TAIWANESE JOURNAL OF MATHEMATICS
Vol. 1, No. 2, pp. 209-215, June 1979

SEMILINEAR ELLIPTIC EQUATIONS IN UNBOUNDED SYMMETRIC DOMAINS

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Abstract. In this article we prove the existence of a minimizer of semilinear elliptic equations in axial symmetric domains.

1. Introduction

Throughout this article, let $N \geq 3$, $1 < p < \frac{N+2}{N-2}$, and $z = (x, y)$ be the generic point of $\mathbf{R}^N$ with $x \in \mathbf{R}^{N-1}$, $y \in \mathbf{R}$. By an axial symmetric domain $\Omega \subset \mathbf{R}^N$, we mean that $z = (x, y) \in \Omega$ if and only if $(|x|, 0, \cdots, 0, y) \in \Omega$. By an axial symmetric function u in Ω, we mean that there is a function $f : [0, \infty) \times \mathbf{R} \to \mathbf{R}$ such that $u(x, y) = f(|x|, y)$ for $(x, y) \in \Omega$.

Let $\Omega \subset \mathbf{R}^N$ be a domain. Consider the problem

$$(1) \qquad \begin{cases} -\Delta u + u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

Let $H_s(\Omega)$ be the H^1-closure of the space $\{u \in C_0^\infty(\Omega) \,|\, u \text{ is axial symmetric}\}$,

$$\alpha_s(\Omega) = \inf\left\{\int_\Omega (|\bigtriangledown u|^2 + u^2) \mid u \in H_s(\Omega), \int_\Omega |u|^{p+1} = 1\right\},$$

$$\alpha(\Omega) = \inf\left\{\int_\Omega (|\bigtriangledown u|^2 + u^2) \mid u \in H_0^1(\Omega), \int_\Omega |u|^{p+1} = 1\right\},$$

$$\alpha = \alpha(R^N) = \inf\left\{\int_{R^N} (|\bigtriangledown u|^2 + u^2) \mid u \in H^1(R^N), \int_{R^N} |u|^{p+1} = 1\right\}.$$

Definition 1. *$\Omega \subset \mathbf{R}^N$ is solvable if there is a solution of equation (1), otherwise Ω is unsolvable.*

Received February 26, 1996
Communicated by S.-B. Hsu.
1991 *Mathematics Subject Classification*: 35J20, 35J25.
Key words and phrases: Semilinear elliptic equation, unbounded symmetric domain.

Many mathematicians have studiedthe solvability and unsolvability of $\Omega \subset \mathbf{R}^N$ as follows:

Example 2. *If Ω is bounded or $\Omega = \mathbf{R}^N$, then $\alpha(\Omega)$ admits a minimizer and that Ω is solvable.*

Proof. Taking a minimizing sequence for $\alpha(\Omega)$, then apply a compactness imbedding theorem. ■

Example 3. *If Ω is the upper half plane $\mathbf{R}^N_+$ or the upper half strip $S = \omega \times \mathbf{R}^n_+$, where $\omega \subset R^m$ and $N = m + n$, then Ω is unsolvable.*

Proof. Esteban-Lions [3] have derived an integral identity to prove it. ■

Theorem 4. *If $\Omega_1, \Omega_2 \subset \mathbf{R}^N$ such that $\Omega_1 \cap \Omega_2$ is bounded, $\alpha(\Omega_1) \leq \alpha(\Omega_2)$ and $\alpha(\Omega_1)$ admits a minimizer, then $\alpha(\Omega_1 \cup \Omega_2)$ admits a minimizer.*

Proof. See Lien-Tzeng-Wang [4; Theorem 5.1] ■

Example 5. *If $S = \omega \times \mathbf{R}^n_+$, $B(0, r)$ is a ball of radius r and $\Omega_r = S \cup B(0, r)$, then there is $r_0 > 0$ such that Ω_r is solvable provided that $r \geq r_0$.*

Proof. Note $\alpha(S) > \alpha$ and $\lim_{r\to\infty} \alpha(B(0, r)) = \alpha$. Then apply Theorem 2. ■

Example 6. *The hyperboloid $|x|^2 - y^2 = l^2$ in R^N divides R^N into two axial symmetric domains A^l and A_l such that*

1) *A^l contains the y-axis and satisfying, for any $r > 0$ there is $a_r > 0$ such that*

$$\{(x, y) \in R^N \mid |x| \leq l\} \cup \{\{(x, y) \in R^N \mid |x| < r,\ |y| > a_r\} \subset A^l.$$

2) *A_l satisfies*

$$\lim_{r\to\infty} \inf\{|x| \mid (x, y) \in A_l,\ |y| \geq r\} = \infty.$$

Example 7. *There is $l_0 > 0$ such that if $l \geq l_0$, then A^l is solvable.*

Proof. First establish a decomposition lemma of a (PS)-sequence to get good energy levels $(\alpha(A^l), 2^{(p-1)/(p+1)}\alpha(A^l))$. Then raise higher the energy to be in the good level through that the center of mass done as Coron [1] and that the length of l. ■

But for the solvability of A_l, it is nontrivial. In this article we shall establish a surprising result (see Theorem 11) in Section 2 and then in Section 3 use it to prove the solvability of A_l as follows:

Main Theorem. *A_l is solvable and $\alpha_s(A_l)$ admits a minimizer.*

2. An Analysis Theorem

In order to prove the Main Theorem , we need the following two known results:

Proposition 8. *Let $m \geq 1,\ k \geq 2,\ \omega$ be a smooth bounded open set in $\mathbf{R}^m$, and $E = \omega \times \mathbf{R}^k$. Denote by (x, y) a generic point in $\mathbf{R}^m \times \mathbf{R}^k$ and consider the space $H_s(E)$ consisting of functions in $H_0^1(E)$ which are spherically symmetric in y-variable. Then the Sobolev imbedding from $H_s(E)$ into $L^q(E)$ is compact for every $q \in \left(2, \frac{2N}{N-2}\right)$ with $N = m + k$.*

Proof. By Esteban [2]. ∎

Lemma 9. *If $\{v_k\} \subset H_s(\Omega)$ is a minimizing sequence for J, then $\{\alpha_s(\Omega)^{\frac{1}{p-1}} v_k\}$ is a (PS)$_d$−sequence of I, where $d = (\frac{1}{2} - \frac{1}{p+1})\alpha_s(\Omega)^{(p+1)/(p-1)}$,*

$$I(u) = \frac{1}{2}\int_\Omega (|\bigtriangledown u|^2 + u^2) - \frac{1}{p+1}\int_\Omega |u|^{p+1} \quad \textit{for} \ \ u \in H_s(\Omega),$$

and

$$J(u) = \frac{1}{2}\int_\Omega (|\bigtriangledown u|^2 + u^2)$$

Proof. By routine computation. ∎

Lemma 10. *Let $B_r = \{(x, y) \in \mathbf{R}^N |\ \ |x| > r\}$ and $r > 0$. Then $\alpha(B_r) = \alpha$ for each $r > 0$.*

Proof See Lien-Tzeng-Wang [4]. ∎

However we have the following surprising result:

Theorem 11.

$$\lim_{r\to\infty} \alpha_s(B_r) = \infty.$$

Proof. Assume $\lim_{r\to\infty} \alpha_s(B_r) = \eta < \infty$. For $n = 1, 2, \cdots$, take $\alpha_n = \alpha_s(B_n)$. By a proof similar to that in Lien-Tzeng-Wang [4, Theorem 4.8], we obtain that $\alpha_s(B_n)$ admits a minimizer u_n. Then by the Maximum Principle

$$\alpha_1 < \alpha_2 < \cdots,$$

$$\lim_{n\to\infty} \alpha_n = \eta,$$

$$\left\{\| u_n \|_{H_s(B_n)}\right\} \text{ is bounded},$$

$$\int_{B_n} |u_n|^{p+1} = 1 \text{ for } n = 1, 2, \cdots.$$

Embed $H_s(B_n)$ into $H_s(R^N)$ by letting $u_n = 0$ outside B_n and consider the concentration function $Q_n(t)$ of u_n :

$$Q_n(t) = \sup_{y'\in\mathbf{R}} \int_{\mathbf{R}^{N-1}\times(y'-t,y'+t)} |u_n(x,y)|^{p+1} dxdy \quad \text{for } t > 0.$$

Then for $n = 1, 2, \cdots$

$Q_n(t)$ is an increasing function of t,
$\lim_{t\to\infty} Q_n(t) = 1$,
$\lim_{t\to 0^+} Q_n(t) = 0$.

By the Helly Theorem, we may choose a subsequence $\{Q_n\}$ such that

$$\lim_{n\to\infty} Q_n(t) = Q(t) \quad \text{for } t > 0,$$

where Q is a nondecreasing function in t with $0 \leq Q \leq 1$. Claim that $\lim_{t\to\infty} Q(t) \neq 0$. For otherwise, assume $\lim_{t\to\infty} Q(t) = 0$, then $Q \equiv 0$ and consequently $\lim_{n\to\infty} Q_n(t) = 0$ for $t > 0$. Take q and r such that $p + 1 < q < r < \frac{2N}{N-2}$. By the Hölder Inequality and the Sobolev Imbedding Theorem,

$$\begin{aligned}
\int_{\mathbf{R}^N} |u_n|^q &= \sum_{j=-\infty}^{\infty} \int_{\mathbf{R}^{N-1}\times[2j-1,2j+1]} |u_n|^q \\
&\leq \sum_{j=-\infty}^{\infty} \left[\int_{\mathbf{R}^{N-1}\times[2j-1,2j+1]} |u_n|^{p+1}\right]^{\frac{r-q}{r-p-1}} \left[\int_{\mathbf{R}^{N-1}\times[2j-1,2j+1]} |u_n|^r\right]^{\frac{q-p-1}{r-p-1}} \\
&\leq cQ_n(1)^{\frac{r-q}{r-p-1}} \sum_{j=-\infty}^{\infty} \left[\int_{\mathbf{R}^{N-1}\times[2j-1,2j+1]} (|\nabla u_n|^2 + u_n^2)\right]^{\frac{r(q-p-1)}{2(r-p-1)}}.
\end{aligned}$$

Since $\lim_{r\to q}\frac{r(q-p-1)}{2(r-p-1)}=\frac{q}{2}>\frac{p+1}{2}>1$, we can choose r so close to q that

$$\frac{r(q-p-1)}{2(r-p-1)}>1.$$

We have

$$\sum_{j=-\infty}^{\infty}\left[\int_{R^{N-1}\times[2j-1,2j+1]}(|\bigtriangledown u_n|^2+u_n^2)\right]^{\frac{r(q-p-1)}{2(r-p-1)}}$$
$$\le\left[\sum_{j=-\infty}^{\infty}\int_{R^{N-1}\times[2j-1,2j+1]}(|\bigtriangledown u_n|^2+u_n^2)\right]^{\frac{r(q-p-1)}{2(r-p-1)}}$$
$$=\left[\int_{R^N}(|\bigtriangledown u_n|^2+u_n^2)\right]^{\frac{r(q-p-1)}{2(r-p-1)}}$$
$$=\alpha_n^{\frac{r(q-p-1)}{2(r-p-1)}}.$$

Therefore

$$\int_{\mathbf{R}^N}|u_n|^q\le c\alpha_n^{\frac{r(q-p-1)}{2(r-p-1)}}Q_n(1)^{\frac{r-q}{r-q-1}}=o(1)\quad as\ \ n\to\infty.$$

By the interpolation property, $\| u_n \|_{L^{p+1}}=o(1)$ as $n\to\infty$, a contradiction. Therefore $\lim_{t\to\infty}Q(t)=\beta>0$. Consequently there is $R_0>0$ such that $Q(R_0)>\frac{\beta}{2}$. Take $n_0>0$ such that $n\ge n_0$ implies $Q_n(R_0)>\frac{\beta}{2}$. Choose $\{y_n\}_{n=n_0}^{\infty}\subset\mathbf{R}$ such that

$$\int_{\mathbf{R}^{N-1}\times[y_n-R_0,y_n+R_0]}|u_n(x,y)|^{p+1}\ge\frac{\beta}{2}\,.$$

Let $\widetilde{u_n}(x,y)=u_n(x,y+y_n)$. Then

$$\int_{\mathbf{R}^{N-1}\times[-R_0,R_0]}|\widetilde{u_n}|^{p+1}\ge\frac{\beta}{2}\quad\text{for}\ \ n\ge n_0\,. \tag{2}$$

By Proposition 8, if necessary, replace R_0 by R_0+1, then we can take a subsequence $\{\widetilde{u_n}\}$ and $\widetilde{u}$ such that

$$\lim_{n\to\infty}\widetilde{u_n}=\widetilde{u}\quad in\ \ L^{p+1}(\ \mathbf{R}^{N-1}\times[-R_0,R_0]).$$

By (2), $\not\equiv 0$. But since $\widetilde{u_n}(x)\in H_s(B_n)$, we have

$$\lim_{n\to\infty}\widetilde{u_n}(z)=0\quad\text{for}\ \ z\in\mathbf{R}^N,$$

a contradiction. Therefore

$$\lim_{r\to\infty}\alpha_s(B_r)=\infty.$$

■

3. Solvability of A_l

Note that by Lemma 10 and the Maximum Principle, $\alpha(A_l)$ does not admit any minimizer. However, in the following we will prove that $\alpha_s(A_l)$ admits a minimizer.

Main Theorem. *A_l is solvable and $\alpha_s(A_l)$ admits a minimizer.*

Proof. Take $r_1 > 0$ such that $\mathrm{I}_{r_1} = \{(x,y) \in \Omega \mid |x| < r_1\} \neq \emptyset$. For $r \geq r_1$, decompose

$$\Omega = \mathrm{I}_{r+1} \cup \mathrm{II}_r,$$

where

$$\begin{aligned} \mathrm{I}_s &= \{(x,y) \in \Omega \mid |x| < s\}, \\ \mathrm{II}_r &= \{(x,y) \in \Omega \mid |x| > r\}. \end{aligned}$$

Then $\alpha_s(\mathrm{I}_r)$ is decreasing in r and $\alpha_s(\mathrm{II}_r)$ is increasing in r. Let

$$B_r = \{(x,y) \in R^N \mid |x| > r\}.$$

By Theorem 11

$$\lim_{r\to\infty} \alpha_s(B_r) = \infty.$$

Take $r_2 \geq r_1$ such that

$$\alpha_s(B_{r_2}) \geq \alpha_s(\mathrm{I}_{r_1}).$$

Therefore

$$\alpha_s(\mathrm{I}_{r_2+1}) \leq \alpha_s(\mathrm{I}_{r_1}) \leq \alpha_s(B_{r_2}) \leq \alpha_s(\mathrm{II}_{r_2}).$$

Since

$$\lim_{r\to\infty} \inf\{|x| \mid (x,y) \in \Omega, |y| \geq r\} = \infty,$$

I_{r_2+1} is bounded and axial symmetric. Therefore $\alpha_s(\mathrm{I}_{r_2+1})$ admits a minimizer. By Theorem 4, $\alpha_s(A_l)$ admits a minimizer. ■

Remark 1. *By the Main Theorem and the Maximum Principle, let A_l be as in the Main Theorem, we have*

$$\alpha_s(A_l) > \alpha(A_l).$$

Similar proof as in the Main Theorem can be applied to obtain the following:

Corollary 12. *For $r > 0$, let either*

1. $\Omega = \{(x,y) \in \mathbf{R}^{N-1} \times \mathbf{R} \mid \ |x|^2 - r < y < |x|^2 + r\}$, *or*

2. $\Omega = \{(x,y) \mid \ 0 < y < |x|^2 + 2r\}$.

Then $\alpha_s(\Omega)$ *admits a minimizer.*

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