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EQUIVARIANT EXPONENTIALLY NASH VECTOR BUNDLES

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Abstract. Let G be a compact affine exponentially Nash group and let η be a $C^\infty G$ vector bundle over a compact affine exponentially Nash G manifold X. We prove that η admits a unique strongly exponentially Nash G vector bundle structure ζ, and that η admits a non-strongly exponentially Nash G vector bundle structure if $\dim X \geq 1$, rank $\eta \geq 1$ and X has a 0-dimensional orbit. Moreover we show that every exponentially Nash G vector bundle structure of η which is not necessarily strongly exponentially Nash is exponentially Nash G vector bundle isomorphic to ζ if the action on X is transitive.

1. Introduction

Nash manifolds have been studied over the field $I\!R$ of real numbers with the standard structure $\mathbf{R}_{stan} := (I\!R, <, +, \cdot, 0, 1)$ (e.g. [16], [17], [18], [20]). Moreover they are considered over any real closed field (e.g. [2], [4]). Since every real closed field admits quantifier elimination [22], the family of semialgebraic sets coincides with that of definable sets (with parameters) in $\mathbf{R}_{stan}$. Let $\mathbf{R}_{exp}$ be the structure $(I\!R, <, +, \cdot, exp, 0, 1)$ obtained by adding the exponential function $exp : I\!R \longrightarrow I\!R$ to $\mathbf{R}_{stan}$. In [7] exponentially Nash manifolds and equivariant exponentially Nash manifolds are defined in $\mathbf{R}_{exp}$, which are generalizations of the usual Nash ones, and equivariant exponentially Nash manifold structures of equivariant C^∞ manifolds are studied.

By [24] $\mathbf{R}_{exp}$ is model complete, namely any subset of $I\!R^n$ definable in $\mathbf{R}_{exp}$ is the image of a subset of $I\!R^n \times I\!R^m$ definable in $\mathbf{R}_{exp}$ without quantifier by the natural projection $I\!R^n \times \mathrm{R}^m \longrightarrow I\!R^n$ for some $m \in I\!N$. In $\mathbf{R}_{stan}$,

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each non-polynomially bounded function $\mathbb{R} \longrightarrow \mathbb{R}$ is not definable, where a polynomially bounded function means a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $|f(x)| \le x^N, x > x_0$ for some $N \in \mathbb{N}$ and some $x_0 \in \mathbb{R}$. Moreover if there exists a non-polynomially bounded function definable in an 0-minimal expansion $(\mathbb{R}, <, +, \cdot, 0, 1, \cdots)$ of $\mathbf{R}_{stan}$, then in this structure the exponential function is definable [13].

In the present paper, we define exponentially Nash vector bundles (See Definition 2.3) and exponentially Nash G vector bundles (See Definition 2.6), and we investigate exponentially Nash G vector bundle structures of $C^\infty G$ vector bundles.

Let G be an affine exponentially Nash group (See Definition 2.5) and let X be an affine exponentially Nash G manifold (See Definition 2.5). We say that a $C^\infty G$ vector bundle η over X *admits an exponentially Nash G vector bundle structure* (resp. *a strongly exponentially Nash G vector bundle structure*) if η is $C^\infty G$ vector bundle isomorphic to some exponentially Nash G vector bundle (resp. strongly exponentially Nash G vector bundle (See Definition 2.8)) over X. The corresponding notion of strongly exponentially Nash G vector bundles in the non-equivariant algebraic category (resp. in the non-equivariant (standard) Nash category, in the equivariant (standard) Nash category) was introduced by [1] (resp. [2], [9]). It is known that there exists a non-strongly algebraic vector bundle over $\mathbb{R}^2$ (resp. a non-strongly Nash vector bundle over $\mathbb{R}^2$, a non-strongly Nash G vector bundle over a positive-dimensional representation of G when G is a compact affine Nash group) [21] (resp. [2, 12.7.9.], [9]).

Theorem. *Let G be a compact affine exponentially Nash group and let η be a $C^\infty G$ vector bundle over a compact affine exponentially Nash G manifold X.*

(1) *η admits exactly one strongly exponentially Nash G vector bundle structure ξ up to exponentially Nash G vector bundle isomorphism.*

(2) *If $\dim X \ge 1$, X has a 0-dimensional orbit, and rank $\eta \ge 1$, then η admits a non-strongly exponentially Nash G vector bundle structure.*

(3) *If the action on X is transitive, then any exponentially Nash G vector bundle structure of η (which is not necessarily strongly exponentially Nash) is exponentially Nash G vector bundle isomorphic to ξ.* □

We obtain the following as a corollary of Theorem.

Corollary. *Any C^∞ vector bundle of positive rank over a compact affine exponentially Nash manifold of positive dimension admits a non-strongly exponentially Nash vector bundle structure.* □

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2. Exponentially Nash G vector bundles

We recall definitions of exponentially definable sets (cf. [11]) and locally exponentially definable sets [7].

Let $\Re_n = I\!R[x_1, \cdots, x_n, exp(x_1), \cdots, exp(x_n)]$. A subset X of $I\!R^n$ is called $\Re_n$*-semianalytic* if

$$X = \cup_{i=1}^{k}\{x \in I\!R^n | f_i(x) = 0, g_j(x) > 0, 1 \leq j \leq a_i, a_i \in I\!N\},$$

where $f_i, g_j \in \Re_n$. A subset $Y \subset I\!R^n$ is said to be *exponentially definable* if Y is the image of an $\Re_{n+m}$-semianalytic set by the natural projection $\pi : I\!R^n \times I\!R^m \longrightarrow I\!R^n$ for some $m \in I\!N$. We say that a subset $X' \subset I\!R^n$ is *locally exponentially definable* if for any $x \in X'$ there exists an open exponentially definable neighborhood U of x in $I\!R^n$ such that $X' \bigcap U$ in exponentially definable.

Let $X \subset I\!R^n$ and $Y \subset I\!R^m$ be exponentially definable sets (resp. locally exponentially definable sets). A map $f : X \longrightarrow Y$ is said to be *exponentially definable* (resp. *locally exponentially definable*) if the graph of $f \subset X \times Y (\subset I\!R^n \times I\!R^m)$ is exponentially definable (resp. locally exponentially definable).

The next proposition is a collection of basic properties of exponentially definable sets.

Proposition 2.1 (cf. [7]). (1) *Any exponentially definable set consists of only finitely many connected components.*

Let $X \subset I\!R^n$ and $Y \subset I\!R^m$ be exponentially definable sets.

(2) *The closure $Cl(X)$ and the interior $Int(X)$ of X are exponentially definable.*

(3) *The distance function $d(x, X)$ from x to X defined by $d(x, X) = \inf\{\|x - y\| | y \in X\}$ is a continuous exponentially definable function, where $\| \cdot \|$ denotes the standard norm of $I\!R^n$.*

(4) *Let $f : X \longrightarrow Y$ be an exponentially definable map. Then $f(A)$ is exponentially definable if so is $A \subset X$, and $f^{-1}(B)$ is exponentially definable if so is $B \subset Y$.*

(5) *Let $Z \subset I\!R^l$ be an exponentially definable set and let $f : X \longrightarrow Y$ and $h : Y \longrightarrow Z$ be exponentially definable maps. Then the composition $h \circ f : X \longrightarrow Z$ is also exponentially definable.*

(6) *The set of exponentially definable functions on X forms a ring.*

(7) *Any two disjoint closed exponentially definable subsets of $I\!R^k$ can be separated by a continuous exponentially definable function on $I\!R^k$.* □

Let $U \subset I\!R^n$ and $V \subset I\!R^m$ be open exponentially definable sets (resp. open locally exponentially definable sets). We say that a C^ω map $f : U \longrightarrow V$ is *an exponentially Nash map* (resp. *a locally exponentially Nash map*) if it is exponentially definable (resp. locally exponentially definable). An exponentially Nash map (resp. A locally exponentially Nash map) $f : U \longrightarrow V$ is called *an exponentially Nash diffeomorphism* (resp. *a locally exponentially Nash diffeomorphism*) if there exists an exponentially Nash map (resp. a locally exponentially Nash map) $h : V \longrightarrow U$ such that $f \circ h = id$ and $h \circ f = id$.

Remark 2.2. (1) Any usual Nash map between open semialgebraic sets is exponentially Nash.

(2) The function $f : I\!R \longrightarrow I\!R$ defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ e^{-\frac{1}{x}} & \text{if} \quad x > 0 \end{cases}$$

is C^∞ and exponentially definable but not analytic. Remark that any C^∞ semialgebraic map between open semialgebraic sets is analytic.

(3) Every non-constant periodic function $I\!R \longrightarrow I\!R$ (eg. $h : I\!R \longrightarrow I\!R, h(x) = \sin x$) is not exponentially Nash.

Definition 2.3 ([7]). (1) We say that an n-dimensional C^ω manifold with a finite system of charts $\{\phi_i : U_i \longrightarrow I\!R^n\}$ is *an exponentially Nash manifold* if for each i and j $\phi_i(U_i \cap U_j)$ is an open exponentially definable subset of $I\!R^n$, and that the map $\phi_j \circ \phi_i^{-1}|\phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is an exponentially Nash diffeomorphism. We call these charts *exponentially Nash*.

(2) An exponentially definable subset of $I\!R^n$ is called *an exponentially Nash submanifold of dimension d* if it is a C^ω submanifold of dimension d of $I\!R^n$. Remark that an exponentially Nash submanifold is of course an exponentially Nash manifold by the similar way of 1.3.9. [20].

(3) Let X (resp. Y) be an exponentially Nash manifold with exponentially Nash charts $\{\phi_i : U_i \longrightarrow I\!R^n\}_i$ (resp. $\{\psi_j : V_j \longrightarrow I\!R^m\}_j$). A C^ω map $f : X \longrightarrow Y$ is said to be *an exponentially Nash map* if for any i and j $\phi_i(f^{-1}(V_j) \cap U_i)$ is open and exponentially definable in $I\!R^n$, and that the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \longrightarrow I\!R^m$ is an exponentially Nash map.

(4) Let X and Y be exponentially Nash manifolds. We say that X is *exponentially Nash diffeomorphic* to Y if one can find exponentially Nash maps $f : X \longrightarrow Y$ and $h : Y \longrightarrow X$ such that $f \circ h = id$ and $h \circ f = id$.

(5) An exponentially Nash manifold is said to be *affine* if it is exponentially Nash diffeomorphic to some exponentially Nash submanifold of $I\!R^l$.

(6) A topological vector bundle (E, p, X) of rank k is called *an exponentially Nash vector bundle* if the following three conditions are satisfied:

(a) The total space E is an exponentially Nash manifold and the base space X is an affine exponentially Nash manifold.

(b) The projection p is an exponentially Nash map.

(c) There exists a family of finitely many local trivializations $\{\phi_i : U_i \times I\!R^k \longrightarrow p^{-1}(U_i)\}_i$ such that $\{U_i\}_i$ is an open exponentially definable cover of X, and that for any i and j the map $\phi_i^{-1} \circ \phi_j | (U_i \cap U_j) \times I\!R^k : (U_i \cap U_j) \times I\!R^k \longrightarrow (U_i \cap U_j) \times I\!R^k$ is an exponentially Nash map.

We call these local trivializations *exponentially Nash.*

(7) Let $\eta = (E, P, X)$ (resp. $\xi = (F, q, X)$) be an exponentially Nash vector bundle of rank n (resp. m) and let $\{\phi_i : U_i \times I\!R^n \longrightarrow p^{-1}(U_i)\}_i$ (resp. $\{\psi_j : V_j \times I\!R^m \longrightarrow q^{-1}(V_j)\}_j$) be exponentially Nash local trivializations of η (resp. ξ). A vector bundle map $f : \eta \longrightarrow \xi$ is said to be *an exponentially Nash vector bundle map* if for any i and j the map $(\psi_j)^{-1} \circ f \circ \phi_i | (U_i \cap V_j) \times I\!R^n : (U_i \cap V_j) \times I\!R^n \longrightarrow (U_i \cap V_j) \times I\!R^m$ is an exponentially Nash map.

(8) A C^ω section $s : X \longrightarrow E$ of η is said to be *exponentially Nash* if for any i $(\phi_i)^{-1} \circ s | U_i : U_i \longrightarrow U_i \times I\!R^n$ is exponentially Nash.

It is proved in [7] using [10] that any compact affine exponentially Nash manifold X of positive dimension admits an infinite family of nonsingular algebraic sets $\{Y_n\}_{n \in N}$ of some $I\!R^k$ such that each Y_n is exponentially Nash diffeomorphic to X and that Y_n is not birationally isomorphic to Y_m for $n \neq m$.

Remark 2.4. (1) Every usual Nash manifold is of course an exponentially Nash one.

(2) An affine exponentially Nash manifold is not always subanalytic (eg. $\{(x, y) \in I\!R^2 | x > 0, y = \exp(-(1/x))\}$). Remark that every affine Nash manifold in $I\!R^n$ is semialgebraic in $I\!R^n$.

(3) Let $I\!R^>$(resp. $I\!R^\geq$) denote $\{x \in I\!R | x > 0\}$ (resp. $\{r \in I\!R | x \geq 0\}$). The functions $f_1 : I\!R^> \longrightarrow I\!R$ and $f_2 : I\!R^\geq \longrightarrow R$ defined by $f_1(x) = logx$ and, $f_2(x) = x^\alpha, \alpha \in I\!R$, respectively, are exponentially Nash functions but not Nash ones unless α is rational.

Definition 2.5 ([7]). (1) *An exponentially Nash group* (resp. *An affine exponentially Nash group*) is a group G such that G itself is an exponentially Nash manifold (resp. an affine exponentially Nash manifold), and that the multiplication $G \times G \longrightarrow G$ and the inversion $G \longrightarrow G$ are exponentially Nash.

(2) Let G be an exponentially Nash group. *A representation of* G means an exponentially Nash group homomorphism $G \longrightarrow GL(I\!R^n)$ for some n. Here

an exponentially Nash group homomorphism means a group homomorphism which is an exponentially Nash map. We use a representation as a representation space.

(3) An exponentially Nash submanifold in a representation of G is called *an exponentially Nash G submanifold* if it is G invariant.

(4) An exponentially Nash manifold X is said to be *an exponentially Nash G manifold* if X admits a G action whose action map $G \times X \longrightarrow X$ is exponentially Nash.

(5) *Exponentially Nash G maps, exponentially Nash G diffeomorphisms*, and *affine exponentially Nash G manifolds* are defined in a similar way.

In the equivariant Nash category, it is known that any compact equivariant C^∞ manifold of positive dimension such that some connected component of it consists of at least two orbits admits a continuous family of nonaffine equivariant Nash manifold structures [8].

Definition 2.6. Let G be an exponentially Nash group.

(1) An exponentially Nash vector bundle $\eta = (E, p, X)$ is said to be *an exponentially Nash G vector bundle* if the following three conditions are satisfied:

(a) The total space E is an exponentially Nash G manifold and the base space X is an affine exponentially Nash G manifold.
(b) The projection p is an exponentially Nash G map.
(c) For any $x \in X$ and $g \in G$, the map $p^{-1}(x) \longrightarrow p^{-1}(gx)$ is linear.

(2) Let η and ξ be two exponentially Nash G vector bundles. An exponentially Nash vector bundle map $f : \eta \longrightarrow \xi$ is called *an exponentially Nash G vector bundle map* if f is a G map.

(3) Two exponentially Nash G vector bundles η and ξ are said to be *exponentially Nash G vector bundle isomorphic* if there exist exponentially Nash G vector bundle maps $f : \eta \longrightarrow \xi$ and $h : \xi \longrightarrow \eta$ such that $f \circ h = id$ and $h \circ f = id$.

(4) An exponentially Nash section $s : X \longrightarrow E$ of η is called *an exponentially G section* if it is G map.

We recall universal G vector bundles, and we define strongly exponentially Nash G vector bundles.

Definition 2.7. Let Ω be an n-dimensional representation of G and let B be the representation map $G \longrightarrow GL_n(\mathbb{R})$ of Ω. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \longrightarrow B(g)^{-1}AB(g) \in M(\Omega)$. For any positive integer k, we define the vector bundle

$\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$$\begin{aligned} G(\Omega, k) &= \{A \in M(\Omega) | A^2 = A, A = A', TrA = k\}, \\ E(\Omega, k) &= \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\}, \\ u &: E(\Omega, k) \longrightarrow G(\Omega, k) : u((A, v)) = A, \end{aligned}$$

where A' denotes the transposed matrix of A. Then $\gamma(\Omega, k)$ is an algebraic one. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic G vector bundle. We call it *the universal G vector bundle associated with* Ω *and* k. Since $G(\Omega, k)$ and $E(\Omega, k)$ are nonsingular [15], $\gamma(\Omega, k)$ is an exponentially Nash G vector bundle.

Definition 2.8. Let G be an affine exponentially Nash group. An exponentially Nash G vector bundle $\eta = (E, p, X)$ of rank k is said to be *strongly exponentially Nash* if there exist some representation Ω of G and an exponentially Nash G map $f : X \longrightarrow G(\Omega, k)$ such that η is exponentially Nash G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$.

The following two propositions are obtained in a similar way of the usual equivariant Nash cases (cf. [8]).

Proposition 2.9. *Let G be a compact affine exponentially Nash group and let X be an affine exponentially Nash G submanifold in a representation Ω of G. Then there exists an exponentially Nash G tubular neighborhood (U, p) of X in Ω, namely U is an affine exponentially Nash G submanifold in Ω and the orthogonal projection $p : U \longrightarrow X$ is an exponentially Nash G map.* □

Proposition 2.10. *Let G be a compact affine exponentially Nash group. Any compact affine exponentially Nash G manifold X with boundary ∂X admits an exponentially Nash G collar, that is, there exists an exponentially Nash G imbedding $\phi : \partial X \times [0, 1] \longrightarrow X$ such that $\phi|_{\partial X \times 0} = id_{\partial X}$, where the action on the closed unit interval $[0, 1]$ is trivial.* □

3. Proof of our result

To prove Theorem (1), we prepare the following two propositions and a theorem proved by A.G. Wasserman [23]. By the similar way of Proposition 3.1 [6] and Proposition 3.3 [6], we have Proposition 3.1 and Proposition 3.2, respectively.

Proposition 3.1. *Let G be an affine exponentially Nash group and let X be an affine exponentially Nash G manifold. If η_1 and η_2 are strongly* *exponentially Nash G vector bundles over X, then the exponentially Nash G vector bundle $Hom(\eta_1, \eta_2)$ is strongly exponentially Nash.* □

Proposition 3.2. *Let G be a compact affine exponentially Nash group and let η be a strongly exponentially Nash G vector bundle over a compact affine exponentially Nash G manifold. Then every C^rG section $(r < \infty)$ of η can be C^r approximated by an exponentially Nash G one.* □

Theorem 3.3 [23]. *Let G be a compact Lie group and let X be a $C^\infty G$ manifold. Suppose that η is a $C^\infty G$ vector bundle over a $C^\infty G$ manifold Y. If two $C^\infty G$ maps $f_1, f_2 : X \longrightarrow Y$ are $C^\infty G$ homotopic, then $f_1^*(\eta)$ and $f_2^*(\eta)$ are $C^\infty G$ vector bundle isomorphic.* □

Proof of Theorem (1). Since G and X are compact, there exist a representation Ω of G and a $C^\infty G$ map $f : X \longrightarrow G(\Omega, k) \subset M(\Omega)$ such that η is $C^\infty G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η. Thus $i \circ f : X \longrightarrow M(\Omega)$ is C^1 approximated by a polynomial map $h : X \longrightarrow M(\Omega)$. Here i denotes the inclusion $G(\Omega, k) \longrightarrow M(\Omega)$. By Lemma 4.1 [5], we may assume that h is a G map. One can find an exponentially Nash G tubular neighborhood (U, q) of $G(\Omega, k)$ in $M(\Omega)$ by Proposition 2.9. If the approximation is sufficiently close then the image of h lies in U. Hence an exponentially Nash G map $q \circ h$ is an approximation of f. In particular $q \circ h$ is $C^\infty G$ homotopic to f. Therefore $\xi := (q \circ h)^*(\gamma(\Omega, k))$ is a strongly exponentially Nash G vector bundle structure of η by Theorem 3.3.

Let ξ_1 and ξ_2 be two strongly exponentially Nash G vector bundles over X which are $C^\infty G$ vector bundle isomorphic. Then a $C^\infty G$ vector bundle isomorphism between ξ_1 and ξ_2 defines a $C^\infty G$ section s of $Hom(\xi_1, \xi_2)$. By Proposition 3.1 and 3.2, s is approximated by an exponentially Nash G section σ of $Hom(\xi_1, \xi_2)$. Since $Iso(\xi_1, \xi_2)$ is open in $Hom(\xi_1, \xi_2)$ and X is compact, σ determines an exponentially Nash G vector bundle isomorphism $\xi_1 \longrightarrow \xi_2$ if the approximation is sufficiently close. □

We prove the following theorem which is more general than Theorem (2).

Theorem 3.4. *Let G be a compact affine exponentially Nash group and let η be a $C^\infty G$ vector bundle of positive rank over a compact affine exponentially Nash G manifold X of positive dimension. If there exist a representation of Ξ of G and G invariant open exponentially definable subsets U and V of X with $\overline{V} \subset \cup \neq X$ such that $\eta|U$ is exponentially Nash G vector bundle isomorphic to $U \times \Xi$, then η admits a non-strongly exponentially Nash G vector bundle structure.*

The following lemma is useful to prove the existence of nonaffine exponentially Nash G manifolds, which is a generalization of the usual Nash case (I.22.XV [20]).

Proposition 3.5 [7]. *Let M and N be exponentially Nash manifolds and let $h : M \longrightarrow N$ be a locally exponentially Nash map. If N is affine then h is an exponentially Nash map.* □

Proof of Theorem 3.4. By Theorem (1) one can find a unique strongly exponentially Nash G vector bundle structure of η over X. Hence we may assume that η is a strongly exponentially Nash G vector bundle over X. Since the total space of a strongly exponentially Nash G vector bundle over X is affine, we only have to find an exponentially Nash G vector bundle structure of η whose total space is nonaffine.

By Proposition 2.10 there exists an exponentially Nash G collar of $\partial\overline{V}$ in $\overline{V} \subset U, \phi : \partial\overline{V} \times [0,1] \longrightarrow \overline{V}$. Let $D(\epsilon)\,(0 < \epsilon < 1)$ denote $\phi(\partial\overline{V} \times (0,\epsilon))$.

Take an order-preserving exponentially Nash diffeomorphism $f : \mathbb{R} \longrightarrow (0,1)$ (e.g. the inverse map of the composition of $f_1 : (0,1) \longrightarrow (-1,1)$, $f_1(x) = 2x - 1$ with $f_2 : (-1,1) \longrightarrow \mathbb{R}$, $f_2(x) = x/(1-x^2)$). Let

$$N_1 = (0,\, f(1)),\ N_2 = (f(0),\, 1),\ N_3 = (f(0),\, f(1)).$$

Define the exponentially Nash maps $h_1 : N_3 \longrightarrow N_1, h_2 : N_3 \longrightarrow N_2$ by

$$h_1(t) = f((f^{-1}(t))^2) \ \text{ and } \ h_2(t) = f(2f^{-1}(t) - (f^{-1}(t))^2).$$

Then h_1 and h_2 are exponentially Nash imbeddings such that $h_1(N_3) = h_2(N_3) = N_3$.

Let

$$U_1 = D(f(1)),\ U_2 = U - \overline{D(f(0))},\ U_3 = D(f(1)) - \overline{D(f(0))}.$$

Then each U_i is an open affine exponentially Nash G submanifold of X. We define exponentially Nash G vector bundle maps H_1 and H_2 as follows:

$$H_1 : U_3 \times \Xi \longrightarrow U_1 \times \Xi,\ H_1(x,t) = (x, (h_1(p \circ \phi^{-1}(x)))t),$$
$$H_2 : U_3 \times \Xi \longrightarrow U_2 \times \Xi,\ H_2(x,t) = (x, (h_2(p \circ \phi^{-1}(x)))t),$$

where $p : \partial\overline{V}\times(f(0), f(1)) \longrightarrow (f(0), f(1))$ denotes the natural projection. Let W be the quotient space of the disjoint union $\coprod_{i=1}^3 (U_i \times \Xi)$, and the equivalence relation $(x,t) \sim H_1(x,t) \sim H_2(x,t), (x,t) \in U_3 \times \Xi$. Then $\xi_1 = (W, p', U)$ is an exponentially Nash G vector bundle, where p' is the natural projection $W \longrightarrow U$. Replacing the local trivialization $U \times \Xi$ over U by ξ_1 over U, we get the exponentially Nash G vector bundle $\xi' = (F, q, X)$, where q is the natural projection $F \longrightarrow X$. Clearly ξ' is $C^\infty G$ vector bundle isomorphic to η.

We now prove that F is nonaffine. To prove this, we use Proposition 3.5. Fix $z \in \partial\overline{V}$ and $0 \neq t_0 \in \Xi$. Let $\psi : (f(0), f(1)) \longrightarrow F$ be the composition

$$(f(0)),\ f(1)) \longrightarrow \partial\overline{V} \times (f(0), f(1)) \longrightarrow U_3 \longrightarrow U_3 \times \Xi \longrightarrow F,$$

where the first map is $x \longrightarrow (z, x)$, the second is $\phi|(\partial\overline{V} \times (f(0), f(1)))$, the third is $x \longrightarrow (x, t_0)$, and the last is the natural imbedding from $U_3 \times \Xi$ into F. Then ψ is an imbedding. We extend ψ as widely as possible as an exponentially Nash map. Let $l_i\,(i = 1, 2, 3)$ be the natural imbedding $U_i \times \Xi \longrightarrow F$ and let $V_i\,(i = 1, 2, 3)$ denote its image. Then

$$p_1 \circ l_1^{-1} \circ \psi(x) = (h_1(x))t_0 \text{ and}$$
$$p_2 \circ l_2^{-1} \circ \psi(x) = (h_2(x))t_0 \text{ on } (f(0), f(1)),$$

where $p_i (i = 1, 2)$ is the projection $U_i \times \Xi \longrightarrow \Xi$. We extend ψ to $(f(0), f(1 + \varepsilon))$ for small positive ε. It suffices to consider $p_2 \circ l_2^{-1} \circ \psi(x) = (h_2(x))t_0$ because the image of ψ lies in V_2 and $\lim_{t \to f(1)} \psi(t) \in V_2$. Now $p_2 \circ l_2^{-1} \circ \psi(x) = (f(2f^{-1}(x) - (f^{-1}(x))^2))t_0$ on $(f(0), f(1))$. Thus $p_2 \circ l_2^{-1} \circ \psi(x)$ and ψ are extensible to $(f(0), f(2))$ and

$$p_2 \circ l_2^{-1} \circ \psi(x) = (f(2f^{-1}(x) - (f^{-1}(x))^2))t_0 \text{ on } [f(1), f(2)).$$

Clearly we can extend ψ to $[f(0), f(1)]$, and $\psi((f(0), f(2)) \subset \psi([f(0), f(1)])$. Hence

$$\psi_0^{-1} \circ \psi(x) = f(2 - f^{-1}(x)) \text{ on } [f(1), f(2)),$$

where ψ_0 denotes the homeomorphism $\psi : [f(0), f(1)] \longrightarrow \psi([f(0), f(1)])$. In the same way, ψ can be defined on $(f(-1), f(0)]$ satisfying

$$\psi_0^{-1} \circ \psi(x) = f(-f^{-1}(x)) \text{ for } x \in (f(-1), f(0)].$$

Repeating this argument, we obtain

$$\psi_0^{-1} \circ \psi(x) = \begin{cases} \qquad\vdots & \\ f(-(2 + f^{-1}(x))) & \text{on } [f(-3), f(-2)] \\ f(2 + f^{-1}(x)) & \text{on } [f(-2), f(-1)] \\ f(-f^{-1}(x)) & \text{on } [f(-1), f(0)] \\ x & \text{on } [f(0), f(1)] \\ f(2 - f^{-1}(x)) & \text{on } [f(1), f(2)] \\ f(-(2 - f^{-1}(x))) & \text{on } [f(2), f(3)] \\ f(2 + (2 - f^{-1}(x))) & \text{on } [f(3), f(4)] \\ \qquad\vdots & \end{cases}$$

Thus ψ is extensible on (0,1), ψ is locally exponentially Nash, and the image of ψ is $\psi([f(0), f(1)])$. Moreover for any $e \in (f(0), f(1))$, $(\psi_0^{-1} \circ \psi)^{-1}(e)$ is discrete and consists of infinitely many elements. Since ψ is locally exponentially Nash but not exponentially Nash and by Proposition 3.5, F is nonaffine. Therefore ξ' is a non-strongly exponentially Nash G vector bundle structure of η. □

Proof of Theorem (2). By Theorem (1) we may assume that η is a strongly exponentially Nash G vector bundle over X. The assumption of Theorem (2) implies that there exists an orbit $G(x) = \{x_1, \cdots, x_n\}$. Let $B(a, r)$ denote the open ball in X of radius r and center $a \in X$. We can find a positive real number r such that the disjoint unions $U := \coprod_{i=1}^n B(x_i, r)$ and $V := \coprod_{i=1}^n B(x_i, r/2)$ are exponentially Nash G tubular neighborhoods of $G(x)$ by means of Proposition 2.9. Hence shrinking r, if necessary, $\eta|U$ is exponentially Nash G vector bundle isomorphic to $U \times \Xi$ for some representation Ξ of G. Therefore Theorem (2) follows from Theorem 3.4. □

Proof of Theorem (3). Let ξ be the strongly exponentially Nash G vector bundle structure of η constructed in (1) and let ξ' be another exponentially Nash G vector bundle structure of η which is not necessarily strongly exponentially Nash. Let $x \in X$. By the assumption, $\xi|x$ is isomorphic to $\xi'|x$ as a G_x representation. Since ξ and ξ' are exponentially Nash G vector bundle structures of η over X, there exists a G_x invariant open exponentially definable neighborhood U of x in X such that $\xi|U$ is exponentially Nash G_x vector bundle isomorphic to $\xi'|U$. Translating this isomorphism, we have an exponentially Nash G vector bundle isomorphism between ξ and ξ'. □

Finally, we consider exponentially Nash group structures of compact centerless Lie groups.

It is known in [3] that every compact Lie group admits a unique algebraic group structure up to algebraic group isomorphism. Thus in particular it admits an affine Nash group structure. Notice that all connected one-dimensional Nash groups and locally Nash groups are classified by [12] and [19], respectively. In particular the standard unit circle S^1 admits a nonaffine exponentially Nash group structure.

Let G be a compact centerless Lie group and let G' be an exponentially Nash group structure of G. Then the adjoint representation $Ad : G' \longrightarrow Gl_n(\mathbb{R})$ is exponentially definable by a similar proof of Lemma 2.2 [14], and it is analytic. Here n denotes the dimension of G. Thus Ad is an exponentially Nash map and its kernel is the center of G'. Hence the image G'' of Ad is an affine exponentially Nash group and Ad is an exponentially Nash group isomorphism from G' to G''. Therefore we have the following remark.

Remark 3.6. *Let G be a compact centerless Lie group. Then G does not admit any nonaffine exponentially Nash group structure.* □

References

1. R. Benedetti and A. Tognoli, On real algebraic vector bundles, *Bull. Sci. Math.* **(2)** (1980), 89–112.
2. J. Bochnak, M. Coste, and M. F. Roy, *Géométie algébrique réelle*, Ergebnisse der Mathematik, und ihrer Grenzebiete, Springer-Berlag, 1987.
3. C. Chevalley, Theory of Lie groups, *Princeton Univ. Press, Princeton, N. J.*, 1946.
4. M. Coste and M. Shiota, Nash triviality in families of Nash manifolds, *Invent. Math.* **108** (1992), 349-368.
5. K. H. Dovermann, M. Masuda, and T. Petrie, Fixed point free algebraic actions on varieties diffeomorphic to $\mathbb{R}^n$, *Progress in Math.* **80**, Birkhäuser (1990), 49–80.
6. T. Kawakami, Algebraic G vector bundles and Nash G vector bundles, *Chinese J. Math.* **22** (1994), 275–289.
7. T. Kawakami, Exponentially Nash manifolds and equivariant exponentially Nash manifolds, Preprint.
8. T. Kawakami, Nash G manifold structures of compact or compactifiable $C^\infty G$ manifolds, *J. Math. Soc. Japan.* **48** (1996), 321–331.
9. T. Kawakami, Nash structures of $C^\infty G$ vector bundles, Preprint.
10. T. Kawakami, Nonisomorphic algebraic models of Nash manifolds and compactifiable C^∞ manifolds, *Osaka J. Math.* **31** (1994), 831–835.
11. T. L. Loi, Analytic cell decomposition of sets definable in the structure $\mathbb{R}_{exp}$, *Ann. Polon. Math.* **59** (1994), 255–266.
12. J. J. Madden and C. H. Stanton, One-dimensional Nash groups, *Pacific J. Math.* **154** (1992), 331–344.
13. C. Miller, Exponentiation is hard to avoid, *Proc. Amer. Math. Soc.* **122** (1994), 257–259.
14. A. N. Nesin and A. Pillay, Some model theory of compact Lie groups, *Trans. Amer. Math. Soc.* **326** (1991), 453–463.
15. R. Palais, Equivariant real algebraic differential topology, Part 1, Smoothness categories and Nash manifolds, *Notes. Brandeis Univ.* (1972).
16. M. Shiota, Abstract Nash manifolds, *Proc. Amer. Math. Soc.* **96** (1986), 155–162.

17. M. Shiota, Approximation theorems for Nash mappings and Nash manifolds, *Trans. Amer. Math. Soc.* **293** (1986), 319–337.

18. M. Shiota, Classification of Nash manifolds, *Ann. Inst. Fourier, Grenoble* **33** (1983), 209–232.

19. M. Shiota, Geometry of subanalytic and semialgebraic sets, to appear.

20. M. Shiota, Nash manifolds, Lecture Note in Mathematics vol. 1269, Springer-Verlag, 1987.

21. A. Tognoli, Algebraic geometry and Nash functions, Academic Press, 1978.

22. A. Tarski, A decision method for elementary algebra and geometry, 2nd. ed., revised, Berkely and Los Angels, 1951,

23. A. G. Wasserman, Equivariant differential topology, Topology **8** (1969), 127–150.

24. A. Wilkie, Model completeness results for expansions of the real field 2: the exponential function, to appear.

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