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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SINGULAR SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS IN R^n

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Abstract. In this paper we consider the quasilinear elliptic equation

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) + f(u) = 0 \tag{1}$$

where $n > m > 1$. We obtain a necessary and sufficient condition for the existence of positive radial solutions $u = u(r)$ on $[r_0, \infty)$, where $r_0 > 0$. If f satisfies a further condition, then Eq. (1) possesses infinitely many singular ground state solutions $u(r)$ satisfying $u(r) \sim r^{\frac{-(n-m)}{m-1}}$ at ∞ and $u(r) \to \infty$ as $r \to 0^+$. We also obtain some important conclusions via our main results.

1. Introduction

In this paper we consider the quasilinear elliptic equation

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) + f(u) = 0 \tag{1.1}$$

in $\Omega_{r_0} = R^n \backslash D_{r_0}$, where $n > m > 1$, $r_0 \geq 0$, $D_{r_0} = \{(x_1, \cdots, x_n) \in R^n | x_1^2 + \cdots + x_n^2 \leq r_0^2\}$ and $f \in C[0, \infty)$. We are concerned with the problem of finding positive solutions u of (1.1) in Ω_{r_0}. When $m = 2$, (1.1) is known as the Lane-Emden type equation and plays an important role in astrophysics. When $r_0 = 0$, Ni and Serrin [12] used the generalized Pohozaev identity to derive a nonexistence theorem for singular ground state solution of (1.1) in $R^n \setminus \{0\}$. They proved that : if f is positive at infinity, nonpositive near

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zero, and satisfies some suitable nonlinearities, then (1.1) does not possess any radial solution in $R^n\backslash\{0\}$ which is positive near the origin and tends to 0 at infinity. The study of equations of the form (1.1) has been the subject of many papers. Here we mention only a part of this literature. See [1], [3], [4], [5], [7], [9], [10], [11], and [12].

The main purpose of this paper is to study the existence of positive radial solutions $u = u(|x|)$ of (1.1) in Ω_{r_0}. Let $r = |x|$. Then, in this case, (1.1) reduces to the ordinary differential equation

$$(r^{n-1}|u'|^{m-2}u')' + r^{n-1}f(u) = 0, \quad r > r_0, \quad u > 0. \tag{1.2}$$

For the case $m = 2$, (1.2) reduces to the semilinear elliptic equation

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \quad r > r_0. \tag{1.3}$$

For equations (1.3), T. Makino [9] obtained several interesting results. Now we consider the more general equation (1.2) and obtain the following Theorem 1.1.

We shall assume throughout this paper that

$$(f_a) \qquad n > m > 1, \quad f \in C[0,\infty) \text{ and } f(u) > 0 \text{ for all } u > 0.$$

Our first result is a necessary and sufficient condition for the existence of positive solutions $u = u(r)$ on $[r_0, \infty)$, where $r_0 > 0$.

Theorem 1.1. *If* (1.2) *possesses a positive solution* $u = u(r)$ *on* $[r_0, \infty)$, *then*

$$\int_0^1 f(u)u^{-\frac{m(n-1)}{n-m}}du < \infty. \tag{1.4}$$

Conversely, if inequality (1.4) *holds, then for any positive constant* C *there exists* $r_0 > 0$ *such that* (1.2) *possesses a positive solution* $u = u(r)$ *on* $[r_0, \infty)$ *and* u *satisfies*

$$u(r) \sim r^{-\frac{n-m}{m-1}} \text{ at } \infty \tag{1.5}$$

and

$$\lim_{r\to\infty} r^{\frac{n-1}{m-1}}u'(r) = -\frac{n-m}{m-1}C. \tag{1.6}$$

In this paper we use the notation "$g \sim h$ at ∞ (at 0)" to denote that "there exists two positive constants C_1, C_2 such that $C_1h \geq g \geq C_2h$ at ∞

(at 0)". Now if f satisfies a further condition, then (1.2) possesses infinitely many singular ground state solutions $u(r)$. By a singular ground state we mean a positive classical solution in $R^n \backslash \{0\}$ which tends to zero at ∞ and tends to ∞ at the origin. Our second result is

Theorem 1.2. *Assume that f satisfies the inequality* (1.4) *and*

$$nF(u) \geq \frac{n-m}{m} u f(u) \quad \text{for all} \quad u > 0. \tag{1.7}$$

where $F(u) = \int_0^u f(t)dt$. Then (1.1) *possesses infinitely many singular ground state solutions u which satisfy* (1.5) – (1.6) *and*

$$u(r) \to \infty \quad as \ r \to 0^+. \tag{1.8}$$

We will obtain some important conclusions via Theorem 1.2 in the following remarks.

Remark 1.3. Let $f(u) = \alpha u^p + \beta u^q$, where $\alpha > 0$ and $\beta > 0$ are two positive constants, and $1 < p < q$. Then (1.1) becomes

$$\operatorname{div}(|\nabla u|^{m-2} \nabla u) + \alpha u^p + \beta u^q = 0 \text{ in } R^n. \tag{1.9}$$

When $\beta = 0$, then (1.9) becomes

$$\operatorname{div}(|\nabla u|^{m-2} \nabla u) + \alpha u^p = 0 \quad \text{in} \quad R^n. \tag{1.10}$$

Let u be a positive radial solution of (1.9) or (1.10) which tends to zero at ∞. Then from [11] we know that

$$C_1 r^{-\frac{n-m}{m-1}} \leq u(r) \leq C_2 r^{-\frac{m}{p-m+1}} \quad \text{at} \quad \infty. \tag{1.11}$$

Using the similar method in [LN] we can prove that

$$\begin{aligned} &\text{either} \quad u(r) \sim r^{-\frac{m}{p-m+1}} \quad \text{at} \quad \infty \text{ (slow decay)}, \\ &\text{or} \quad u(r) \sim r^{-\frac{n-m}{m-1}} \quad \text{at} \quad \infty \text{ (fast decay)}. \end{aligned} \tag{1.12}$$

From Theorem 1.2 we obtain the existence of the singular ground state with fast decay solutions. We have

Corollary 1.4 *Suppose that α and β are any two positive constants, and $\frac{(m-1)n}{n-m} < p < q \leq \frac{(m-1)n+m}{n-m}$ in* (1.9) *(or $\frac{(m-1)n}{n-m} < p \leq \frac{(m-1)n+m}{n-m}$ in* (1.10)*). Then equation* (1.9) *(or equation* (1.10)*) possesses infinitely many singular ground state with fast decay solutions which satisfy* (1.5), (1.6) *and* (1.8).

Remark 1.5 When $m = 2$, equations (1.9) and (1.10) reduce to the following equations (1.14) and (1.15), respectively.

$$\triangle u + \alpha u^p + \beta u^q = 0 \text{ in } R^n, \ \alpha > 0, \beta > 0, \tag{1.14}$$

$$\triangle u + \alpha u^p = 0 \text{ in } R^n, \ \alpha > 0. \tag{1.15}$$

If $\frac{n}{n-2} < p < q \leq \frac{n+2}{n-2}$, then from [13] we know that (1.14) has no positive radial entire solutions. From the following Corollary 1.6, we obtain that the singular ground state solutions do exist. From Corollary 1.4, we obtain that

Corollary 1.6. *Suppose that $\frac{n}{n-2} < p < q \leq \frac{n+2}{n-2}$ in* (1.14) *(or $\frac{n}{n-2} < p \leq \frac{n+2}{n-2}$ in* (1.15)*). Then for any positive constant C, Eq.* (1.14) *(or Eq.*(1.15)*) possesses a singular ground state solution $u(r)$ which satisfies*

$$u(r) \sim r^{-(n-2)} \text{ at } \infty, \tag{1.16}$$

$$\lim_{r\to\infty} r^{n-1}u'(r) = -(n-2)C. \tag{1.17}$$

We organize this paper as follows. In Section 2, we study the initial value problem of (1.2) and obtain a theorem about the estimates and nonexistence of solutions of (1.2). Applying the information of Section 2, we give a detailed proof of Theorem 1.1 in Section 3. Finally in Section 4, we give a complete proof of Theorem 1.2.

2. Preliminaries

In this section we consider the initial-value problem

$$\begin{cases} (r^{n-1}|u'|^{m-2}u')' + r^{n-1}f(u) = 0, & r > r_0; \\ u(r_0) = u_0 > 0, \ u'(r_0) = u_0', \ u > 0 \text{ on } [r_0, \infty), \end{cases} \tag{2.1}$$

where $r_0 > 0$ and m and f satisfy the assumption (f_a) in Section 1. We have the following theorem.

Theorem 2.1. *Let $u(r)$ be a positive solution of* (2.1) *defined on $[r_0, \infty)$. Then there exists $r_1 > r_0$ such that*

$$\frac{u'(r)}{u(r)} \geq -\frac{n-m}{m-1}\left(\frac{1}{r}\right) \text{ for all } r \geq r_1, \tag{2.2}$$

therefore $u(r)$ satisfies

$$u(r) \geq u(r_1)\left(\frac{r_1}{r}\right)^{\frac{n-m}{m-1}} \quad \text{for all } r \geq r_1. \tag{2.3}$$

On the other hand, if the initial data in (2.1) satisfy

$$\frac{u_0'}{u_0} < -\frac{n-m}{m-1}\left(\frac{1}{r_0}\right), \tag{2.4}$$

then (2.1) does not possess any positive solution on $[r_0, \infty)$.

From (2.3) in Theorem 2.1 we obtain that every positive solution u(r) of (2.1) defined on $[r_0, \infty)$ cannot be more rapid than that of the fundamental p-harmonic singularity. On the other hand, if the initial data satisfies (2.4), then every positive solution u(r) of (2.1) must have a finite zero on $[r_0, \infty)$.

The proof of (2.3) in Theorem 2.1 can be easily obtained by standard method. See, for example, [6, Lemma 1.1]. For the sake of completeness, here we give the proof of the second part of Theorem 2.1. First we need the following lemma.

Lemma 2.1. *If u is a positive solution of* (2.1) *defined on* $[r_0, \infty)$*, then $u(r)$ is bounded on* $[r_0, \infty)$ *and $u(r)$ is strictly decreasing to 0 as $r \to \infty$.*

Proof. From Eq. (2.1), it is easy to see that $u(r)$ is bounded on $[r_0, \infty)$ and there exists $r_1 \geq r_0$ such that $u'(r) < 0$ for all $r \geq r_1$. Hence $u(r)$ is strictly decreasing to u_∞. We have to prove that $u_\infty = 0$. Suppose that this is not true, i.e., $u_\infty > 0$. Then for all $r \geq r_1$ we have

$$u'(r) = -\left[\left(\frac{r_1}{r}\right)^{n-1}\Big(-u'(r_1)\Big)^{m-1} + \int_{r_1}^{r}\left(\frac{s}{r}\right)^{n-1} f\Big(u(s)\Big)ds\right]^{\frac{1}{m-1}} \tag{2.5}$$

and

$$\begin{aligned}
() \quad u(r) &= u(r_1) - \int_{r_1}^{r}\left[\left(\frac{r_0}{t}\right)^{n-1}\Big(-u'(r_1)\Big)^{m-1} + \int_{r_1}^{t}\left(\frac{s}{t}\right)^{n-1}\Big(u(s)\Big)ds\right]^{\frac{1}{m-1}} dt \\
&\leq u(r_1) - m^{\frac{n-1}{m-1}}\int_{r_1}^{r}\left[\int_{r_1}^{t}\left(\frac{s}{t}\right)^{n-1} ds\right]^{\frac{1}{m-1}} dt \\
&\leq u(r_1) - Cr^{\frac{m}{m-1}} \quad \text{for r large} \\
&< 0 \quad \text{for r large.}
\end{aligned} \tag{2.6}$$

This contradicts $u(r) > 0$ for all r. The proof of this Lemma is complete. Q.E.D.

Lemma 2.2. *If the initial data in* (2.1) *satisfies*

$$\frac{u_0'}{u_0} < -\frac{n-m}{m-1}\frac{1}{r_0}, \tag{2.7}$$

then (2.1) *does not possess any positive solution on* $[r_0, \infty)$.

Proof. Suppose that the conclusion of the lemma is false. Then (2.1) has a positive solution $u(r)$ on $[r_0, \infty)$. Let

$$v(r) = -\frac{ru'(r)}{u(r)}. \tag{2.8}$$

Then v satisfies

$$\begin{cases} rv' + \dfrac{n-m}{m-1}v - v^2 = \dfrac{1}{m-1}\dfrac{r^2 f(u)}{u}|u'|^{2-m} > 0, \ r > r_0; \\ v(r_0) = -\dfrac{r_0 u_0'}{u_0} > \dfrac{n-m}{m-1}. \ \text{(by the assumption (2.7))} \end{cases} \tag{2.9}$$

Let

$$w(r) = \frac{\frac{n-m}{m-1}}{1 - Tr^{\frac{n-m}{m-1}}}, \quad T = \frac{v(r_0) - \frac{n-m}{m-1}}{v(r_0) r_0^{\frac{n-m}{m-1}}} > 0. \tag{2.10}$$

Then w blows up as $r \to T_0 = T^{-\frac{n-m}{m-1}}$ and w satisfies

$$\begin{cases} rw' + \dfrac{n-m}{m-1}w - w^2 = 0, \quad r > r_0; \\ w(r_0) = v(r_0). \end{cases} \tag{2.11}$$

By the comparison theorem we obtain

$$v(r) \geq w(r) > 0 \ \text{ for all } \ r \geq r_0.$$

This proves that $v(r)$ also blows up as $r \to T_0$. This contradiction completes the proof of Lemma 2.2. Q.E.D.

Now we are in a position to proof Theorem 2.1.

Proof of Theorem 2.1. Suppose that $u(r)$ is a positive solution of (2.1) defined on $[r_0, \infty)$. Then applying Lemma 1.1 of [6], we can obtain the result

(2.3). The rest of the proof of Theorem 2.1 is just the consequence of Lemma 2.2. The proof is complete. Q.E.D.

3. Proof of Theorem 1.1

In this section, we give a complete proof of Theorem 1.1.

Proof of Theorem 1.1. First we shall prove that if (1.2) admits a positive solution $u = u(r)$ on $[r_0, \infty)$, then (1.4) holds. From Lemma 2.1, there exists $r_1 \geq r_0$ such that $u'(r) < 0$ for all $r \geq r_1$. Let

$$J(\epsilon) = \int_{\epsilon}^{u(r_1)} f(u) u^{-\frac{m(n-1)}{n-m}} du. \tag{3.1}$$

We want to prove that $J(\epsilon)$ is bounded as $\epsilon \to 0^+$. If we change the integration variable from u to r along the solution $u = u(r)$, then

$$\begin{aligned} J(\epsilon) &= \int_{r(\epsilon)}^{r_1} f(u(r))(u(r))^{-\frac{m(n-1)}{n-m}} u'(r) dr \\ &= \int_{r_1}^{r(\epsilon)} f(u(r))(u(r))^{-\frac{n(m-1)}{n-m}} v(r) r^{-1} dr, \end{aligned} \tag{3.2}$$

where $r(\epsilon)$ stands for the solution of $u(r) = \epsilon$ and $v(r) = -\frac{ru'(r)}{u(r)}$. From Theorem 2.1 and Lemma 2.1 we have $r(\epsilon) \to \infty$ as $\epsilon \to 0^+$ and $0 < v(r) \leq \frac{n-m}{m-1}$ for all $r \geq r_1$. Then we have

$$\begin{aligned} & f(u) u^{-\frac{n(m-1)}{n-m}} v r^{-1} = \{r^2 \tfrac{f(u)}{u}\} u^{-\frac{nm+m-2n}{n-m}} v r^{-3} \\ &= \left\{ (m-1)|u'|^{m-2} \left[r v' + \left(\frac{n-m}{m-1} - v \right) v \right] \right\} u^{-\frac{nm+m-2n}{n-m}} v r^{-3} \text{ (by (2.10))} \\ &\leq (m-1) \left(\frac{n-m}{m-1} \right)^{m-1} \left[v' + \left(\frac{n-m}{m-1} - v \right) v r^{-1} \right] u^{-\frac{m(m-1)}{n-m}} r^{-m} \text{ (by (2.2))}. \end{aligned} \tag{3.3}$$

Let

$$X(r) = u^{-\frac{m(m-1)}{n-m}} r^{-m}. \tag{3.4}$$

Then

$$\frac{dX}{dr} = \frac{X}{r} \left(-\frac{m(m-1)}{n-m} \right) \left(\frac{n-m}{m-1} - v(r) \right). \tag{3.5}$$

Integrating (3.5) from r_1 to r we obtain

$$
\begin{aligned}
&\Big(u(r)\Big)^{-\frac{m(m-1)}{n-m}} r^{-m} \\
(3.6)\qquad &= \Big(u(r_1)\Big)^{-\frac{m(m-1)}{n-m}} r_1^{-m} \exp\left(-\frac{m(m-1)}{n-m}\int_{r_1}^{r}\Big(\frac{n-m}{m-1}-v(s)\Big)s^{-1}ds\right).
\end{aligned}
$$

From (3.3) and (3.6) we conclude that

$$
\begin{aligned}
& f(u(r))\Big(u(r)\Big)^{-\frac{n(m-1)}{n-m}} v(r)r^{-1} \\
&\quad\le C_1\left[v'(r)+\left(\frac{n-m}{m-1}-v(r)\right)v(r)r^{-1}\right] \\
(3.7)\qquad &\qquad\cdot\exp\left(-\frac{m(m-1)}{n-m}\int_{r_1}^{r}\Big(\frac{n-m}{m-1}-v(s)\Big)s^{-1}ds\right) \\
&\quad\le C_1\frac{d}{dr}\left\{\left[v(r)-\frac{(n-m)(m^2-2m+n)}{m(m-1)^2}\right]\right. \\
&\qquad\left.\cdot\exp\left(-\frac{m(m-1)}{n-m}\int_{r_1}^{r}\left(\frac{n-m}{m-1}-v(s)\right)s^{-1}ds\right)\right\},
\end{aligned}
$$

where $C_1=(m-1)(\frac{n-m}{m-1})^{m-1}(u(r_1))^{-\frac{m(m-1)}{n-m}}{r_1}^{-m}$. Thus from (3.2) and (3.7) we have

$$
\begin{aligned}
J(\epsilon)\le\ & C_1\left[\left(-\frac{(n-m)(m^2-2m+n)}{m(m-1)^2}+v(r(\epsilon))\right)\right. \\
&\cdot\exp\left(-\frac{m(m-1)}{n-m}\int_{r_1}^{r(\epsilon)}\Big(\frac{n-m}{m-1}-v(s))s^{-1}ds\right) \\
(3.8)\qquad &\left.-v(r_1)+\frac{(n-m)(m^2-2m+n)}{m(m-1)^2}\right] \\
\le\ & \frac{(n-m)(m^2-2m+n)}{m(m-1)^2}C_1 \\
&\left(\text{using}:\quad v\le\frac{n-m}{m-1}\le\frac{n-m}{m-1}\frac{(m^2-2m+n)}{m(m-1)}\right).
\end{aligned}
$$

Hence $J(\epsilon)$ is bounded as $\epsilon\to 0^+$. Thus (1.4) holds.

Now if (1.4) holds, then for any positive constant C we want to construct a positive solution u which exists for sufficiently large r and satisfies (1.5) and (1.6). First we make the following change of variables:

$$
y=r^{-\frac{n-m}{m-1}},\quad z=-r^{\frac{n-1}{m-1}}\frac{du}{dr}. \tag{3.9}
$$

Then $y,\ z$ satisfies

$$
\left\{
\begin{aligned}
\frac{dy}{du} &= \frac{n-m}{m-1} z^{-1}, \\
\frac{dz}{du} &= -\frac{1}{m-1} f(u) y^{-\frac{m(n-1)}{n-m}} z^{1-m}.
\end{aligned}
\right.
\tag{3.10}
$$

We want to prove that $(y(u), z(u))$ exists for u small and

$$
y(u) \sim \frac{u}{C}, \quad z(u) \to \frac{n-m}{m-1} C \qquad \text{as } u \to 0^+. \tag{3.11}
$$

Let

$$
\begin{aligned}
Y = \Bigg\{ \Big(y(u), z(u) \Big) \in C[0,u_0] \times C[0,u_0] \ \Bigg| \\
\frac{u}{C} \le y(u) \le \frac{2u}{C},\ \frac{1}{2}\frac{n-m}{m-1} C \le z(u) \le \frac{n-m}{m-1} C,\ 0 \le u \le u_0 \Bigg\},
\end{aligned}
\tag{3.12}
$$

where u_0 is a positive constant to be determined later. It is easy to see that Y is a closed convex subset of $C[0,u_0] \times C[0,u_0]$ in the usual topology. We define a mapping T on Y by

$$
\begin{aligned}
T\begin{pmatrix} y \\ z \end{pmatrix}(u) &= \begin{pmatrix} \dfrac{n-m}{m-1} \displaystyle\int_0^u \frac{1}{z(s)} ds, \\ \dfrac{n-m}{m-1} C - \dfrac{1}{m-1} \displaystyle\int_0^u f(s) (y(s))^{-\frac{m(n-1)}{n-m}} (z(s))^{1-m} ds \end{pmatrix} \\
&\equiv \begin{pmatrix} y^*(u) \\ z^*(u) \end{pmatrix}.
\end{aligned}
\tag{3.13}
$$

We shall verify that T is a compact continuous mapping from Y into Y. Let $(y(u), z(u)) \in Y,\ 0 \le u \le u_0$. We have

$$
\begin{aligned}
&\frac{u}{C} \le y^*(u) \le \frac{2u}{C}, \\
&\frac{n-m}{m-1} C - \frac{2^{m-1}}{m-1} \left(\frac{m-1}{n-m} \right)^{m-1} C^{\frac{m(m-2)+n}{n-m}} g(u) \le z^*(u) \le \frac{n-m}{m-1} C,
\end{aligned}
\tag{3.14}
$$

where

$$
g(u) \equiv \int_0^u f(s) s^{-\frac{m(n-1)}{n-m}} ds.
$$

Choose $u_0 > 0$ so small that

$$g(u_0) \le \frac{n-m}{2^m}\left(\frac{n-m}{m-1}\right)^{m-1} C^{-\frac{m(m-1)}{n-m}}. \tag{3.15}$$

This is possible since (1.4) holds. Hence $(y^*(u), z^*(u)) \in Y$ for $0 \le u \le u_0$. This proves that T maps Y into Y. It is easy to see that T is continuous. To prove that T is compact, we let $\{(y_m, z_m)\} \subset Y$ be a sequence of functions. From (3.14) we obtain that $\left\{T\begin{pmatrix} y_m \\ z_m \end{pmatrix}\right\}$ is uniformly bounded. Moreover if $0 \le u \le u+h \le u_0$, then

$$\begin{aligned}
|y_m^*(u+h) - y_m^*(u)| &= \frac{n-m}{m-1}\int_u^{u+h} \frac{1}{z_m(s)} ds \le \frac{2h}{C} \\
|z_m^*(u+h) - z_m^*(u)| &= \frac{1}{m-1}\int_u^{u+h} f(s)(y_m(s))^{-\frac{m(n-1)}{n-m}}(z_m(s))^{1-m} ds \\
&\le \frac{2^{m-1}}{m-1}\left(\frac{m-1}{n-m}\right)^{m-1} C^{\frac{m^2-2m+n}{n-m}}[g(u+h) - g(u)].
\end{aligned} \tag{3.16}$$

From (3.14) and (3.16) we conclude that $\left\{T\begin{pmatrix} y_m \\ z_m \end{pmatrix}\right\}$ is a uniformly bounded and equicontinuous sequence of functions. This proves that T is a compact mapping. Schauder's fixed point theorem then ensures that there is a $\begin{pmatrix} \overline{y} \\ \overline{z} \end{pmatrix} \in Y$ such that

$$\begin{pmatrix} \overline{y} \\ \overline{z} \end{pmatrix} = T\begin{pmatrix} \overline{y} \\ \overline{z} \end{pmatrix}.$$

This $\begin{pmatrix} \overline{y}(u) \\ \overline{z}(u) \end{pmatrix}, 0 \le u \le u_0$, is a solution of (3.10). Since $\frac{d\overline{y}}{du} = \frac{n-m}{m-1}\frac{1}{\overline{z}(u)} > 0$, $\overline{y}(u)$ is strictly monotone, and also so is $r(u) = (\overline{y}(u))^{-\frac{m-1}{n-m}}$. Moreover $r(u) \to \infty$ as $u \to 0^+$. Thus the inverse function $u = u(r)$ is well-defined and is a solution of (1.2) for all $r \ge r_0$, where $r_0 = (\overline{y}(u_0))^{-\frac{m-1}{n-m}}$. From (3.11) we obtain (1.5) and (1.6). The proof of Theorem 1.1 is complete. Q.E.D.

4. Proof of Theorem 1.2

In this section, we give a complete proof of Theorem 1.2. First, we need the following lemma.

Lemma 4.1 *Assume that f satisfies the inequality* (1.4). *Let u be a positive solution of* (1.2) *on $[r_0, \infty)$ which was obtained in Theorem* 1.1. *Then for all $r \geq r_0$, we have the Pohozaev identity*

$$\begin{aligned} &\int_r^{\infty} [nF(u(s)) - \frac{n-m}{m} u(s) f(u(s))] s^{n-1} ds \\ &= -r^n \left[\frac{m-1}{m} |u'(r)|^m + F(u(r)) + \frac{n-m}{m} \frac{|u'(r)|^{m-2} u'(r) u(r)}{r} \right]. \end{aligned} \tag{4.1}$$

Proof. Let

$$V(r) = r^n \left[\frac{m-1}{m} |u'(r)|^m + F(u(r)) + \frac{n-m}{m} \frac{|u'(r)|^{m-2} u'(r) u(r)}{r} \right].$$

From the proof of Theorem 1.1, we know that $u'(r) < 0$ for all $r \geq r_0$. Using this observation and (1.2), we can calculate $V'(r)$ and obtain

$$\frac{dV(r)}{dr} = [nF(u(r)) - \frac{n-m}{m} u(r) f(u(r))] r^{n-1}. \tag{4.2}$$

Integrating (4.2) from r to ∞ and using the fact of (1.5) – (1.6), we finally obtain (4.1). This completes the proof of Lemma 4.1. Q.E.D.

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. For any positive constant C, from Theorem 1.1 we obtain that : there exists a positive r_0, which is depending on f and C, such that (1.2) possesses a positive solution $u(r)$ on $[r_0, \infty)$ and u satisfies (1.5) – (1.6) and $u'(r) < 0$ for all $r \geq r_0$. Now we extend this solution $u(r)$ backward into the region $r < r_0$. Let

$$\xi = \inf\{\delta > 0 | u(r) \text{ satisfies (1.2) and } u'(r) < 0 \text{ for all } r \in (\delta, \infty)\}$$

First we claim that $\xi = 0$. Assume $\xi > 0$. From (1.2) we obtain that, for all $\xi < r < r_0$,

$$\begin{aligned} r^{n-1} |u'(r)|^{m-2} u'(r) &= r_0^{n-1} |u'(r_0)|^{m-2} u'(r_0) + \int_r^{r_0} s^{n-1} f(u(s)) ds \\ &> r_0^{n-1} |u'(r_0)|^{m-2} u'(r_0). \end{aligned} \tag{4.3}$$

Since $u'(r) < 0$ for all $\xi < r < r_0$, we have

$$0 > u'(r) > \left(\frac{r_0}{r}\right)^{\frac{n-1}{m-1}} u'(r_0) > -\infty.$$

Hence $u'(r)$ remains bounded as $r \to \xi^+$. So we obtain that $u(\xi)$ remains bounded. From (4.3) we also obtain that $\lim_{r\to\xi^+} u'(r)$ exists and $u'(\xi) \leq 0$. From the definition of ξ and the assumption $\xi > 0$, we must have $u'(\xi) = 0$. Therefore, $u(r)$ is a classical positive solution of (1.2) on $[\xi, \infty)$ and $u'(\xi) = 0$. From (4.1) in Lemma 4.1, we obtain that $0 \leq \int_\xi^\infty [nF(u(s)) - \frac{n-m}{m} u(s) f(u(s))] s^{n-1} ds = -\xi^n F(u(\xi)) < 0$, a contradiction. Hence $\xi = 0$. Again by (4.1) in Lemma 4.2, it is easy to conclude that $u(r)$ must be singular at $r = 0$. The proof is complete. Q.E.D.

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