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ON A CRITERION FOR MULTIVALENTLY STARLIKENESS

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Abstract. In this paper, we point out an error in [3, Main theorem] and obtain some sufficient conditions for multivalently starlikeness.

1. Introduction

Let $p \in N = \{1, 2, 3, \cdots\}$ and $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$.

A function $f(z)$ in $A(p)$ is called p-valently starlike if and only if

$$Re\frac{zf'(z)}{f(z)} > 0 \quad (z \in E).$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are p-valently starlike in E.

There are many papers in which various sufficient conditions for multivalently starlikeness were obtained. Recently, Nunokawa [3, Main theorem] gave the following:

Theorem A. *Let $f(z) \in A(p)$ and suppose that*

$$1 + \frac{zf''(z)}{f'(z)} \neq ib \quad (z \in E),$$

where b is a real number and

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$$|\,b\,| \geq 3^{1/2}p. \tag{1}$$

Then $f(z) \in S(p)$.

For $p = 1$, the above result was also proved by Mocanu [2]. However, we find that Theorem A is not true when $p \geq 2$.

Counterexample. Let $p \geq 2$ and $f_0(z)$ be defined by

$$f_0(z) = p\int_0^z \frac{t^{p-1}(1+t)^p}{(1-t)^{3p}}dt \in A(p).$$

Then

$$1 + \frac{zf_0''(z)}{f_0'(z)} = p\left(\frac{1+z}{1-z} + \frac{2z}{1-z^2}\right). \tag{2}$$

Note that the univalent function

$$w = \frac{1+z}{1-z} + \alpha\frac{2z}{1-z^2} \quad (\alpha > 0)$$

maps E onto the complex plane minus the half-lines Re $w = 0$, Im $w \geq (\alpha(\alpha+2))^{1/2}$ and Re $w = 0$, Im $w \leq -(\alpha(\alpha+2))^{1/2}$ (see [2, p. 233]). From (2), we have

$$1 + \frac{zf_0''(z)}{f_0'(z)} \neq ib \quad (z \in E),$$

where b is real and $|\,b\,| \geq 3^{1/2}p$.

On the other hand, it is well-known that if $f(z) \in S(p)$, then for $|\,z\,| = r < 1$,

$$\frac{p(1-r)r^{p-1}}{(1+r)^{2p+1}} \leq |\,f'(z)\,| \leq \frac{p(1+r)r^{p-1}}{(1-r)^{2p+1}}.$$

Since

$$|\,f_0'(r)\,| > \frac{p(1+r)r^{p-1}}{(1-r)^{2p+1}} \quad (p \geq 2,\ 0 < r < 1),$$

it follows that $f_0(z) \notin S(p)$.

In this paper, we shall correct and extend the main theorem of [3].

2. Results

We need the following lemma due to Miller and Mocanu [1].

Lemma. *Let $g(z)$ be analytic and univalent in E and let $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(E)$, with $\varphi(w) \neq 0$ when $w \in g(E)$. Set*

$$Q(z) = zg'(z)\varphi(g(z)), \quad h(z) = \theta(g(z)) + Q(z)$$

and suppose that

(i) *$Q(z)$ is univalent and starlike in E, and*

(ii) $\operatorname{Re}\dfrac{zh'(z)}{Q(z)} = \operatorname{Re}\left\{\dfrac{\theta'(g(z))}{\varphi(g(z))} + \dfrac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in E).$

If $P(z)$ is analytic in E, with $P(0) = g(0)$, $P(E) \subseteq D$ and

$$\theta(P(z)) + zP'(z)\varphi(P(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z), \tag{3}$$

then

$$P(z) \prec g(z),$$

where the symbol $\prec$ denotes subordination, and $g(z)$ is the best dominant of (3).

Applying the above lemma, we derive

Theorem 1. *If $f(z) \in A(p)$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \neq ib \quad (z \in E), \tag{4}$$

where $\alpha > 0$, b is a real number and

$$|\, b \,| \geq (\alpha(\alpha + 2p))^{1/2}, \tag{5}$$

then $f(z) \in S(p)$.

Proof. Let us put

$$P(z) = \frac{zf'(z)}{f(z)}, \tag{6}$$

where $P(0) = p$. From (4) and using the same argument as [3, p. 133], we easily have $P(z) \neq 0$ in E.

From (6) we obtain

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=P(z)+\alpha\frac{zP'(z)}{P(z)}. \tag{7}$$

Using (4), (5) and (7), we deduce that

$$P(z)+\alpha\frac{zP'(z)}{P(z)}\prec p\left(\frac{1+z}{1-z}+\frac{\alpha}{p}\frac{2z}{1-z^2}\right). \tag{8}$$

Set

$$g(z)=p\frac{1+z}{1-z},\quad \theta(w)=w,\quad \varphi(w)=\frac{\alpha}{w}$$

and $D=\{w:w\neq 0\}$ in the lemma. Then the function

$$Q(z)=\frac{\alpha zg'(z)}{g(z)}=\frac{2\alpha z}{1-z^2}$$

is univalent and starlike in E. Also,

$$h(z)=g(z)+Q(z)=p\frac{1+z}{1-z}+\frac{2\alpha z}{1-z^2}$$

and

$$\mathrm{Re}\frac{zh'(z)}{Q(z)}=\mathrm{Re}\left\{\frac{p}{\alpha}\frac{1+z}{1-z}+\frac{1+z^2}{1-z^2}\right\}>0\quad(z\in E).$$

In view of (8), the lemma yields

$$P(z)\prec p\frac{1+z}{1-z}.$$

This shows that $f(z)\in S(p)$ and the proof is complete.

From Theorem 1, we easily have the following results.

Corollary 1. *If $\alpha>0$ and $f(z)\in A(p)$ satisfies*

$$\left|\mathrm{Im}\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{zf''(z)}{f'(z)}\right\}\right|<(\alpha(\alpha+2p))^{1/2}\quad(z\in E),$$

then $f(z)\in S(p)$.

Corollary 2. *If $\alpha>0$ and $f(z)\in A(p)$ satisfies*

$$\left|(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|<p+\alpha\quad(z\in E),$$

then $f(z) \in S(p)$.

For $\alpha = 1$, Corollary 2 leads to the assertion: If $f(z) \in A(p)$ satisfies $\left|1 + \frac{zf''(z)}{f'(z)} - p\right| < p + 1 \quad (z \in E)$, then $f(z) \in S(p)$.

Corollary 3. *If* $f(z) \in A(p)$ *satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \neq ib \quad (z \in E),$$

where b *is real and*

$$|b| \geq (1 + 2p)^{1/2},$$

then $f(z) \in S(p)$.

Next, we derive

Theorem 2. *If* $\alpha > 0$ *and* $f(z) \in A(p)$ *satisfies*

$$\left|(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - p\right| < p + \frac{\alpha}{2} \quad (z \in E), \tag{9}$$

then

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p \quad (z \in E). \tag{10}$$

Proof. According to Corollary 2 the function $f(z)$ belongs to $S(p)$.

Let

$$P(z) = \frac{zf'(z)}{f(z)}.$$

Then by (7), the assumption (9) becomes

$$\left|P(z) + \alpha\frac{zP'(z)}{P(z)} - p\right| < p + \frac{\alpha}{2} \quad (z \in E). \tag{11}$$

Set

$$g(z) = p(1 + z), \quad \theta(w) = w, \quad \varphi(w) = \alpha/w$$

and $D = \{w : w \neq 0\}$ in the lemma. Then we have

$$Q(z) = \frac{\alpha z g'(z)}{g(z)} = \frac{\alpha z}{1 + z},$$

$$h(z) = g(z) + Q(z) = p(1+z) + \frac{\alpha z}{1+z}$$

and

$$\mathrm{Re}\frac{zh'(z)}{Q(z)} = \mathrm{Re}\left\{\frac{p}{\alpha}(1+z) + \frac{1}{1+z}\right\} > \frac{1}{2} \quad (z \in E).$$

For $|z| = 1$ and $z \neq -1$,

$$|\, h(z) - p \,| \geq p + \alpha \mathrm{Re}\frac{1}{1+z} = p + \frac{\alpha}{2}.$$

Thus it follows from (11) that

$$P(z) + \alpha\frac{zP'(z)}{P(z)} \prec h(z).$$

The lemma now leads to $P(z) \prec p(1+z)$, which gives (10).

When $p = 1$, Mocanu [2] proved Theorem 2 by a different method.

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