)$ is *Lorentzian*.

The curvature tensor $\tilde{R}$ of $\tilde{M}^n_i(4c)$ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX \\ &\quad - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ\}.\end{aligned} \tag{1.1}$$

Let $\mathbf{C}^n$ denote the complex n-plane with complex coordinates $z_1, \ldots, z_n$. The $\mathbf{C}^n$ endowed with $g_{i,n}$, i.e., the real part of the Hermitian form

$$b_{i,n}(z,w) = -\sum_{k=1}^{i} \bar{z}_k w_k + \sum_{j=i+1}^{n} \bar{z}_j w_j, \quad z, w \in \mathbf{C}^n,$$

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defines a flat indefinite complex space form with complex index i. We simply denote the pair $(\mathbf{C}^n, g_{i,n})$ by $\mathbf{C}^n_i$.

Consider the differentiable manifold:

$$S^{2n+1}_2(c) = \{z \in \mathbf{C}^{n+1}_1 \,;\, b_{1,n+1}(z,z) = c^{-1} > 0\},$$

which is an indefinite real space form of constant sectional curvature c. The Hopf fibration

$$\pi : S^{2n+1}_2(c) \to CP^n_1(4c) : z \mapsto z \cdot \mathbf{C}^*$$

is a submersion and there exists a unique pseudo-Riemannian metric of complex index one on $CP^n_1(4c)$ such that π is a Riemannian submersion.

The pseudo-Riemannian manifold $CP^n_1(4c)$ is a Lorentzian complex space form of positive holomorphic sectional curvature $4c$.

Analogously, if $c < 0$, consider

$$H^{2n+1}_2(c) = \{z \in \mathbf{C}^{n+1}_2 \,;\, b_{2,n+1}(z,z) = c^{-1} < 0\},$$

which is an indefinite real space form of constant sectional curvature $c < 0$. The Hopf fibration

$$\pi : H^{2n+1}_2(c) \to CH^n_1(4c) : z \mapsto z \cdot \mathbf{C}^*$$

is a submersion and there exists a unique pseudo-Riemannian metric of complex index 1 on $CH^n_1(4c)$ such that π is a Riemannian submersion.

The pseudo-Riemannian manifold $CH^n_1(4c)$ is a Lorentzian complex space form of negative holomorphic sectional curvature $4c$.

It is well-known that a complete simply-connected Lorentzian complex space form $\tilde{M}^n_1(4c)$ is holomorphically isometric to $\mathbf{C}^n_1$, $CP^n_1(4c)$, or $CH^n_1(4c)$, according to $c = 0, c > 0$ or $c < 0$, respectively.

The history of minimal surfaces goes back to J. L. Lagrange (1736-1813) who initiated in 1760 the study of minimal surfaces in Euclidean 3-space (cf. [12]). Since then the theory of minimal surfaces have attracted many mathematicians for more than two centuries. In particular, minimal surfaces in real space forms have been studied very extensively during the last two centuries (see, [3, pages 207–249] and [14, 15] for details).

In this article, we apply the method in [7, 8, 9] to investigate Lorentzian minimal surfaces in Lorentzian complex space forms. In sections 2 and 3 we provide some basic notations, formulas and results. In section 4, we prove that, for Lorentzian minimal surfaces in Lorentzian complex space forms, the equation of Ricci is a consequence of the equations of Gauss and Codazzi. Two existence results are given in section 5. In section 6, we classify Lorentzian minimal surfaces in the Lorentzian complex plane $\mathbf{C}^2_1$. In the last two sections, we classify minimal slant surfaces in the Lorentzian complex projective plane $CP^2_1(4)$ and in the Lorentzian complex hyperbolic plane $CH_1^2(-4)$. In particular, our results obtained in the last two sections show that if a minimal slant surface in $CP_1^2(4)$ or in $CH_1^2(-4)$ contains no open subset of constant curvature, then it is of Klein-Gordon type which arises from the solutions of certain nonlinear Klein-Gordon equations.

2. Preliminaries

2.1. Basic formulas, equation and definitions

Let M be a Lorentzian surface of a Lorentzian Kähler surface $\tilde{M}_1^2$ equipped with an almost complex structure J and metric $\tilde{g}$. Let $\langle\ ,\ \rangle$ denote the inner product associated with $\tilde{g}$. Denote the induced metric on M by g.

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connection on M and $\tilde{M}_1^2$, respectively. Then the formulas of Gauss and Weingarten are given respectively by (cf. [1, 11, 13])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h, A and D are the second fundamental form, the shape operator and the normal connection.

The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle \tag{2.3}$$

for X, Y tangent to M and ξ normal to M.

For each normal vector ξ of M at $x \in M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_xM. The mean curvature vector is defined by

$$H = \frac{1}{2}\operatorname{trace} h. \tag{2.4}$$

A Lorentzian surface in $\tilde{M}_1^2$ is called *minimal* if its mean curvature vector vanishes at each point on M.

For a Lorentzian surface M in a Lorentzian complex space form $\tilde{M}_1^2(4c)$, the equations of Gauss, Codazzi and Ricci are given respectively by

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= \langle\, \tilde{R}(X,Y)Z, W\,\rangle + \langle h(X,W), h(Y,Z)\rangle \\ &\quad - \langle h(X,Z), h(Y,W)\rangle,\end{aligned} \tag{2.5}$$

$$(\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{2.6}$$

$$\left\langle R^D(X,Y)\xi,\eta\right\rangle = \langle\, \tilde{R}(X,Y)\xi,\eta\,\rangle + \langle [A_\xi, A_\eta]X, Y\rangle\,, \tag{2.7}$$

where X, Y, Z, W are vector tangent to M, and $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.8}$$

2.2. Special Legendre curves in light cone

A vector v is called *space-like* (respectively, *time-like*) if $\langle v,v\rangle > 0$ (respectively, $\langle v,v\rangle < 0$). A vector v is called *null* or *light-like* if it is a nonzero vector and it satisfies $\langle v,v\rangle = 0$.

The *light cone* $\mathcal{L}C$ in $\mathbf{C}^n_i$ $(n \geq 3, i = 1, 2)$ is defined by

$$\mathcal{L}C = \{z \in \mathbf{C}^n_i : \langle z,z\rangle = 0\}.$$

A unit speed curve $z(s)$ lying in $\mathcal{L}C$ is called *Legendre* if $\langle \mathrm{i}z', z\rangle = 0$ holds identically. For a unit speed Legendre curve z in $\mathcal{L}C$, we have

$$\langle z,z\rangle = \left\langle z,z'\right\rangle = \left\langle z,\mathrm{i}z'\right\rangle = \left\langle \mathrm{i}z,z''\right\rangle = \left\langle z',z''\right\rangle = 0.$$

The Legendre curve z in $\mathcal{L}C$ is called *special Legendre* if $\langle \mathrm{i}z', z''\rangle = 0$ holds.

The *squared curvature* κ^2 of a unit speed special Legendre curve z is defined by $\kappa^2 = \langle z'', z''\rangle$ and its *Legendre torsion* $\hat{\tau}$ is defined by $\hat{\tau} = \epsilon_z \langle z'', \mathrm{i}z'''\rangle$, where $\epsilon_z = 1$ or -1 according to z is space-like or time-like (see [5, 6] for more details).

2.3. Existence and uniqueness theorems

We need the following results from [11] for Section 6.

Theorem A. *Let (M^n, g) be a simply connected Lorentzian n-manifold and TM denote the tangent bundle of M^n. If σ is a TM-valued symmetric bilinear form on M satisfying*

(1) $\langle \sigma(X,Y), Z\rangle$ *is totally symmetric,*

(2) $(\bar{\nabla}\sigma)(X,Y,Z) = \nabla_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ *is totally symmetric,*

(3) $R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + \sigma(\sigma(Y,Z),X) - \sigma(\sigma(X,Z),Y)$,

then there exists a Lagrangian isometric immersion L from (M,g) into the complete simply-connected Lorentzian complex space form $\tilde{M}^n_1(4c)$ whose second fundamental form h is given by $h(X,Y) = J\sigma(X,Y)$.

Theorem B. *Let $L_1, L_2\colon M^n \to \tilde M^n_1(4c)$ be Lagrangian isometric immersions of a Lorentzian n-manifold M^n with second fundamental forms h^1, h^2, respectively. If*

$$\left\langle h^1(X,Y), JL_{1\star}Z\right\rangle = \left\langle h^2(X,Y), JL_{2\star}Z\right\rangle,$$

for all vector fields X, Y, Z tangent to M^n, then there exists an isometry ϕ of $\tilde M^n_1(4c)$ such that $L_1 = L_2 \circ \phi$.

3. Basics Results for Lorentzian Surfaces

Let M be a Lorentzian surface in a Lorentzian Kähler surface $(\tilde M^2_1, g, J)$. For each tangent vector X of M, we put

$$JX = PX + FX, \tag{3.1}$$

where PX and FX are the tangential and the normal components of JX.

On the Lorentzian surface M there exists a *pseudo-orthonormal* local frame $\{e_1, e_2\}$ on M such that

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0,\ \langle e_1, e_2\rangle = -1. \tag{3.2}$$

For a pseudo-orthonormal frame $\{e_1, e_2\}$ satisfying (3.2), it follows from (3.1), (3.2), and $\langle JX, JY\rangle = \langle X, Y\rangle$ that

$$Pe_1 = (\sinh\alpha)e_1,\ \ Pe_2 = -(\sinh\alpha)e_2 \tag{3.3}$$

for some function α. This function α is called the *Wirtinger angle* of M.

When the Wirtinger angle α is constant on M, the Lorentzian surface M is called a *slant surface* (cf. [2, 10]). In this case, α is called the *slant angle*; the slant surface is then called α-slant.

A α-slant surface is called *Lagrangian* if $\alpha = 0$ (see [3, 4] for recent survey on Lagrangian surfaces). Obviously, slant surfaces (in particular, Lagrangian surfaces) in Lorentzian Kähler surfaces are Lorentzian surfaces.

If we put

$$e_3 = (\operatorname{sech}\alpha)Fe_1,\ \ e_4 = (\operatorname{sech}\theta)Fe_2, \tag{3.4}$$

then we find from (3.1)-(3.4) that

$$Je_1 = \sinh\alpha e_1 + \cosh\alpha e_3,\ \ Je_2 = -\sinh\alpha e_2 + \cosh\alpha e_4, \tag{3.5}$$

$$Je_3 = -\cosh\alpha e_1 - \sinh\alpha e_3,\ \ Je_4 = -\cosh\alpha e_2 + \sinh\alpha e_4, \tag{3.6}$$

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = 0, \ \ \langle e_3, e_4\rangle = -1. \tag{3.7}$$

We call such a frame $\{e_1, e_2, e_3, e_4\}$ an *adapted pseudo-orthonormal frame* for the Lorentzian surface M in $\tilde{M}^2_1$.

We need the following lemmas (see [8]).

Lemma 3.1. *If M is a Lorentzian surface in Lorentzian Kähler surface $\tilde{M}^2_1$, then every tangent plane of M is not J-invariant.*

Lemma 3.2. *If M is a Lorentzian surface in a Lorentzian Kähler surface $\tilde{M}^2_1$, then with respect to an adapted pseudo-orthonormal frame we have*

$$\nabla_X e_1 = \omega(X)e_1, \ \ \nabla_X e_2 = -\omega(X)e_2, \tag{3.8}$$

$$D_X e_3 = \Phi(X)e_3, \ \ D_X e_4 = -\Phi(X)e_4 \tag{3.9}$$

for some 1-forms ω, Φ on M.

It is easy to see that $\Phi = \omega$ holds for Lagrangian surfaces in $\tilde{M}^2_1$.

For a Lorentzian surface M in $\tilde{M}^2_1$, we put

$$h(e_i, e_j) = h^3_{ij}e_3 + h^4_{ij}e_4, \tag{3.10}$$

where e_1, e_2, e_3, e_4 is an adapted pseudo-orthonormal frame and h is the second fundamental form of M.

The following lemma is fundamental in our study.

Lemma 3.3. ([3.1]). *If M is a Lorentzian surface in a Lorentzian Kähler surface $\tilde{M}^2_1$, then with respect to an adapted pseudo-orthonormal frame $\{e_1, e_2, e_3, e_4\}$ we have*

$$\begin{cases} A_{e_3}e_j &= h^4_{j2}e_1 + h^4_{1j}e_2, \\ A_{e_4}e_j &= h^3_{j2}e_1 + h^3_{1j}e_2, \end{cases} \tag{3.11}$$

$$e_j\alpha = (\omega_j - \Phi_j)\coth\alpha - 2h^3_{1j}, \tag{3.12}$$

$$e_1\alpha = h^4_{12} - h^3_{11}, \ \ e_2\alpha = h^4_{22} - h^3_{12}, \tag{3.13}$$

$$\omega_j - \Phi_j = (h^3_{1j} + h^4_{j2})\tanh\alpha, \tag{3.14}$$

for $j = 1, 2$, where $\omega_j = \omega(e_j)$ and $\Phi_j = \Phi(e_j)$.

4. Fundamental Equations of Lorentzian Minimal Surfaces

The three fundamental equations of Gauss, Codazzi and Ricci are independent in general. However, for Lorentzian minimal surfaces in $\tilde{M}_1^2(4c)$ we have

Theorem 4.1. *The equation of Ricci is a consequence of the equations of Gauss and Codazzi for Lorentzian minimal surfaces in a Lorentzian complex space form* $\tilde{M}_1^2(4c)$.

Proof. Assume that M is a Lorentzian minimal surface in a Lorentzian complex space form $\tilde{M}_1^2(4c)$. Without loss of generality, we may assume that M is equipped with the Lorentzian metric tensor:

$$g = -m^2(x,y)(dx \otimes dy + dy \otimes dx) \tag{4.1}$$

for some positive function $m(x,y)$. Then we have

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \frac{2m_x}{m}\frac{\partial}{\partial x},\ \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0,\ \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{2m_y}{m}\frac{\partial}{\partial y}. \tag{4.2}$$

The Gaussian curvature G of M is given by

$$G = \frac{2mm_{xy} - 2m_x m_y}{m^4}. \tag{4.3}$$

If we put

$$e_1 = \frac{1}{m}\frac{\partial}{\partial x},\quad e_2 = \frac{1}{m}\frac{\partial}{\partial y}, \tag{4.4}$$

then $\{e_1, e_2\}$ is a pseudo-orthonormal frame satisfying (3.2). From (4.2) and (4.4) we find

$$\begin{aligned} \nabla_{e_1} e_1 &= \frac{m_x}{m^2} e_1,\ \nabla_{e_2} e_1 = -\frac{m_y}{m^2} e_1, \\ \nabla_{e_1} e_2 &= -\frac{m_x}{m^2} e_2,\ \nabla_{e_2} e_2 = \frac{m_y}{m^2} e_2. \end{aligned} \tag{4.5}$$

Let e_3, e_4 be the normal vector fields defined by (3.4). Then $\{e_1, e_2, e_3, e_4\}$ is an adapted pseudo-orthonormal frame. Since M is minimal and Lorentzian, it follows from (2.4) and (3.2) that

$$h(e_1,e_1) = \beta e_3 + \gamma e_4,\ h(e_1,e_2) = 0,\ \ h(e_2,e_2) = \lambda e_3 + \mu e_4 \tag{4.6}$$

for some functions $\beta, \gamma, \lambda, \mu$.

From (1.1), (3.2), (3.5) and (3.7), we find

$$\langle\, \tilde{R}(e_1, e_2)e_2, e_1 \,\rangle = c(3\sinh^2\alpha - 1). \tag{4.7}$$

In view of (1.1), (3.5), (3.7), (4.3), (4.6) and (4.7), equation (2.5) of Gauss can be expressed as

$$\gamma\lambda + \beta\mu = c(3\sinh^2\alpha - 1) + \frac{2(mm_{xy} - m_x m_y)}{m^4}. \tag{4.8}$$

Using Lemma 3.3 and (4.5) we find

$$\begin{aligned} D_{e_1}e_3 &= \left(\frac{m_x}{m^2} - \beta\tanh\alpha\right)e_3, \;\; D_{e_2}e_3 = -\left(\frac{m_y}{m^2} + \mu\tanh\alpha\right)e_3, \\ D_{e_1}e_4 &= \left(\beta\tanh\alpha - \frac{m_x}{m^2}\right)e_4, \;\; D_{e_2}e_3 = \left(\frac{m_y}{m^2} + \mu\tanh\alpha\right)e_4. \end{aligned} \tag{4.9}$$

It follows from (4.5), (4.6) and (4.9) that

$$\begin{aligned} (\bar\nabla_{e_1}h)(e_1, e_2) &= (\bar\nabla_{e_2}h)(e_1, e_2) = 0, \\ (\bar\nabla_{e_1}h)(e_2, e_2) &= \left(\frac{\lambda_x}{m} + \frac{\lambda m_x}{m^2} - \beta\lambda\tanh\alpha\right)e_3 \\ &\quad + \left(\frac{\mu_x}{m} - \frac{\mu m_x}{m^2} + \beta\mu\tanh\alpha\right)e_4 + \frac{2m_x}{m^2}(\lambda e_3 + \mu e_4), \\ (\bar\nabla_{e_2}h)(e_1, e_1) &= \left(\frac{\beta_y}{m} - \frac{\beta m_y}{m^2} - \beta\mu\tanh\alpha\right)e_3 \\ &\quad + \left(\frac{\gamma_y}{m} + \frac{\gamma m_y}{m^2} + \gamma\mu\tanh\alpha\right)e_4 + \frac{2m_y}{m^2}(\beta e_3 + \gamma e_4). \end{aligned} \tag{4.10}$$

On the other hand, we derive from (1.1) and (3.5) that

$$\begin{aligned} (\tilde{R}(e_1, e_2)e_2)^{\perp} &= 3c\,\sinh\alpha\cosh\alpha e_4, \\ (\tilde{R}(e_2, e_1)e_1)^{\perp} &= -3c\,\sinh\alpha\cosh\alpha e_3. \end{aligned} \tag{4.11}$$

Thus, by using (4.10), (4.11), and the equation of Codazzi we find

$$\lambda_x = \beta\lambda m\tanh\alpha - \frac{3\lambda m_x}{m}, \tag{4.12}$$

$$\mu_x = 3cm\sinh\alpha\cosh\alpha - \beta\mu m\tanh\alpha - \frac{\mu m_x}{m}, \tag{4.13}$$

$$\beta_y = -3cm\sinh\alpha\cosh\alpha + \beta\mu m\tanh\alpha - \frac{\beta m_y}{m}, \tag{4.14}$$

$$\gamma_y = -\gamma\mu m \tanh\alpha - \frac{3\gamma m_y}{m}. \tag{4.15}$$

Also, from (4.4) (4.5), (4.6), (4.9) and Lemma 3.3 we obtain

$$A_{e_3}e_1 = \gamma e_2,\ A_{e_3}e_2 = \mu e_1,\ A_{e_4}e_1 = \beta e_2,\ A_{e_4}e_2 = \lambda e_1, \tag{4.16}$$

$$\beta = -\frac{\alpha_x}{m},\ \ \mu = \frac{\alpha_y}{m}. \tag{4.17}$$

Substituting (4.17) into equation (4.8) of Gauss yields

$$\gamma\lambda = c(3\sinh^2\alpha - 1) + \frac{2(mm_{xy} - m_x m_y)}{m^4} + \frac{\alpha_x\alpha_y}{m^2}. \tag{4.18}$$

Now, by applying (1.1), (3.5) and (3.6), we get

$$\langle\ \tilde{R}(e_1, e_2)e_3, e_4\ \rangle = c(3\sinh^2\alpha + 1). \tag{4.19}$$

On the other hand, by using (4.5), (4.9), (4.16), and (4.17), we find

$$\left\langle R^D(e_1, e_2)e_3, e_4\right\rangle = \frac{2(mm_{xy} - m_x m_y)}{m^4} + \frac{2\alpha_x\alpha_y}{m^2}\operatorname{sech}^2\alpha + \frac{2\alpha_{xy}}{m^2}\tanh\alpha, \tag{4.20}$$

$$\langle [A_{e_3}, A_{e_4}]e_1, e_2\rangle = \gamma\lambda + \frac{\alpha_x\alpha_y}{m^2}. \tag{4.21}$$

Therefore, the equation of Ricci is given by

$$\begin{aligned}\frac{2(mm_{xy} - m_x m_y)}{m^4} + \frac{2\alpha_x\alpha_y}{m^2}\operatorname{sech}^2\alpha + \frac{2\alpha_{xy}}{m^2}\tanh\alpha \\ = c(3\sinh^2\alpha + 1) + \gamma\lambda + \frac{\alpha_x\alpha_y}{m^2}.\end{aligned} \tag{4.21}$$

On the other hand, we derive from (4.14) and (4.17) that

$$\alpha_{xy} = \alpha_x\alpha_y\tanh\alpha + 3cm^2\sinh\alpha\cosh\alpha. \tag{4.23}$$

After substituting (4.23) into (4.22) we know that equation (4.22) of

$$\begin{aligned}\frac{2(mm_{xy} - m_x m_y)}{m^4} + \frac{2\alpha_x\alpha_y}{m^2}\operatorname{sech}^2\alpha + \frac{2\alpha_x\alpha_y}{m^2}\tanh^2\alpha \\ = c(1 - 3\sinh^2\alpha) + \gamma\lambda + \frac{\alpha_x\alpha_y}{m^2},\end{aligned} \tag{4.24}$$

Since this equation of Ricci can be simplified as equation (4.18) of Gauss, we conclude that the equation of Ricci is a consequence of Gauss and Codazzi for

Lorentzian minimal surfaces in Lorentzian complex space forms. This completes the proof of the theorem. ∎

Corollary 4.1. *Every minimal slant surface in a Lorentzian complex space form* $\tilde{M}_1^2(4c)$ *with* $c \neq 0$ *is Lagrangian.*

Proof. If M is a minimal slant surface in a Lorentzian complex space form $\tilde{M}_1^2(4c)$, then α is constant. So, by applying (4.13) we have $3mc \sinh\alpha \cos\alpha = 0$. But this is impossible unless $\alpha = 0$ or $c = 0$. Therefore, if $c \neq 0$, then the surface is Lagrangian. ∎

In view of (4.17), equations (4.12)-(4.15) reduce to

$$\lambda_x = -\lambda(\ln(m^3\cosh\alpha))_x, \;\; \gamma_y = -\gamma(\ln(m^3\cosh\alpha))_y, \tag{4.25}$$

$$\alpha_{xy} = \alpha_x\alpha_y \tanh\alpha + 3cm^2 \sinh\alpha\cosh\alpha. \tag{4.26}$$

We derive from (4.25) that

$$\gamma = \frac{\varphi(x)\,\mathrm{sech}\,\alpha}{m^3}, \;\; \lambda = \frac{\psi(y)\,\mathrm{sech}\,\alpha}{m^3} \tag{4.27}$$

for some functions φ, ψ.

In view of (4.17) and (4.27), equation (4.8) of Gauss becomes

$$\alpha_x\alpha_y = cm^2(1 - 3\sinh^2\alpha) - \frac{2mm_{xy} - 2m_x m_y}{m^2} + \frac{\varphi(x)\psi(y)\,\mathrm{sech}^2\alpha}{m^4}. \tag{4.28}$$

Hence, the second fundamental form of M in $\tilde{M}_1^2(4c)$ satisfies

$$\begin{aligned} &h\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big) = -m\alpha_x e_3 + \frac{\varphi(x)}{m}\,\mathrm{sech}\,\alpha e_4, \\ &h\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big) = 0, \\ &h\Big(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big) = \frac{\psi(y)}{m}\,\mathrm{sech}\,\alpha e_3 + m\alpha_y e_4. \end{aligned} \tag{4.29}$$

In term of (4.29), the equation of Gauss and Codazzi are given by (4.26) and (4.28). Consequently, by applying Theorem 4.1 together with the fundamental existence theorem of submanifolds, we obtain the following.

Corollary 4.2. *Suppose that* $\alpha(x, y)$ *and* $m(x, y) \neq 0$ *are solutions of the following PDE system:*

$$\alpha_{xy} = \alpha_x\alpha_y \tanh\alpha + 3cm^2 \sinh\alpha\cosh\alpha, \tag{4.30}$$

$$\alpha_x\alpha_y = cm^2(1-3\sinh^2\alpha) - \frac{2mm_{xy}-2m_xm_y}{m^2} + \frac{\varphi(x)\psi(y)\operatorname{sech}^2\alpha}{m^4} \tag{4.31}$$

for some functions $\varphi(x)$ and $\psi(y)$ defined on open intervals I_1 and I_2, respectively. Let g_m be the Lorentzian metric on $I_1 \times I_2$ defined by

$$g_m = -m^2(dx \otimes dy + dy \otimes dx).$$

Then there exists a Lorentzian minimal immersion $\phi : (I_1 \times I_2, g_m) \to \tilde{M}^2_1(4c)$ with Wirtinger angle α.

5. Classification OF Lorentzian Minimal Surfaces in $\mathbf{C}^2_1$

In this section we completely classify Lorentzian minimal surfaces in $\mathbf{C}^2_1$.

Theorem 5.1. *Let $z(x)$ and $w(y)$ be two null curves defined on open intervals I_1 and I_2 respectively in the Lorentzian complex plane $\mathbf{C}^2_1$. If $\langle z(x), w(y)\rangle \neq 0$ for $(x, y) \in I_1 \times I_2$, then*

$$\psi(x, y) = z(x) + w(y) \tag{5.1}$$

defines a Lorentzian minimal surface in $\mathbf{C}^2_1$.

Conversely, locally every Lorentzian minimal surface in $\mathbf{C}^2_1$ is congruent to the translation surface defined above.

Proof. Let $z(x), x \in I_1$, and $w(y), y \in I_2$, be null curves in the Lorentzian complex plane $\mathbf{C}^2_1$ satisfying $\langle z(x), w(y)\rangle \neq 0$ for $(x, y) \in I_1 \times I_2$. Let ψ be the map defined by (5.1). Then we have

$$\psi_x = z'(x), \;\; \psi_y = w'(y), \tag{5.2}$$

$$\psi_{xx} = z''(x), \;\; \psi_{xy} = 0, \;\; \psi_{yy} = w''(y). \tag{5.3}$$

From (5.2) and the assumption on z, w we find $\langle z', z'\rangle = \langle w', w'\rangle = 0$. Thus, the induced metric of ψ is the following Lorentzian metric:

$$g = \big\langle z'(x), w'(y)\big\rangle\,(dx \otimes dy + dy \otimes dx). \tag{5.4}$$

Since $\psi_{xy} = 0$, it follows from (5.4) and the formula of Gauss that the trace of the second fundamental form of ψ vanishes identically. Hence, ψ defines a Lorentzian minimal surface in $\mathbf{C}^2_1$.

Conversely, let us assume that M is a Lorentzian minimal surface in $\mathbf{C}^2_1$. We may suppose that locally M is an open portion of the xy-plane equipped with the Lorentzian metric :

$$g = -m^2(x,y)(dx \otimes dy + dy \otimes dx) \tag{5.5}$$

for some nonzero function $m(x,y)$. From (5.5) we derive that

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \frac{2m_x}{m}\frac{\partial}{\partial x},\ \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0,\ \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{2m_y}{m}\frac{\partial}{\partial y}. \tag{5.6}$$

If we put

$$e_1 = \frac{1}{m}\frac{\partial}{\partial x},\ \ e_2 = \frac{1}{m}\frac{\partial}{\partial y}, \tag{5.7}$$

then $\{e_1, e_2\}$ is a pseudo-orthonormal frame in M satisfying (3.2). Let e_3, e_4 be the normal vector fields defined by (3.4).

Since M is minimal and Lorentzian, it follows from (2.4) and (3.2) that

$$h(e_1,e_1) = \beta e_3 + \gamma e_4,\ h(e_1,e_2) = 0,\ \ h(e_2,e_2) = \lambda e_3 + \mu e_4 \tag{5.8}$$

for some functions $\beta, \gamma, \lambda, \mu$.

On the other hand, we obtain from (3.5), (5.7) that

$$e_3 = \frac{1}{m}(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)\psi_x,\ e_4 = \frac{1}{m}(\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha)\psi_y. \tag{5.9}$$

Hence, (5.6)-(5.9) and the formula of Gauss yield

$$\begin{aligned} \psi_{xx} &= \frac{1}{m}\big(2m_x + \beta(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)\big)\psi_x + \frac{\gamma}{m}(\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha)\psi_y, \\ \psi_{xy} &= 0, \\ \psi_{yy} &= \frac{\lambda}{m}(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)\psi_x + \frac{1}{m}\big(2m_y + \mu(\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha)\big)\psi_y. \end{aligned} \tag{5.10}$$

Solving the second equation in (5.10) gives

$$\psi(x,y) = z(x) + w(y) \tag{5.11}$$

for some $\mathbf{C}^2_1$-valued functions $z(x), w(y)$. Thus, we find from (5.5) and (5.1) that

$$\big\langle z'(x), z'(x)\big\rangle = \big\langle w'(y), w'(y)\big\rangle = 0,\ \ \big\langle z'(x), w'(y)\big\rangle = -m^2(x,y). \tag{5.12}$$

These imply that $z(x), w(y)$ are null curves satisfying $\langle z', w'\rangle \neq 0$. Consequently, every Lorentzian minimal surface in $\mathbf{C}^2_1$ is locally congruent to the translation surface described in the theorem. ∎

6. Lagrangian Minimal Surfaces of Klein-gordon Type

In this section, we give the following existence results.

Proposition 6.1. *Let F be a nonconstant real-valued function defined on a simply-connected open subset U of $\mathbf{R}^2$ which satisfies the following nonlinear Klein-Gordon equation:*

$$(\ln F)_{uv} = -\frac{1}{F} - F^2. \tag{6.1}$$

Put $g_F = -F^{-1}(du \otimes dv + dv \otimes du)$. Then, up to rigid motions, there exists a unique Lagrangian minimal immersion $L_F : (U, g_F) \to CP^2_1(4)$ whose second fundamental form satisfies

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = FJ\frac{\partial}{\partial v}, \;\; h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0, \;\; h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = FJ\frac{\partial}{\partial u}. \tag{6.2}$$

Proof. A direct computation shows that the Levi-Civita connection of g_F satisfies

$$\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u} = -(\ln F)_u\frac{\partial}{\partial u}, \; \nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial v} = 0, \; \nabla_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v} = -(\ln F)_v\frac{\partial}{\partial v}. \tag{6.3}$$

If we define a symmetric bilinear form σ by

$$\sigma\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = F\frac{\partial}{\partial v}, \;\; \sigma\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0, \;\; \sigma\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = F\frac{\partial}{\partial u}, \tag{6.4}$$

then it follows from (6.1), (6.3), (6.4) and the definitions of g_F that $\langle \sigma(X,Y), Z\rangle$ and $(\bar{\nabla}\sigma)(X,Y,Z)$ are totally symmetric. Moreover, a direct computation shows that the curvature tensor R and σ satisfy condition (iii) of Theorem A in section 2. Therefore, according to Theorems A and B, up to rigid motions there exists a unique Lagrangian immersion $L_F : (U, g_F) \to CP^2_1(4)$ whose second fundamental form is given by $J\sigma$.

The minimality of the immersion follows from the expression of g_F and (6.2). ∎

We call such a Lagrangian minimal surface associated with a solution of the nonlinear Klein-Gordon equation (6.1) a *Lagrangian minimal surface of Klein-Gordon type in $CP^2_1(4)$*.

Similarly, we have the following

Proposition 6.2. *Let $K(u,v)$ be a nonconstant real-valued function on a simply-connected open subset U of $\mathbf{R}^2$ which satisfies the nonlinear Klein-Gordon equation:*

$$(\ln K)_{uv} = \frac{1}{K} - K^2. \tag{6.5}$$

Put $g_K = -K^{-1}(du \otimes dv + dv \otimes du)$. Then, up to rigid motions, there exists a unique Lagrangian minimal immersion $L_K : (U, g_K) \to CH^2_1(-4)$ whose second fundamental form satisfies

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = KJ\frac{\partial}{\partial v}, \;\; h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0, \;\; h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = KJ\frac{\partial}{\partial u}. \tag{6.6}$$

Proof. This can be done in the same way as Proposition 6.1. ∎

Similarly, we call a Lagrangian minimal surface in $CH^2_1(4)$ associated with a solution of the nonlinear Klein-Gordon equation (6.5) a *Lagrangian minimal surface of Klein-Gordon type in $CH^2_1(-4)$*.

Remark 6.1. The nonlinear Klein-Gordon equations (6.1) and (6.5) admit infinitely many solutions. Consequently, there exist infinitely many Lagrangian minimal surfaces of Klein-Gordon type in $CP^2_1(4)$ and in $CH^2_1(-4)$.

7. Classification of Minimal Slant Surfaces in CP^2_1

Let $\pi : S^5_2(1) \to CP^2_1(4)$ denote the Hopf fibration.

Theorem 7.1. *Let $L : M \to CP^2_1(4)$ be a minimal slant surface in the Lorentzian complex projective plane $CP^2_1(4)$. Then we have:*

(1) *If M is of constant curvature, then M is congruent to one of the following three types of surfaces:*

(1.a) *a totally geodesic Lagrangian surface of $CP^2_1(4)$;*

(1.b) *a curvature one Lagrangian minimal surface defined by $\pi \circ \tilde{L}$ with*

$$\tilde{L}(x,y) = z'(x) - \frac{2z(x)}{x+y}, \tag{7.1}$$

where $z(x), x \in I$, is a unit speed space-like special Legendre curve lying in the light cone $\mathcal{LC} \subset \mathbf{C}^3_1$ with null squared curvature $\kappa^2(s)$, i.e., $\langle z''(x), z''(x)\rangle = 0$ on I;

$(1.c)$ *a flat Lagrangian minimal surface defined by* $\pi \circ \tilde{L}$ *with*

$$\tilde{L}(x,y) = \frac{1}{\sqrt{3}}\left(\sqrt{2}e^{\frac{\mathrm{i}}{2a}(x-a^2y)}\cosh\left(\frac{\sqrt{3}}{2a}(x+a^2y)\right), e^{\frac{\mathrm{i}}{a}(a^2y-x)}, \right. \tag{7.2}$$
$$\left. \sqrt{2}e^{\frac{\mathrm{i}}{2a}(x-a^2y)}\sinh\left(\frac{\sqrt{3}}{2a}(x+a^2y)\right)\right),$$

where a *is a nonzero real number.*

(2) *If* M *contains no open subset of constant curvature, then* M *is a Lagrangian minimal surface of Klein-Gordon type in* $CP^2_1(4)$.

Proof. Let $L : M \to CP^2_1(4)$ be a minimal slant surface. It follows from Corollary 4.1 that M is Lagrangian. Thus, we get $\alpha = 0$.

As in section 4, we may assume that M is equipped with the Lorentzian metric:

$$g = -m^2(x,y)(dx \otimes dy + dy \otimes dx) \tag{7.3}$$

for some positive function $m(x,y)$. Then we have

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \frac{2m_x}{m}\frac{\partial}{\partial x},\ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0,\ \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \frac{2m_y}{m}\frac{\partial}{\partial y}. \tag{7.4}$$

The Gaussian curvature of M is then given by

$$G = \frac{2mm_{xy} - 2m_xm_y}{m^4}. \tag{7.5}$$

If we put

$$e_1 = \frac{1}{m}\frac{\partial}{\partial x},\ \ e_2 = \frac{1}{m}\frac{\partial}{\partial y}, \tag{7.6}$$

then $\{e_1, e_2\}$ is a pseudo-orthonormal frame satisfying (3.2). Since M is minimal and Lagrangian, it follows from (2.4) and (3.2) that

$$h(e_1,e_1) = \beta e_3 + \gamma e_4,\ h(e_1,e_2) = 0,\ \ h(e_2,e_2) = \lambda e_3 + \mu e_4 \tag{7.7}$$

for some functions $\beta, \gamma, \lambda, \mu$, where $e_3 = Je_1$ and $e_4 = Je_2$.

Since $\alpha = 0$, we derive from section 4 that

$$\beta = \mu = 0, \tag{7.8}$$

$$\lambda_x = -\frac{3\lambda m_x}{m},\ \ \gamma_y = -\frac{3\gamma m_y}{m} \tag{7.9}$$

$$\gamma\lambda = \frac{2mm_{xy} - 2m_x m_y}{m^4} - 1. \tag{7.10}$$

Case (a). $\gamma = \lambda = 0$. In this case, M is a totally geodesic Lagrangian surface which has constant curvature one. This gives case (1.a) of the theorem.

Case (b). $\gamma \neq 0$ *and* $\lambda = 0$. In this case, M is of constant curvature one. Thus, we may assume that the metric tensor of M is given by

$$g = \frac{-2(dx \otimes dy + dy \otimes dx)}{(x+y)^2}. \tag{7.11}$$

So, we have

$$m = \frac{\sqrt{2}}{x+y}. \tag{7.12}$$

Since $\alpha = \beta = \lambda = \mu = 0$, (7.7) and (7.9) reduce to

$$h(e_1, e_1) = \gamma e_4, \; h(e_1, e_2) = h(e_2, e_2) = 0, \tag{7.13}$$

$$(\ln \gamma)_y = \frac{3}{x+y}. \tag{7.14}$$

Solving (7.14) gives

$$\gamma = \varphi(x)(x+y)^3 \tag{7.15}$$

for some functions φ. Now, it follows from (7.4), (7.1), (7.12), (7.13), (7.15) and the formula of Gauss that the horizontal lift $\tilde{L} : M \to S^5_2(1)$ of L satisfies

$$\begin{aligned} \tilde{L}_{xx} &= \frac{-2\tilde{L}_x}{x+y} + \mathrm{i}\sqrt{2}\varphi(x)(x+y)^2 \tilde{L}_y, \\ \tilde{L}_{xy} &= \frac{2\tilde{L}}{(x+y)^2}, \;\; \tilde{L}_{yy} = \frac{-2\tilde{L}_y}{x+y}. \end{aligned} \tag{7.16}$$

Solving the last equation in (7.16) gives

$$\tilde{L}(x, y) = w(x) - \frac{2z(x)}{x+y} \tag{7.17}$$

for some vector functions $z(x), w(x)$. Substituting this into the second equation in (7.16) gives $w(x) = z'(x)$. Hence, (7.17) becomes

$$\tilde{L}(x, y) = z'(x) - \frac{2z(x)}{x+y} \tag{7.18}$$

After substituting this into the first equation in (7.16), we find

$$z'''(x) = 2\sqrt{2}\,\mathrm{i}\varphi(x)z(x). \tag{7.19}$$

Since $\langle \tilde{L}, \tilde{L}\rangle = 1$, we derive from (7.18) that

$$\langle z(x), z(x)\rangle = 0, \;\; \big\langle z'(x), z'(x)\big\rangle = 1. \tag{7.20}$$

Therefore, z is a unit speed space-like curve lying in the light cone $\mathcal{LC}$ of $\mathbf{C}^3_1$.

On the other hand, because $\tilde{L}$ is a horizontal lift of a Lagrangian immersion, we also have $\langle \tilde{L}_x, \mathrm{i}\tilde{L}_y\rangle = 0$. Hence, we may obtain from (7.19) that

$$\big\langle z(x), \mathrm{i}z'(x)\big\rangle = 0, \;\; \big\langle z(x), \mathrm{i}z''(x)\big\rangle = 0. \tag{7.21}$$

Differentiating the second equation in (7.2) yields

$$\big\langle z'(x), \mathrm{i}z''(x)\big\rangle = \big\langle iz(x), z'''(x)\big\rangle. \tag{7.22}$$

Combining this with (7.19) and using (7.20) give $\langle z'(x), \mathrm{i}z''(x)\rangle = 0$. Thus, $z(x)$ is a special Legendre curve in $\mathcal{LC}$.

Finally, from $\langle \tilde{L}_x, \tilde{L}_x\rangle = 0$ and (7.18), we find

$$\big\langle z''(x), z''(x)\big\rangle = 0. \tag{7.23}$$

This shows that the squared curvature κ^2 of $z(x)$ vanishes identically. Consequently, we obtain case (1.b) of the theorem.

Case (c). $\gamma = 0$ and $\lambda \neq 0$. By interchanging x and y, this reduces to case (b).

Case (d). $\gamma\lambda \neq 0$. In this case, (7.7) and (7.9) reduce to

$$h(e_1, e_1) = \gamma e_4, \;\; h(e_1, e_2) = 0, \;\; h(e_2, e_2) = \lambda e_3, \tag{7.24}$$

$$(\ln \lambda)_x = -3(\ln m)_x, \;\; (\ln \gamma)_y = -3(\ln m)_y. \tag{7.25}$$

Solving (7.25) gives

$$\gamma = \frac{f(x)}{m^3}, \;\; \lambda = \frac{k(y)}{m^3} \tag{7.26}$$

for some functions $f(x), k(y)$. Since $\gamma\lambda \neq 0$, we must have $f(x)k(y) \neq 0$.

Substituting (7.26) into (7.10) gives

$$f(x)k(y) = 2m^2(mm_{xy} - m_x m_y) - m^6. \tag{7.27}$$

Case (d.1). *M is of constant curvature $G = \varepsilon$.* It follows from (7.5) and (7.27) that

$$f(x)k(y) = (\varepsilon - 1)m^6. \tag{7.28}$$

Since $fk \neq 0$, (7.28) shows that $\varepsilon \neq 1$. Hence, we have $m^6 = f(x)k(y)/(\varepsilon - 1)$. Therefore, $m(x, y)$ is the product of two functions of single variable, which implies that $mm_{xy} = m_x m_y$. Hence, it follows from (7.5) that $G = 0$. Consequently, the surface is given by case (1.c) of the theorem according to [8].

Case (d.2). *M contains no open subset of constant curvature.* It follows from (7.6), (7.24) and (7.25) that

$$h(e_1, e_1) = \gamma e_4, \;\; h(e_1, e_2) = 0, \;\; h(e_2, e_2) = \lambda e_3, \tag{7.29}$$

$$e_2\gamma = 3\gamma\omega_2, \;\; e_1\lambda = -3\lambda\omega_1, \tag{7.30}$$

where the connection form ω is defined in Lemma **??**. Let us put

$$\eta = \gamma^{1/3}, \;\; \delta = \lambda^{1/3}. \tag{7.31}$$

By applying Lemma 3.2 we find $[e_1/\eta, e_2/\delta] = 0$. Hence, there exist coordinates u, v such that

$$e_1 = \eta \frac{\partial}{\partial u}, \;\; e_2 = \delta \frac{\partial}{\partial v}. \tag{7.32}$$

So, we know from (3.2) and (7.32) that the metric tensor is given by

$$g = -\frac{du \otimes dv + dv \otimes du}{F}, \;\; F = \eta\delta. \tag{7.33}$$

Since M has nonconstant curvature, $\eta\delta$ is a nonconstant function. Hence, the Levi-Civita connection satisfies

$$\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = -(\ln F)_u \frac{\partial}{\partial u}, \; \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v} = 0, \; \nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v} = -(\ln F)_v \frac{\partial}{\partial v}. \tag{7.34}$$

Therefor, by applying (7.24), (7.3), (7.32), (7.34) and the formula of Gauss, we obtain the following PDE system for the horizontal lift $\tilde{L} : M \to S^5_2(1)$:

$$\begin{aligned} \tilde{L}_{uu} &= -(\ln F)_u \tilde{L}_u + \mathrm{i} F \tilde{L}_v, \;\; \tilde{L}_{uv} = \frac{\tilde{L}}{F}, \\ \tilde{L}_{vv} &= \mathrm{i} F \tilde{L}_u - (\ln F)_v \tilde{L}_v. \end{aligned} \tag{7.35}$$

The compatibility condition of this system is given by the nonlinear Klein-Gordon equation:

$$(\ln F)_{uv} = -\frac{1}{F} - F^2. \tag{7.36}$$

Hence, the surface is a Lagrangian minimal surface of Klein-Gordon type in $CP^2_1(4)$ as described in Proposition 6.1. Consequently, we obtain case (2). ■

Example 7.1. There exist infinitely many unit speed space-like special Legendre curve lying in the light cone $\mathcal{LC} \subset \mathbf{C}^3_1$ with null squared curvature. The simplest such examples are the following.

$$z(x) = \left(a + \left(\frac{1}{4a} + \mathrm{i}b\right)s^2, a - \left(\frac{1}{4a} - \mathrm{i}b\right)s^2, s\right),$$

where a, b are nonzero real numbers. It is easy to check that this special Legendre curve has null Legendre torsion, i.e., $\hat{\tau} = 0$.

Example 7.2. Another example of unit speed space-like special Legendre curve lying in the light cone $\mathcal{LC} \subset \mathbf{C}^3_1$ with null squared curvature is the following.

$$z(x) = \frac{1}{\sqrt{3}}\left(e^{\frac{\mathrm{i}s}{2}}\cosh\left(\frac{\sqrt{3}s}{2}\right) - \mathrm{i}\sqrt{3}e^{\frac{\mathrm{i}s}{2}}\sinh\left(\frac{\sqrt{3}s}{2}\right), 2e^{\frac{\mathrm{i}s}{2}}\sinh\left(\frac{\sqrt{3}s}{2}\right), e^{-\mathrm{i}s}\right).$$

This special Legendre curve has constant Legendre torsion $\hat{\tau} = -1$.

8. Classification of Minimal Slant Surfaces in CH^2_1

Let $\pi : H^5_2(-1) \to CH^2_1(-4)$ denote the Hopf fibration.

Theorem 8.1. *Let $L : M \to CH^2_1(-4)$ be a minimal slant surface in the Lorentzian complex hyperbolic plane $CH^2_1(-4)$. Then we have:*

(1) *If M is of constant curvature, then M is congruent to one of the following three types of surfaces:*

  (1.a) *a totally geodesic Lagrangian surface of $CH^2_1(-4)$;*

  (1.b) *a Lagrangian minimal surface of constant curvature -1 given by $\pi \circ \tilde{L}$ with*

$$\tilde{L}(x, y) = z'(x) - \sqrt{2}z(x)\tanh\left(\frac{x+y}{\sqrt{2}}\right), \tag{8.1}$$

where $z(x), x \in I$, is a unit speed time-like special Legendre curve in the light cone $\mathcal{LC} \subset \mathbf{C}_2^3$ with constant squared curvature $\kappa^2 = 2$;

$(1.c)$ *a flat Lagrangian minimal surface defined by $\pi \circ \tilde{L}$ with*

$$\tag{8.2} \begin{aligned} \tilde{L}(x,y) = \frac{1}{\sqrt{3}} \Bigg(& \sqrt{2} e^{-\frac{\mathrm{i}}{2a}(x+a^2 y)} \cosh\left(\frac{\sqrt{3}}{2a}(x - a^2 y) \right), e^{\mathrm{i}(ay+\frac{x}{a})}, \\ & \sqrt{2} e^{-\frac{\mathrm{i}}{2a}(x+a^2 y)} \sinh\left(\frac{\sqrt{3}}{2a}(x - a^2 y) \right) \Bigg), \end{aligned}$$

where a is a nonzero real number.

(2) *If M contains no open subset of constant curvature, then M is a Lagrangian minimal surface of Klein-Gordon type in $CH_1^2(-4)$.*

Proof. Assume that M is a minimal slant surface in $CH_1^2(-4)$. Then, according to Corollary 4.1, M is Lagrangian. Thus, we get $\alpha = 0$.

We may assume that M is equipped with the Lorentzian metric:

$$\tag{8.3} g = -m^2(x,y)(dx \otimes dy + dy \otimes dx)$$

for some positive function $m(x,y)$. So, we have (4.2) and (4.3). If we put

$$\tag{8.4} e_1 = \frac{1}{m}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{m}\frac{\partial}{\partial y}$$

as before, then $\{e_1, e_2\}$ is a pseudo-orthonormal frame satisfying (3.2). Since M is minimal and Lorentzian, we have as before that

$$\tag{8.5} h(e_1,e_1) = \gamma e_4, \; h(e_1,e_2) = 0, \;\; h(e_2,e_2) = \lambda e_3,$$

$$\tag{8.6} \lambda_x = -\frac{3\lambda m_x}{m}, \;\; \gamma_y = -\frac{3\gamma m_y}{m}$$

$$\tag{8.7} \gamma\lambda = \frac{2mm_{xy} - 2m_x m_y}{m^4} + 1$$

for some functions γ, λ, where $e_3 = Je_1$ and $e_4 = Je_2$.

Case (a). $\gamma = \lambda = 0$. In this case, M is a totally geodesic Lagrangian surface which has constant curvature -1. Thus, we get case (1.a) of the theorem.

Case (b). $\gamma \neq 0$ *and* $\lambda = 0$ *on* M. In this case, M is of constant curvature -1. Thus, we may assume that the metric tensor is given by

$$g = -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)(dx \otimes dy + dy \otimes dx). \tag{8.8}$$

So, we have

$$m = \operatorname{sech}\left(\frac{x+y}{\sqrt{2}}\right). \tag{8.9}$$

Since $\lambda = 0$, (8.5) and (8.6) reduce to

$$h(e_1, e_1) = \gamma e_4,\ h(e_1, e_2) = h(e_2, e_2) = 0, \tag{8.10}$$

$$(\ln \gamma)_y = \frac{3}{\sqrt{2}} \tanh\left(\frac{x+y}{\sqrt{2}}\right). \tag{8.11}$$

Solving (8.1) gives

$$\gamma = \varphi(x) \cosh^3\left(\frac{x+y}{\sqrt{2}}\right) \tag{8.12}$$

for some nonzero functions φ.

It follows from (4.2), (8.8), (8.9), (8.10), (8.12) and the formula of Gauss that the horizontal lift $L : M \to H^5_2(-1)$ of M in $CH^2_1(-4)$ satisfies

$$\begin{aligned} \tilde{L}_{xx} &= -\sqrt{2} \tanh\left(\frac{x+y}{\sqrt{2}}\right) \tilde{L}_x + \mathrm{i}\varphi(x) \cosh^2\left(\frac{x+y}{\sqrt{2}}\right) \tilde{L}_y, \\ \tilde{L}_{xy} &= -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) \tilde{L}, \\ \tilde{L}_{yy} &= -\sqrt{2} \tanh\left(\frac{x+y}{\sqrt{2}}\right) \tilde{L}_y. \end{aligned} \tag{8.13}$$

Solving the last two equations in (8.13) gives

$$\tilde{L} = z'(x) - \sqrt{2} z(x) \tanh\left(\frac{x+y}{\sqrt{2}}\right) \tag{8.14}$$

After substituting this into the first equation in (8.13), we find

$$z'''(x) - 2z'(x) + \mathrm{i}\varphi(x) z(x) = 0. \tag{8.15}$$

Since $\langle \tilde{L}, \tilde{L} \rangle = -1$, we derive from (8.14) that

$$\langle z(x), z(x) \rangle = 0,\ \langle z'(x), z'(x) \rangle = -1. \tag{8.16}$$

Hence, z is a unit speed time-like curve lying in the light cone $\mathcal{LC}$ of $\mathbf{C}^3_2$.

On the other hand, because $\tilde{L}$ is a horizontal lift of a Lagrangian immersion, we also have $\langle \tilde{L}_x, \mathrm{i}\tilde{L}_y \rangle = 0$. Hence, we may obtain from (8.15) that

$$\big\langle z(x), \mathrm{i}z'(x) \big\rangle = 0, \;\; \big\langle z(x), \mathrm{i}z''(x) \big\rangle = 0. \tag{8.17}$$

Differentiating the second equation in (8.17) yields

$$\big\langle z'(x), \mathrm{i}z''(x) \big\rangle = \big\langle iz(x), z'''(x) \big\rangle . \tag{8.18}$$

Combining this with (8.15) and using (8.16) and (8.17) give $\langle z'(x), \mathrm{i}z''(x) \rangle = 0$. Therefore, $z(x)$ is a special Legendre curve in $\mathcal{LC}$.

Finally, from $\langle \tilde{L}_x, \tilde{L}_x \rangle = 0$, (8.14), and (8.16), we find

$$\kappa^2 = \big\langle z''(x), z''(x) \big\rangle = 2. \tag{8.19}$$

Consequently, we obtain case (1.b) of the theorem.

Case (c). $\gamma = 0$ and $\lambda \neq 0$. By interchanging x and y, this reduces to case (b).

Case (d). $\gamma\lambda \neq 0$. In this case, (8.5) and (8.6) reduce to

$$h(e_1, e_1) = \gamma e_4, \;\; h(e_1, e_2) = 0, \;\; h(e_2, e_2) = \lambda e_3, \tag{8.20}$$

$$(\ln \lambda)_x = -3(\ln m)_x, \;\; (\ln \gamma)_y = -3(\ln m)_y. \tag{8.21}$$

Solving (8.2) gives

$$\gamma = \frac{f(x)}{m^3}, \;\; \lambda = \frac{k(y)}{m^3} \tag{8.22}$$

for some functions $f(x), k(y)$. Since $\gamma\lambda \neq 0$, we have $f(x)k(y) \neq 0$.

Substituting (8.22) into (8.7) gives

$$f(x)k(y) = 2m^2(mm_{xy} - m_x m_y) + m^6. \tag{8.23}$$

Case (d.1). *M is of constant curvature ε.* It follows from (4.3) and (8.23) that

$$f(x)k(y) = (\varepsilon + 1)m^6. \tag{8.24}$$

Since $fk \neq 0$, we have $\varepsilon \neq -1$. Thus, m is the product of two functions of one variable. So, we get $mm_{xy} = m_x m_y$. Hence, it follows from (4.3) that $G = 0$. Consequently, the surface is given by case (1.c) of the theorem according to [8].

Case (d.2). *M contains no open subset of constant curvature.* It follows from (8.20) and (8.2) that

$$h(e_1, e_1) = \gamma e_4, \; h(e_1, e_2) = 0, \;\; h(e_2, e_2) = \lambda e_3, \tag{8.25}$$

$$e_2\gamma = 3\gamma\omega_2, \;\; e_1\lambda = -3\lambda\omega_1. \tag{8.26}$$

Let us put $\eta = \gamma^{1/3}, \delta = \lambda^{1/3}$. Then, by Lemma **??** we get $[e_1/\eta, e_2/\delta] = 0$. Thus, there exist coordinates u, v such that

$$e_1 = \eta\frac{\partial}{\partial u}, \;\; e_2 = \delta\frac{\partial}{\partial v}. \tag{8.27}$$

From (3.2) and (8.27), we know that the metric tensor is given by

$$g = -\frac{du \otimes dv + dv \otimes du}{K}, \;\; K = \eta\delta. \tag{8.28}$$

Since M has nonconstant curvature, $\eta\delta$ is a nonconstant function.

From (8.28) we derive that

$$\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u} = -(\ln K)_u\frac{\partial}{\partial u}, \; \nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial v} = 0, \; \nabla_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v} = -(\ln K)_v\frac{\partial}{\partial v}. \tag{8.29}$$

Hence, by applying (8.20), (8.27), (8.29) and the formula of Gauss, we obtain the following PDE system:

$$\begin{aligned} \tilde{L}_{uu} &= -(\ln K)_u\tilde{L}_u + \mathrm{i}K\tilde{L}_v, \; \tilde{L}_{uv} = -\frac{\tilde{L}}{K}, \\ \tilde{L}_{vv} &= \mathrm{i}K\tilde{L}_u - (\ln K)_v\tilde{L}_v. \end{aligned} \tag{8.30}$$

The compatibility condition of system (8.30) is given by the following nonlinear Klein-Gordon equation:

$$(\ln K)_{uv} = \frac{1}{K} - K^2. \tag{8.31}$$

Therefore, the surface is a Lagrangian minimal surface of Klein-Gordon type as described in Proposition 6.2. Consequently, we obtain case (2) of the theorem. ∎

Example 8.1. There exist many unit speed time-like special Legendre curve in the light cone $\mathcal{LC} \subset \mathbf{C}^3_2$ with constant squared curvature $\kappa^2 = 2$. The simplest such examples are the following.

$$z(x) = \left(\frac{1}{\sqrt{2}}, ae^{\sqrt{2}s} - \left(\frac{1}{8a} - \mathrm{i}c\right)e^{-\sqrt{2}x}, ae^{\sqrt{2}s} + \left(\frac{1}{8a} + \mathrm{i}c\right)e^{-\sqrt{2}x}\right),$$

where a is a nonzero real number.

Recently, the author is able to prove Theorem 4.1 for arbitrary Lorentzian surfaces in any Lorentzian Kähler surface.

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