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ON JACOBSON PROPERTY OF Γ_N-RINGS

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Abstract. Let M be a Γ-ring in the sense of Nobusawa. The ring $M_2 = \begin{pmatrix} R & \Gamma \\ M & \Gamma \end{pmatrix}$ was defined by Kyuno. Let $\mathcal{P}$ be a class of prime rings such that for every prime ring R and any $0 \neq e^2 = e \in R$, $R \in \mathcal{P}$ if and only if $eRe \in \mathcal{P}$. In this paper, the $\mathcal{P}$-Jacobson Γ-rings which include the Jacobson property and Brown-McCoy property as special case are defined. Relationships between $\mathcal{P}$-Jacobson properties of Γ-ring M and the corresponding properties of $\Gamma_{n,m}$-ring $M_{m,n}$, the right operator ring R of Γ-ring M, M-ring Γ and the ring M_2 are established.

1. Introduction

The class of Jacobson rings, which consists of all rings in which the prime radical coincides with the Jacobson radical in all homomorphic images, is an important class in ring theory. For several years, Jacobson rings and their generalizations are extensively studied. In [9, 10] some of the Jacobson and Brown-McCoy properties have been established for Γ-rings. Jacobson Γ-rings are defined to be those Γ-rings in which the prime radical equals the Jacobson radical in all homomorphic images. Brown-McCoy Γ-rings are defined to be those Γ-rings in which the prime radical equals the Brown-McCoy radical in all homomorphic images.

Througout this paper, let $\mathcal{P}$ be a class of prime rings such that for every prime ring R and any $0 \neq e^2 = e \in R$, $eRe \in \mathcal{P}$ if and only if $R \in \mathcal{P}$. We note that many well known classes of rings satisfy this property: the class of prime rings, the class of primitive rings, the class of prime Levitzki semisimple rings, the class of prime subdirect irreducible rings, the class of primitive rings with

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nonzero socle, the class of weak primitive rings, the class of k-primitive rings, the class of prime Johnson rings, and the class of nonsingular prime rings (cf. [6, P.11]) This motivates us to consider the class of $\mathcal{P}$-Jacobson Γ-rings (cf. Definition 3.1).

Let M and Γ be additive abelian groups. If, for all $x, y, z \in M$ and $\alpha, \beta, \mu \in \Gamma$ the following hold:

(1) $x\beta y \in M$;

(2) $x\beta(y\mu z) = (x\beta y)\mu z$;

(3) $x\beta(y+z) = x\beta y + x\beta z; x(\beta+\mu)y = x\beta y + x\mu y; (x+y)\beta z = x\beta z + y\beta z$,

then M is called a Γ-*ring.* A weak Γ_N-*ring* is a pair (M, Γ) such that

(4) M is a Γ-ring and Γ is a M-ring;

(5) $x\beta(y\mu z) = x(\beta y\mu)z = (x\beta y)\mu z$ and $(\alpha x\beta)y\mu = \alpha(x\beta y)\mu = \alpha x(\beta y\mu)$.

If in addition, we have that

(6) $x\mu y = 0$ for all $x, y \in M$ implies $\mu = 0$,

then the pair (M, Γ) is called a Γ_N-*ring.*

The notions of ideals, prime ideals, homomorphisms and subdirect sums of Γ-rings are defined exactly as for rings. "$I \trianglelefteq M$" will denote "I is an ideal of M".

If M is a Γ-ring, let R be the subring of the endomorphism ring of M (operator being taken to act on the right), consisting of all sums of the form $\sum_{i=1}^{n}[\beta_i, x_i](\beta_i \in \Gamma, x_i \in M)$, where $[\beta, x]$ is defined by: $y[\beta, x] = y\beta x$, for all $y \in M$. The ring R is called the *right operator ring* of M. A *left operator ring* L of M is similarly defined.

A Γ-ring M is said to *have right unity* if there exists an element $\sum_{i=1}^{n}[\delta_i, a_i] \in R$ such that $\sum_{i=1}^{n} x\delta_i a_i = x$ for all $x \in M$. It is easily verified in this case, $\sum_{i=1}^{n}[\delta_i, a_i]$ is the unity of the ring R. Left unity is similarly defined.

Let (M, Γ) be a weak Γ_N-ring and $A \subseteq M$, $P \subseteq R$, $Q \subseteq L$ and $\Phi \subseteq \Gamma$. Then we define: $P^* = \{x \in M : [\beta, x] \in P \text{ for all } \beta \in \Gamma\}$, $Q^+ = \{x \in M : [x, \mu] \in Q \text{ for all } \mu \in \Gamma\}$, $A^{*'} = \{r \in R : Mr \subseteq A\}$, $A^{+'} = \{y \in L : yM \subseteq A\}$, $\Gamma(A) = \{\mu \in \Gamma : M\mu M \subseteq A\}$ and $M(\Phi) = \{x \in M : \Gamma x\Gamma \subseteq \Phi\}$.

Let M be a Γ-ring, and let $M_{m,n}$ and $\Gamma_{n,m}$ denote the sets of $m \times n$ matrices with entries from M and of $n \times m$ matrices with entries from Γ, respectively. Then $M_{m,n}$ is a $\Gamma_{n,m}$-ring with matrix addition and the obviously defined muliplication.

Let (M, Γ) be a Γ_N-ring, R and L denote respectively the right and left operator rings of Γ-ring M.

The set $M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix}$ is a ring with respect to the obvious operations of matrix multiplication and addition. For details, see [1, 5]. Moreover, if $I \trianglelefteq M$, then it is easily verified that

$$I_2 = \begin{pmatrix} I^{*'} & \Gamma(I) \\ I & I^{+'} \end{pmatrix} \trianglelefteq M_2.$$

For further details of Γ-rings and their operator rings, we refer to [1] and [4]. Radical classes of Γ-rings, special radical and hereditary classes are defined exactly as for rings. See, for example, [2, 3].

2. $\mathcal{P}$ ideals and $\mathcal{P}$-radical of Γ-rings

As in G. L. Booth [3], a Γ-ring M is called a (right) *$\mathcal{P}$-Γ-ring* or *$\mathcal{P}_{(\Gamma)}$-ring* if (i) the right operator ring R of M belongs to $\mathcal{P}$, (ii) $M\Gamma x = 0$ implies $x = 0$. We will use $\mathcal{P}_{(\Gamma)}$ to denote the class of all $\mathcal{P}$-Γ-rings. $I \trianglelefteq M$ is called a *$\mathcal{P}$-ideal* if $M/I \in \mathcal{P}_{(\Gamma)}$. For any Γ-ring M, the *$\mathcal{P}$-radical* of M is defined as the intersection of all $\mathcal{P}$-ideals of M and is denoted by $R_{\mathcal{P}}(M)$.

In the following only right operator rings are considered. Analogous results for the left operator ring can be proved similarly. We will need the following two lemmas.

Lemma 2.1. *If A is an ideal of the Γ-ring M, R and $[\Gamma, M/A]$ are the right operator rings of Γ-ring M and Γ-ring M/A, respectively, then we have $[\Gamma, M/A] \cong R/A^{*'}$ under the mapping*

$$\sum_i [\gamma_i, x_i + A] \longrightarrow \sum_i [\gamma_i, x_i] + A^{*'}.$$

Moreover, if (M, Γ) is a weak Γ_N-ring, we have $[\Gamma/\Gamma(A), M/A] \cong R/A^{*'}$ under the mapping

$$\sum_i [\gamma_i + \Gamma(A), x_i + A] \longrightarrow \sum_i [\gamma_i, x_i] + A^{*'}.$$

Lemma 2.2. *If M is Γ-ring with right operator ring R. and P is a prime ideal of R or M has right unity and P is an ideal of R, $[\Gamma, M/P^*]$ is the right operator rings of Γ-ring M/P^*, then we have $[\Gamma, M/P^*] \cong R/P$ under the mapping*

$$\sum_i [\gamma_i, x_i + P^*] \longrightarrow \sum_i [\gamma_i, x_i] + P.$$

The proof of Lemmas 2.1 and 2.2 may be easily verified by direct computation.

As an immediate consequence of Lemma 2.2 and [4, Theorem 1], we have the following result.

Proposition 2.3. *Let M be a Γ-ring with right operator ring R. Then the mapping $P \longrightarrow P^*$ defines a one-to-one correspondence between the $\mathcal{P}$-ideals of R and that of M. Moreover, $(P^*)^{*'} = P$.*

Let (M, Γ) be a Γ_N-ring. It is easily verified that an ideal P of M is $\mathcal{P}$-ideal if and only if M/P is a $\mathcal{P}$-$\Gamma/\Gamma(P)$-ring. From [1, Theorem 3.3], we have the following

Proposition 2.4. *Let (M, Γ) be a weak Γ_N-ring. Then the mapping $A \longrightarrow \Gamma(A)$ defines a one-to-one correspondence between the sets of $\mathcal{P}$-ideals of Γ-ring M and those of M-ring Γ. Moreover, $M(\Gamma(A)) = A$.*

Corollary 2.5. *Let (M, Γ) be a weak Γ_N-ring. Then $R_{\mathcal{P}}(\Gamma) = \Gamma(R_{\mathcal{P}}(M))$.*

Proposition 2.6. *Let (M, Γ) be a weak Γ_N-ring and Γ-ring M has right unity. Then $M_2 \in \mathcal{P}$ if and only if $M \in \mathcal{P}_{(\Gamma)}$.*

Proof. Suppose that Γ-ring $M \in \mathcal{P}_{(\Gamma)}$. Then $R \in \mathcal{P}$ and M is a prime Γ-ring. By [1, Theorem 3.5], M_2 is a prime ring. Since $eM_2e \cong R \in \mathcal{P}$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 \in \mathcal{P}$.

Conversely, suppose that $M_2 \in \mathcal{P}$. Since $eM_2e \cong R, R \in \mathcal{P}$. By the primeness of M_2 and [1], Theorem 3.5, M is a prime Γ-ring, thus $M\Gamma x = 0$, $x \in M$, implies $x = 0$. Therefore $M \in \mathcal{P}_{(\Gamma)}$.

Proposition 2.7. *Let (M, Γ) be a Γ_N-ring and let Γ-ring M have right unity. Then a subset P_2 of M_2 is a $\mathcal{P}$-ideal if and only if*

$$P_2 = \begin{pmatrix} A^{*'} & \Gamma(A) \\ A & A^{+'} \end{pmatrix},$$

where A is a $\mathcal{P}$-ideal of Γ-ring M.

Proof. Suppose that P_2 is a $\mathcal{P}$-ideal of M_2. Then P_2 is a prime ideal of M_2. Hence, by [1], Theorem 3.6,

$$P_2 = \begin{pmatrix} A^{*'} & \Gamma(A) \\ A & A^{+'} \end{pmatrix}$$

for some prime ideal A of M. Now

$$M_2/P_2 \cong \begin{pmatrix} R/A^{*'} & \Gamma/\Gamma(A) \\ M/A & L/A^{+'} \end{pmatrix} \in \mathcal{P}.$$

Hence, by Proposition 2.6, $M/A \in \mathcal{P}_{(\Gamma)}$, i.e., A is a $\mathcal{P}$-ideal of M.

Conversely, if A is a $\mathcal{P}$-ideal of M. then $M/A \in \mathcal{P}_{(\Gamma)}$. Hence, by Proposition 2.6, $M_2/P_2 \cong (M/A)_2 \in \mathcal{P}$, i.e., P_2 is a $\mathcal{P}$-ideal of M_2.

The following result generalizing a result of Kyuno [5] to Γ-rings which do not necessarily have left unity.

Corollary 2.8. *Let (M, Γ) be a Γ_N-ring and let Γ-ring M have right unity. Then*

$$R_{\mathcal{P}}(M_2) = \begin{pmatrix} R_{\mathcal{P}}(R) & R_{\mathcal{P}}(\Gamma) \\ R_{\mathcal{P}}(M) & R_{\mathcal{P}}(L) \end{pmatrix}.$$

3. $\mathcal{P}$-Jacobson Γ-rings and Matrix $\Gamma_{n,m}$-rings

Definition 3.1. Γ-ring M is called a $\mathcal{P}$-Jacobson Γ-ring if the prime radical equals the $\mathcal{P}$-radical in all homomorphic images of M.

Clearly, Brown-McCoy Γ-rings and Jacobson Γ-rings are special case of $\mathcal{P}$-Jacobson Γ-rings. It is easy to prove the followings:

Proposition 3.2. *If M is a $\mathcal{P}$-Jacobson Γ-ring and $I \trianglelefteq M$, then M/I is also a $\mathcal{P}$-Jacobson.*

Proposition 3.3. *M is a $\mathcal{P}$-Jacobson Γ-ring if and only if every prime ideal of M is an intersection of $\mathcal{P}$-ideals of M.*

Corollary 3.4. *Γ-ring M is $\mathcal{P}$-Jacobson if and only if every prime homomorphic image of M is $R_{\mathcal{P}}$-semisimple.*

If we denote by $\mathcal{R}$ the class of all $\mathcal{P}$-Jacobson Γ-rings, then, from Definition 3.1, we have

$$\mathcal{R} = \{M : P(M/I) = R_{\mathcal{P}}(M/I) \text{ for every ideal } I \text{ of } M\}.$$

If $R_{\mathcal{P}}$ is a hereditary radical, as in the cass of [7, Theorem 26] we have that $\mathcal{R}$ is a radical class. If we denote by γ the radical associated with the $\mathcal{P}$-Jacobson Γ-ring class $\mathcal{R}$. then γ is the largest radical class such that $\gamma(M) \cap R_{\mathcal{P}}(M) \subseteq P(M)$ for every Γ-ring M.

In the following we assume that $R_{\mathcal{P}}$ is a hereditary radical.

Proposition 3.5. *γ is hereditary i.e., ideals of $\mathcal{P}$-Jacobson Γ-rings are $\mathcal{P}$-Jacobson.*

Proof. Let $\mathcal{R}_p$ denote the class of all $\mathcal{P}$-radical Γ-rings, and $\mathcal{U}_p$ the class of all semiprime Γ-rings. Then $\mathcal{R}_p \cap \mathcal{U}_p$ is a class of semiprime Γ-rings. In a way similar to that of [8, Proposition 2] we can prove that γ is hereditary.

Corollary 3.6. *Let M be a Γ-ring and I an ideal of M. Then $\gamma(I) = I \cap \gamma(M)$.*

We now prove the next theorem which indicates one way to construct new $\mathcal{P}$-Γ-rings from given ones.

Theorem 3.7. *Let M be a Γ-ring with right unity. Then $M \in \mathcal{P}_{(\Gamma)}$ if and only if the $\Gamma_{n,m}$-ring $M_{m,n} \in \mathcal{P}_{(\Gamma)}$.*

Proof. Suppose that $M \in \mathcal{P}_{(\Gamma)}$. Then $R = [\Gamma, M] \in \mathcal{P}$ and M satifies $M\Gamma x = 0$ implies $x = 0$. Denote the right operator ring of $M_{m,n}$ by $[\Gamma_{n,m}, M_{m,n}]$, and recall that $[\Gamma_{n,m}, M_{m,n}] \cong R_n$ (see [4, P. 376]) and R_n is prime. We have that $[\Gamma_{n,m}, M_{m,n}] \in \mathcal{P}$. Also, if $M_{m,n}\Gamma_{n,m}(x_{i,j}) = 0, (x_{i,j}) \in M_{m,n}$, then for all $m \in M,\ \gamma \in \Gamma$, we have that

$$0 = (me_{ik})(\gamma e_{kj})(x_{st}) = \begin{pmatrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ m\gamma x_{j1} & \cdots & m\gamma x_{jn} \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} (i).$$

Therefore, $M\Gamma x_{ij} = 0$ for all $1 \le i \le m$, $1 \le j \le n$, and consequently $x_{ij} = 0$ and $(x_{ij}) = 0$. Hence $\Gamma_{n,m}$-ring $M_{m,n} \in \mathcal{P}_{(\Gamma)}$.

Conversely, suppose that $\Gamma_{n,m}$-ring $M_{m,n} \in \mathcal{P}_{(\Gamma)}$. Then $[\Gamma_{n,m}, M_{m,n}] \cong R_n \in \mathcal{P}$. Hence $R \in \mathcal{P}$. Also, if $M\Gamma x = 0$, $x \in M$, then $M_{m,n}\Gamma_{n,m} x e_{11} = 0$ and consequently $xe_{11} = 0$, i.e. $x = 0$. Hence $M \in \mathcal{P}_{(\Gamma)}$.

Lemma 3.8. *If $I \trianglelefteq M$, then the matrix $\Gamma_{n,m}$-ring $(M/I)_{m,n}$ is isomorphic to the $\Gamma_{n,m}$-ring $M_{m,n}/I_{m,n}$.*

Lemma 3.9[4]. *Let M be an arbitrary Γ-ring. Then the prime ideals of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ are precisely the sets $P_{n,m}$, where P is a prime ideal of the Γ-ring M.*

As a consequence of Lemma 3.8 and Theorem 3.7, we have

Theorem 3.10. *Let M be an Γ-ring with right unity. Then the $\mathcal{P}$-ideals of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ are preciselg the sets $P_{n,m}$, where P is a $\mathcal{P}$-ideal of the Γ-ring M.*

Theorem 3.11. *If M is a Γ-ring, then $\mathcal{R}_{\mathcal{P}}(M_{m,n}) = (\mathcal{R}_{\mathcal{P}}(M))_{m,n}$.*

Proof. By Theorem 3.10, we have that

$$\begin{aligned}\mathcal{R}_{\mathcal{P}}(M_{m,n}) &= \cap\{(I)_{m,n} \,|\, I \text{ is a } \mathcal{P}-\text{ideal of } M\}\\ &= (\cap\{I/I \text{ is a } \mathcal{P}-\text{ideal of } M\})_{m,n}\\ &= [\mathcal{R}_{\mathcal{P}}(M)]_{m,n}.\end{aligned}$$

Theorem 3.12. *Let M be a Γ-ring with right unity. Then Γ-ring M is $\mathcal{P}$-Jacobson if and only if, for any m,n, the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ is $\mathcal{P}$-Jacobson.*

Proof. Suppose that M is a $\mathcal{P}$-Jacobson Γ-ring. By Lemma 3.9, any prime ideal of $M_{m,n}$ is of the form $P_{m,n}$, where P is a prime ideal of the Γ-ring M. Since the Γ-ring M is $\mathcal{P}$-Jacobson, whence $P = \cap P_\alpha$, where $P_\alpha(\alpha \in \Lambda)$ are $\mathcal{P}$-ideals of M. Now $P_{m,n} = (\cap P_\alpha)_{m,n} = \cap(P_\alpha)_{m,n}$, and $M_{m,n}/(P_\alpha)_{m,n} \cong (M/P_\alpha)_{m,n}$, we have $(P_\alpha)_{m,n}(\alpha \in \Lambda)$ are $\mathcal{P}$-ideals of $M_{m,n}$. Thus $M_{m,n}$ is a $\mathcal{P}$-Jacobson $\Gamma_{n,m}$-ring.

Conversely, suppose that $M_{m,n}$ is a $\mathcal{P}$-Jacobson $\Gamma_{n,m}$-ring. Let P be a prime ideal of Γ-ring M. Now $M_{m,n}/P_{m,n} \cong (M/P)_{m,n}$ is prime and so $R_{\mathcal{P}}((M/P)_{m,n}) = 0$. Thus $R_{\mathcal{P}}(M/P) = 0$ and M is a $\mathcal{P}$-Jacobson Γ-ring.

Corollary 3.13. *A ring R is a $\mathcal{P}$-Jacobson ring if and only if, for any m, n, the $R_{n,m}$-ring $R_{m,n}$ is $\mathcal{P}$-Jacobson.*

Corollary 3.14. $\gamma(M_{m,n}) = (\gamma(M))_{m,n}$ for any Γ-ring M.

Proof. This follows immediately from Theorem 3.12 and the corresponding result for ring [7, Lemma 8].

4. $\mathcal{P}$-Jacobson Γ-rings and the Operator Rings

In this section, the relationships between $\mathcal{P}$-Jacobson properties of Γ-ring M and its right operator rings are established. Analogous results for the left operator ring can be proved similarly.

Theorem 4.1. *Let M be a Γ-ring, and let R be the right operator ring of M. Then M is a $\mathcal{P}$-Jacobson Γ-ring if and only if R is a $\mathcal{P}$-Jacobson ring.*

Proof. Suppose that M is a $\mathcal{P}$-Jacobson Γ-ring. For every prime ideal Q of R, there is a prime ideal P of M such that $Q = P^{*'}$. Hence, there exist $\mathcal{P}$-ideals P_α of $M(\alpha \in \Lambda)$ such that $P = \cap\{P_\alpha : \alpha \in \Lambda\}$. Then

$$\begin{aligned} Q &= P^{*'} = \{x \in R : Mx \subseteq P\} \\ &= \cap\{x \in R : Mx \subseteq P_\alpha,\ \alpha \in \Lambda\} = \cap\{P_\alpha^{*'} : \alpha \in \Lambda\}. \end{aligned}$$

By Proposition 2.3, $P_\alpha^{*'}(\alpha \in \Lambda)$ are $\mathcal{P}$-ideals of R. Therefore, R is a $\mathcal{P}$-Jacobson ring.

Conversely, suppose now that R is a $\mathcal{P}$-Jacobson ring, and let P be a prime ideal of Γ-ring M. Then there is a prime ideal A of R such that $P = A^* = \{x \in M : [\Gamma, x] \subseteq A\}$. Since R is a $\mathcal{P}$-Jacobson ring, $A = \cap\{A_\alpha : \alpha \in \Lambda\}$, where $A_\alpha(\alpha \in \Lambda)$ are $\mathcal{P}$-ideals of R. But

$$\begin{aligned} P &= A^* = \{x \in M : [\Gamma, x] \subseteq A\} \\ &= \cap\{x \in M : [\Gamma, x] \subseteq A_\alpha, \alpha \in \Lambda\} = \cap\{A_\alpha^* : \alpha \in \Lambda\}. \end{aligned}$$

By Proposition 2.3, $A_\alpha^*(\alpha \in \Lambda)$ are $\mathcal{P}$-ideals of M. Hence, M is a $\mathcal{P}$-Jacobson Γ-ring. This completes the proof.

Lemma 4.2. *Let M be a semiprime Γ-ring and $A \trianglelefteq M$. If $< \Gamma, A >$ denotes the right operator ring of the Γ-ring A, then $< \Gamma, A > \cong [\Gamma, A]$* (see [3, Lemma 2.3]).

Let α be the radical class of $\mathcal{P}$-Jacobson rings, γ denote the radical class of $\mathcal{P}$-Jacobson Γ-rings. Then we have the following.

Lemma 4.3. *Every α-semisimple ring or γ-semisimple Γ-ring is semiprime.*

Lemma 4.4. *Let M be a Γ-ring with right and left unities, and let R be the right operator ring of M. Then M is γ-semisimple if and only if R is α-semisimple.*

Proof. Suppose that R is α-semisimple. If M is not γ-semisimple, then $0 \neq \gamma(M) \trianglelefteq M$ and the right operator ring $< \Gamma, \gamma(M) >$ of Γ-ring $\gamma(M)$ is a α-radical ring by Theorem 4.1. On the other hand, since R is α-semisimple, R is semiprime and hence M is semiprime. By Lemma 4.3, $< \Gamma, \gamma(M) > \cong$

$[\Gamma, \gamma(M)]$ is a α-radical ring. But $[\Gamma, \gamma(M)] \trianglelefteq R$, this contradicts the α-semisimplicity of R. Thus M is γ-semisimple.

Conversely, suppose now that M is γ-semisimple. If R is not α-semisimple, then $0 \neq \alpha(R) \trianglelefteq R$. Since M has right and left unities, it is easy to prove that $\alpha(R) = [\Gamma, \alpha(R)^*]$. By Proposition 3.2 and the fact (see [4, p. 203])

$$\begin{aligned} < \Gamma, \alpha(R)^* > &\cong [\Gamma, \alpha(R)^*]/([\Gamma, \alpha(R)^*] \cap Ann_R \alpha(R)^*) \\ &= \alpha(R)/([\Gamma, \alpha(R)^*] \cap Ann_R \alpha(R)^*) \end{aligned}$$

we have that $< \Gamma, \alpha(R)^* >$ is an α-radical ring. Hence $\alpha(R)^*$ is a γ-radical Γ-ring by Theorem 4.1. But $0 \neq \alpha(R)^* \trianglelefteq M$, this contradicts the γ-semisimplicity of M. Therefore R is α-semisimple.

Theorem 4.5. *Let M be a Γ-ring with left unity, R be the right operator ring of M. Then, we have*
(1) $\alpha(R) \subseteq (\gamma(M))^{*'}$*; and*
(2) *If M has right unity, then $\gamma(M) \subseteq (\alpha(R))^*$.*

Proof. (1) Since $[\Gamma, M/\mu(M)] \cong R/(\mu(M))^{*'}$ by Lemma 2.1 and $M/\gamma(M)$ is γ-semisimple, by Lemma 4.4, $R/(\gamma(M))^{*'}$ is α-semisimple. It follows that $\alpha(R) \subset (\gamma(M))^{*'}$.
(2) By Lemma 2.2, $[\Gamma, M/(\alpha(R))^*] \cong R/\alpha(R)$ and $R/\alpha(R)$ is α-semisimple. Then, by Lemma 4.4, it follows that $\gamma(M) \subseteq (\alpha(R))^*$.

Corollary 4.6. *Let M be a Γ-ring with right and left unities, R be the right operator ring of M. Then, we have $\alpha(R) = (\gamma(M))^{*'}$.*

Proof. Since Γ-ring M has right and left unities, it is easy to prove that $[\Gamma, \gamma(M)] = (\gamma(M))^{*'}$ and $(\alpha(R))^* = M\alpha(R)$. Thus, we have

$$(\gamma(M))^{*'} \subseteq ((\alpha(R))^*)^{*'} = (M\alpha(R))^{*'} = [\Gamma, M\alpha(R)] \subseteq \alpha(R).$$

By Theorem 4.5, we have $\alpha(R) = (\gamma(M))^{*'}$.

5. $\mathcal{P}$-Jacobson Property of M-ring Γ and the Ring M_2

In this section, let (M, Γ) be Γ_N-ring. Let R and L denote respectively the right and left operator rings of Γ-ring M and $M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix}$.

Theorem 5.1. *Let (M, Γ) be a weak Γ_N-ring. Then the Γ-ring M is $\mathcal{P}$-Jacobson if and only if the M-ring Γ is $\mathcal{P}$-Jacobson.*

Proof. Suppose that Γ-ring M is $\mathcal{P}$-Jacobson. For every prime ideal Φ of M-ring Γ, by Proposition 2.4, there is a prime ideal P of Γ-ring M such that $\Phi = \Gamma(P)$. Hence, there exist $\mathcal{P}$-ideals P_α of $M(\alpha \in \Lambda)$ such that $P = \cap\{P\alpha : \alpha \in \Lambda\}$. But

$$\begin{aligned}\Gamma(P) &= \{\gamma \in \Gamma : M\gamma M \subseteq P\} = \cap\{\gamma \in \Gamma : M\gamma M \subseteq P_\alpha,\, \alpha \in \Lambda\} \\ &= \cap\{\Gamma(P_\alpha) : \alpha \in \Lambda\}.\end{aligned}$$

By Proposition 2.4 and [1, Theorem 3.3], $\Gamma(P_\alpha)$ are $\mathcal{P}$-ideals of Γ. Thus, M-ring Γ is $\mathcal{P}$-Jacobson. The proof of the converse is similar. This completes the proof.

Proposition 5.2. *Let (M, Γ) be a weak Γ_N-ring. Then $\gamma(M) \subseteq M(\gamma(\Gamma))$ and $\gamma(\Gamma) \subseteq \Gamma(\gamma(M))$, where $\gamma(M)$ and $\gamma(\Gamma)$ denote respectively the γ-radical of Γ-ring M and M-ring Γ.*

By Theorem 5.1 and the fact that $(M/\gamma(M), \Gamma/\Gamma(\gamma(M)))$ is weak $(\Gamma/\Gamma(\gamma(M))_N$-ring, it is clear that $\gamma(\Gamma) \subseteq \Gamma(\gamma(M))$. Similarly, it can be shown that $\gamma(M) \subseteq M(\gamma(\Gamma))$.

Theorem 5.3. *Let (M, Γ) be a Γ_N-ring and Γ-ring M have right unity. Then the ring $M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix}$ is a $\mathcal{P}$-Jacobson ring if and only if Γ-ring M is $\mathcal{P}$-Jacobson.*

Proof. Suppose that M_2 is a $\mathcal{P}$-Jacobson ring and I is a prime ideal of M. By [1, Theorem 3.6], $I_2 = \begin{pmatrix} I^{*'} & \Gamma(I) \\ I & I^{+'} \end{pmatrix}$ is a prime ideal of M_2. Thus, $I_2 = \cap P_{\alpha,2}$, where $P_{\alpha,2}(\alpha \in \Lambda)$ are $\mathcal{P}$-ideals of M_2. By Proposition 2.7, $P_{\alpha,2} = \begin{pmatrix} P_\alpha^{*'} & \Gamma(P_\alpha) \\ P_\alpha & P_\alpha^{+'} \end{pmatrix}$, where P_α is a $\mathcal{P}$-ideal of Γ-ring M. From this we have that $I = \cap P_\alpha$, i.e., M is a $\mathcal{P}$-Jacobson Γ-ring. The proof of the converse is similar, and will be omitted.

Theorem 5.4. *Let (M, Γ) be a Γ_N-ring and Γ-ring M have right unity. Then*

$$\alpha(M_2) \subseteq \begin{pmatrix} (\gamma(M))^{*'} & \Gamma(\gamma(M)) \\ \gamma(M) & (\gamma(M))^{+'} \end{pmatrix}.$$

Proof. Since $M_2/\begin{pmatrix} (\gamma(M))^{*'} & \Gamma(\gamma(M)) \\ \gamma(M) & (\gamma(M))^{+'} \end{pmatrix} \cong \begin{pmatrix} R/(\gamma(M))^{*'} & \Gamma/\Gamma(\gamma(M)) \\ M/\gamma(M) & L/(\gamma(M))^{+'} \end{pmatrix}$, by Theorem 5.3 and the facts that $[\Gamma/\Gamma(\gamma(M)), M/\gamma(M)] \cong R/(\gamma(M))^{*'}$ and

$[M/\gamma(M), \Gamma/\Gamma(\gamma(M))] \cong L/\gamma(M)^{+'}$, it follows that

$$\alpha(M_2) \subseteq \begin{pmatrix} (\gamma(M))^{*'} & \Gamma(\gamma(M)) \\ \gamma(M) & (\gamma(M))^{+'} \end{pmatrix}.$$

Theorem 5.5. *Let (M, Γ) be a Γ_N-ring and Γ-ring M have right and left unities. Then we have*

$$\alpha(M_2) = \begin{pmatrix} (\gamma(M))^{*'} & \Gamma(\gamma(M)) \\ \gamma(M) & (\gamma(M))^{+'} \end{pmatrix}.$$

Proof. Since Γ-ring M has right and left unities, we can prove that for any $A \lhd M$, $A^{*'} = [\Gamma, A]$, $A^{+'} = [A, \Gamma]$ and $\Gamma(A) = \Gamma A \Gamma$ and thus

$$\begin{pmatrix} (\gamma(M))^{*'} & \Gamma(\gamma(M)) \\ \gamma(M) & (\gamma(M))^{+'} \end{pmatrix} = \begin{pmatrix} [\Gamma, \gamma(M)] & \Gamma\gamma(M)\Gamma \\ \gamma(M) & [\gamma(M), \Gamma] \end{pmatrix}.$$

By [5], Lemma 4.1, we have $\alpha(M_2) = \begin{pmatrix} [\Gamma, A] & [\Gamma A \Gamma] \\ A & [A, \Gamma] \end{pmatrix}$, where $A \trianglelefteq M$. On the other hand, by the facts that

$$M_2/\alpha(M_2) = M_2/\begin{pmatrix} [\Gamma, A] & \Gamma A \Gamma \\ A & [A, \Gamma] \end{pmatrix} \cong \begin{pmatrix} R/A^{*'} & \Gamma/\Gamma(A) \\ M/A & L/A^{+'} \end{pmatrix}$$

and $M_2/\alpha(M_2)$ is α-semisimple, from Theorem 5.3 we get that M/A is γ-semisimple and thus $\gamma(M) \subseteq A$. Hence we have

$$\begin{pmatrix} (\gamma(M))^{*'} & \Gamma(\gamma(M)) \\ \gamma(M) & (\gamma(M))^{+'} \end{pmatrix} \subseteq \begin{pmatrix} [\Gamma, A] & \Gamma A \Gamma \\ A & [A, \Gamma] \end{pmatrix} = \beta(M_2).$$

Thus, we have $\alpha(M_2) = \begin{pmatrix} (\gamma(M))^{*'} & \Gamma(\gamma(M)) \\ \gamma(M) & (\gamma(M))^{+'} \end{pmatrix}$. The proof is completes.

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