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POMPEIU PROBLEM FOR SETS OF HIGHER CODIMENSION IN EUCLIDEAN AND HEISENBERG SETTINGS

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Abstract. In this survey we present some recent results in Euclidean and Heisenberg spaces for Pompeiu problem with sets of higher codimension. We prove a theorem which demonstrates a higher codimension set, together with a full complement of rotations, will possess the Pompeiu property. We also consider some aspects of the Morera side of the problem.

1. INTRODUCTION

We first present the Pompeiu problem in the setting of $\mathbf{R}^n$. Assume $f \in C(\mathbf{R}^n)$ and consider the following integral conditions

$$\int_{\gamma S} f(y+x)d\mu(x) = 0 \qquad \text{for all } y \in \mathbf{R}^n \text{ and } \gamma \in SO(n),$$

for a given set S with area measure μ. We say the set S has the Pompeiu property if the vanishing of these integrals implies $f \equiv 0$.

In the setting of $\mathbf{R}^2 = \mathbf{C}$, we also consider the closely related Morera problem, which addresses analyticity of the function f. For this problem we consider integration over a simple closed curve, the boundary Γ of a set S, and integrate with respect to the differential form dz rather than area measure. Given $f \in C(\mathbf{C})$ and a simple closed curve $\Gamma = \partial S$, we consider the integral conditions

$$\int_{\gamma\Gamma} f(z+w)dz = 0 \qquad \text{for all } w \in \mathbf{C} \text{ and } \gamma \in SO(n).$$

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The set Γ is said to possess the Morera property if the vanishing of these integrals implies that $\frac{\partial}{\partial \bar{z}} f \equiv 0$. In other words, f is entire. By Green's Theorem, the Morera property of $\Gamma = \partial S$ is equivalent to the Pompeiu property for S. The Morera version of the problem allows us to relate the issue of analyticity to integral conditions.

Returning to the Pompeiu problem, we note that it remains an open problem to describe all sets which possess this property. However there are certain sets known not to possess the Pompeiu property. When S is a ball, we have integral conditions

$$\int_{B_r^n} f(y+x)d\mu_r(x) = 0 \qquad \text{for all } y \in \mathbf{R}^n$$

for μ area measure on the ball. Similarly, for a sphere we have the integral conditions

$$\int_{S_r^{n-1}} f(y+x)d\sigma_r(x) = 0 \qquad \text{for all } y \in \mathbf{R}^n$$

where σ is area measure on the sphere. Note that in both of these cases the set S is rotation invariant. Nothing is added by rotations, and in effect there is a reduced collection of integral information. In [23], Williams has conjectured that these two cases characterize all the cases of sets which do not possess the Pompeiu property. Williams' conjecture considers sets $S \subset \mathbf{R}^n$ which have boundary homeomorphic to the $n-1$ sphere S^{n-1}. In this case he conjectures that S does not possess the Pompeiu property if and only if $S = B^n$. We also note that in these cases of the sphere and the ball, the Pompeiu property can be recovered when considering two balls, or two spheres, of appropriate radii. For details, see [24], and note that such theorems of two radii also come up in Sections 5 and 6.

Above we have addressed both the case of a ball, which is of the same dimension as the ambient space, and its boundary, a sphere. The result for the sphere, of codimension 1, is nearly equivalent to the result for the ball, of codimension 0. In some sense the Pompeiu conjecture may be interpreted as stating that rotations are needed in order to gather enough integral information to conclude the vanishing of f. The main concern in this paper will be balls of higher codimension, and here the results are no longer equivalent to those for the ball or sphere. In higher codimension there are additional rotations within the ambient space, and these yield additional integral information. Using a result of [10], Theorem 3.1 demonstrates that in higher codimension all sets possess the Pompeiu property, including balls. This result is consistent with the principle of rotation invariance.

Here we mention this fundamental result of [10] on the Pompeiu problem which relates the Pompeiu property to smoothness of the boundary

Theorem 1.1. *Let $S \subset \mathbf{R}^n$ be a bounded set whose boundary has a corner, and let μ be area measure on S. Let $f \in C(\mathbf{R}^n)$, and consider the following* *integral conditions*

$$\int_{\gamma S} f(x+y)d\mu(x) = 0 \qquad \textit{for all } y \in \mathbf{R}^n \textit{ and } \gamma \in SO(n).$$

Then S has the Pompeiu property, and we may conclude $f = 0$.

Although smoothness of the boundary plays a role in the Pompeiu problem, according to the Pompeiu conjecture of Williams and other results of [10], smoothness of the boundary is not enough to determine the Pompeiu property. This conjecture of Williams inherently implies that the curvature of the boundary is also important. Since the primary concern in this paper is to address the Pompeiu problem for sets of higher codimension, it is interesting to consider this above issue in the higher codimension setting. Some of these questions are answered in Section 3, yet there are deeper issues that remain to be considered.

Much of the work of the two authors on the Pompeiu problem have dealt with the Heisenberg group setting, see *e.g.,* [11, 12] and [13]. Here, too, we consider the Heisenberg group setting for the higher codimension version of the Pompeiu problem. In Section 5 we summarize our work [15] for balls of higher codimension in $\mathbf{H}^2$. We also address briefly the Morera side of the problem. Then in Section 6 we give a different approach to higher codimension within the context of Heisenberg groups, based on our work [14] for products of Heisenberg groups. Finally, we raise some new questions for the Morera problem on the Heisenberg group.

2. Higher Codimension for Euclidean Space

Our first investigations of this issue of higher codimension in the Pompeiu problem come in the setting of Euclidean space. These results come from the paper [7]. We first relate the global version of the theorem, proved using the Fourier transform. Let B_r^{n-1} be the ball $\{\mathbf{x} = (x_1, \ldots, x_{n-1}, 0) : \|\mathbf{x}\| < r\}$. Note this ball will also be denoted $B_{n-1}(r)$ at other points in the paper. Consider the rotations $\gamma \in SO(n)$.

Theorem 2.1. *Consider $f \in C(\mathbf{R}^n)$. Let μ_r be the area measure on B_r^{n-1}. Assume vanishing of the integrals*

$$\int_{\gamma B_r^{n-1}} f(y+x)d\mu_r(x) = 0 \quad \textit{for all } y \in \mathbf{R}^n \textit{ and all } \gamma \in SO(n).$$

We may then conclude $f \equiv 0$.

Note that by integrating over a ball of codimension 1 and including the rotations, we attain the Pompeiu property for a single ball. This contrasts with the theorems of two radii required when rotations are not included.

We note briefly that we also attain an equivalent result for balls B_r^{n-d} of real codimension d in $\mathbf{R}^n$.

Theorem 2.2. *Consider $f \in C(\mathbf{R}^n)$. Let μ_r be the area measure on B_r^{n-d}. Assume vanishing of the integrals*

$$\int_{\gamma B_r^{n-d}} f(y+x)d\mu_r(x) = 0 \quad \textit{for all } y \in \mathbf{R}^n \textit{ and all } \gamma \in SO(n).$$

We may then conclude $f \equiv 0$.

We also mention that similar results hold for spheres of higher codimension, the boundaries of the balls given above. One advantage we have observed for balls and spheres of higher codimension is the preservation of the Pompeiu property when rotations are included. There is no longer a need for theorems of two radii or sets of exceptional radii.

We next consider theorems of a local nature, which only utilize integration within a local region B_R^n, and therefore depend on a restricted set of integral information. Note that the local theorems we now consider are microlocal in nature, based on the method of Quinto [20]. As a result, they require a starter set at the center where the function is assumed to be zero. This approach is closely related to Quinto's earlier work [19] in which he applied microlocal methods to generalized Radon transforms, and it also stems from work in [17]. We have the following theroems, the first of which extends the above global result to a local setting.

Theorem 2.3. *Let $R > r > 0$. Also let $f \in C(B_n(R))$, and assume $f = 0$ on $B_n(r+\epsilon)$. Then consider vanishing of the integrals*

$$\int_{\gamma S_{n-2}(r)} f(y+x)d\sigma_{\gamma,r}(x) = 0 \qquad \textit{for all } y \in B_n(R-r) \textit{ and } \gamma \in SO(n),$$

where $\sigma_{\gamma,r}$ is area measure on $\gamma S_{n-2}(r)$. We may then conclude $f = 0$ on $B_n(R)$.

We next state the obvious reduction in rotations that can be made for this result. Let us first define a set of rotations $P_{\mathbf{vw}} \subset SO(n)$ as follows. Let P_θ be the rotation by the angle θ between vectors $\vec{v} = \langle 1, 0, \ldots, 0\rangle$ and $\vec{w} = \langle 0, \ldots, 0, 1\rangle$. Then define $P_{\mathbf{vw}} = \{P_\theta : \theta \in [0, \pi)\}$. Note that in this case, where $\mathbf{v} = \mathbf{e}_1$ and $\mathbf{w} = \mathbf{e}_2$, we may abbreviate $P_{\mathbf{vw}}$ as P_{1n}.

Theorem 2.4. *Let $R > r > 0$. Also let $f \in C(B_n(R))$, and assume $f = 0$ on $B_n(r+\epsilon)$. Then consider vanishing of the integrals*

$$\int_{\gamma S_{n-2}(r)} f(y+x)d\sigma_{\gamma,r}(x) = 0 \qquad \textit{for all } y \in B_n(R-r) \textit{ and } \gamma \in P_{1n},$$

where $\sigma_{\gamma,r}$ is area measure on $\gamma S_{n-2}(r)$. We may then conclude $f = 0$ on $B_n(R)$.

It is also possible to attain this result with a significant reduction in the number of rotations required, as shown in the folloiwng theorem. We first describe the rotation $\gamma_p \in P_{1n}$ by letting γ_p be the rotation by an angle of $\pi/2$ from vector $\vec{v} = \langle 1, 0, \ldots, 0\rangle$ to $\vec{w} = \langle 0, \ldots, 0, 1\rangle$.

Theorem 2.5. *Let $R > r > 0$. Also let $f \in C(B_n(R))$, and assume $f = 0$ on $B_n(r + \epsilon)$. Then consider the vanishing of integrals*

$$\int_{S_{n-2}(r)} f(y + x)d\sigma_r(x) = 0 \qquad \text{for all } y \in B_n(R - r),$$

and

$$\int_{\gamma_p S_{n-2}(r)} f(y + x)d\sigma_{p,r}(x) = 0 \qquad \text{for all } y \in B_n(R - r),$$

where γ_p is described above, σ_r is area measure on $S_{n-2}(r)$ and $\sigma_{p,r}$ is area measure on $\gamma_p S_{n-2}(r)$. We may then conclude $f = 0$ on $B_n(R)$.

Note that Theorem 2.3, 2.4, and 2.5 require a starter set at the center where f must be assumed already to equal 0. However the integral conditions in these theorems offer a considerable reduction in the rotations required, as compared to Theorem 2.1. Also note that these local results of Theorem 2.3, 2.4, and 2.5 can extend to global theorems. A local theorem without a zero starter set at the center, comparable to Theorem 2.1, may be possible using the methods of [8]. However this approach still remains to be considered. Thus there appears to be a trade off in which assumptions can be weakened, either the starter set at the center or the reduction in the number of rotations. It remains to be considered whether both can be weakened simultaneously.

We have the beginning of an answer to the question of whether the assumption of $f \equiv 0$ on $B_n(r + \epsilon)$, which is made in Theorems 2.3, 2.4, and 2.5, is necessary in order to attain the conclusion, $f \equiv 0$ on $B_n(R)$, of these theorems. Let us first mention that use of the method of microlocal analysis in proving these theorems requires the zero starter set at the center of the local region. However there is the question of whether other methods would allow the same conclusions based solely on the integral conditions, or with some other weakening of the assumption of $f \equiv 0$ on $B_n(r + \epsilon)$. We can begin to answer these questions through use of certain counterexamples, which follow. The following counterexamples work, first for the integral conditions of Theorems 2.5, the weakest of these three sets of integral conditions, and then for the integral conditions of Theorem 2.4, which has a larger set of rotations in the integral conditions, but is still somewhat restricted. For the first counterexample, to Theorem 2.5, let $\alpha_0 \in \mathbf{R}_+$ satisfy $J_{n-2}(\alpha_0 r\sqrt{n-1}) = 0$

and $\alpha = \langle \alpha_0, \ldots, \alpha_0 \rangle$. Then let $f(\mathbf{x}) = e^{i\alpha\cdot\mathbf{x}}$. We observe that $f(\mathbf{x}) = e^{i\alpha\cdot\mathbf{x}} \not\equiv 0$; however $f(x)$ does satisfy the integral conditions of Theorem 2.5. Observe vanishing of the integral

$$\begin{aligned}\int_{S_{n-2}(r)} e^{i\alpha\cdot\mathbf{x}} d\sigma_r(\mathbf{x}_{n-1}) &= \int_{S_{n-2}(r)} e^{i\alpha_{n-1}\cdot\mathbf{x}_{n-1}} d\sigma_r(\mathbf{x}_{n-1}) \\ &= c\frac{J_{n-2}(r|\alpha_{n-1}|)}{(r|\alpha_{n-1}|)^{n-2}} = 0\end{aligned}$$

and likewise

$$\int_{\gamma_p S_{n-2}(r)} e^{i\alpha\cdot\mathbf{x}} d\sigma_r(\mathbf{x}_{n-1}) = c\frac{J_{n-2}(r|\alpha_{n-1}|)}{(r|\alpha_{n-1}|)^{n-2}} = 0$$

Vanishing also occurs for translation by $\mathbf{y}$ for all $\mathbf{y} \in \mathbf{R}^n$, since translation by $\mathbf{y}$ will carry over to multiplication by $e^{-i\alpha\cdot\mathbf{y}}$. However the assumption $f \equiv 0$ on $B_n(r+\epsilon)$ excludes cases such as this, and the theorem remains valid. Note also that, for any given $\epsilon > 0$, the assumption $f \equiv 0$ on $B_n(\epsilon)$ is sufficient to eliminate such cases as this one. Also note this counterexample can accommodate integral conditions associated to the rotation $\gamma_p S_{n-2}(r)$ for any $\gamma_p \in P_{ij}$ where $1 \leq i < j \leq n$, which is a rotation through an angle of $\pi/2$ from $\mathbf{e}_i$ to $\mathbf{e}_j$.

The next counterexample deals with the set of integral conditions in Theorem 2.4, wherein rotations through $[0, \pi)$ are considered in one specific direction. All of the rotations $\gamma \in P_{1n}$ used in the integral conditions of Theorem 2.4 are rotations of the 2-plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_n$. We choose a vector $\mathbf{v}$ such that $\|\gamma\mathbf{v}\| = \|\mathbf{v}\|$ is invariant under all rotations $\gamma \in P_{1n}$. We furthermore make sure $\lambda = \|\mathbf{v}\|$ satisfies a condition relating to the zeros of $J_{n-2}(x)$. Choose $\mathbf{v} = \langle 0, \lambda_2, \ldots, \lambda_{n-1}, 0 \rangle$ such that $\lambda = \sqrt{\lambda_2^2 + \cdots + \lambda_{n-1}^2}$ satisfies $J_{n-2}(\lambda r) = 0$, and let $f(\mathbf{x}) = e^{i\mathbf{v}\cdot\mathbf{x}}$. We now observe that the integral conditions of Theorem 2.4 are satisfied by $f(x)$, while $f(x) \not\equiv 0$. For any rotation $\gamma \in P_{1n}$, we have

$$\begin{aligned}\int_{\gamma S_{n-2}(r)} e^{i\mathbf{v}\cdot\mathbf{x}} d\sigma_{r,\gamma}(\mathbf{x}) &= \int_{S_{n-2}(r)} e^{i\mathbf{v}\cdot\gamma^{-1}\mathbf{x}} d\sigma_r(\mathbf{x}_{n-1}) \\ &= \int_{S_{n-2}(r)} e^{i\gamma\mathbf{v}\cdot\mathbf{x}} d\sigma_r(\mathbf{x}_{n-1}) \\ &= c\frac{J_{n-2}(r|\mathbf{v}|)}{(r|\mathbf{v}|)^{n-2}} = 0\end{aligned}$$

Translations by $\mathbf{y} \in \mathbf{R}^n$ have the same effect noted above, and thus these integrals also vanish for all translations. Furthermore, let $\mathbf{v}_2 = \langle 0, \lambda, 0, \ldots, 0 \rangle = \lambda\mathbf{e}_2$ with the same condition $J_{n-2}(\lambda r) = 0$ and consider the function $f(\mathbf{x}) = e^{i\mathbf{v}_2\cdot\mathbf{x}}$. In this way the counterexample can work for a much larger set of rotations. It is possible to consider $\gamma S_{n-2}(r)$ for all rotations $\gamma \in P_{\hat{2}}$, where $P_{\hat{2}}$ is defined as follows.

$$P_{\hat{2}} = \cup_{i<j,\neq 2} P_{ij},$$

or similar sets of notations. However, note also that this counterexample will not extend to the case of all rotations $\gamma S_{n-2}(r)$ for all $\gamma \in SO(n)$.

In the case of Theorem 2.3, where the integral conditions include all rotations $\gamma S_{n-2}(r)$ for all $\gamma \in SO(n)$, we do not have a counterexample to illustrate the necessity to assume vanishing on some starter set at the center of the region. Perhaps in this case, due to the extent of rotations used, there may not be a need for the starter set at the center of the local region. It may be possible to prove, using another approach, that these integral conditions of Theorem 2.3 are sufficient, without making an initial assumption of vanishing on such a starter set. We will soon investigate this question using the methods of [8].

We mention that these local results also extend more generally to higher codimension $n-d$ inside of $\mathbf{R}^n$, similar to the global result in Theorem 2.2. Theorem 2.3 and 2.4 extend directly, by the same methods. It is more difficult to address the issue of reduction of rotations for the general case of codimension $n-d$. There are some limited theorems we could state; see [7]. We will continue work to address the minimal rotations needed in the local setting of codimension $n-d$. A related question for ongoing work is whether the number of rotations can be reduced in the global results Theorem 2.1 and Theorem 2.2.

3. Relation to Rotation and Pompeiu Conjecture

As we noted in the introduction, the above results are demonstrations of the principle of rotation invariance that stands behind the Pompeiu conjecture of Williams. Since the balls and spheres of higher codimension are no longer invariant under rotation, this principle has correctly predicted that an individual ball or sphere, when considered with all rotations, now possesses the Pompeiu property. We furthermore demonstrate how the result of Brown, Schreiber, and Taylor cited above, Theorem 1.1, will imply the same result. We thus have an alternative proof of Theorem 2.1 and Theorem 2.2. However, it does not carry over to the reductions of rotations made in Theorem 2.3 and Theorem 2.4.

Proof (2.1). To consider the Pompeiu problem for the ball B_r^{n-1} of codimension 1, we will make a direct association with the Pompeiu problem for the cylinder $C_{r,s} = B_r^{n-1} \times [0, s]$, for any $s > 0$, which is a set of codimension 0. First consider the integral conditions for the Pompeiu problem associated to $C_{r,s}$,

$$(1) \qquad \int_{\gamma C_{r,s}} f(x+y)\,d\mu_{\gamma,r,s}(x) = 0 \qquad \text{for all } y \in \mathbf{R}^n \text{ and all } \gamma \in SO(n).$$

Next observe that the integral conditions of Theorem 2.1 for B_r^{n-1}

$$(2) \qquad \int_{\gamma B_r^{n-1}} f(x+y)d\mu_{\gamma,r}(x) = 0 \qquad \text{for all } y \in \mathbf{R}^n \text{ and all } \gamma \in SO(n)$$

already imply the integral conditions (2). In fact integral conditions (1) require more than integral conditions (2). First notice $C_{r,s}$ can be sliced into an infinite set of translations of B_r^{n-1}.

$$C_{r,s} = \cup_{t\in[0,s]} B_r^{n-1} \times \{t\},$$

where $B_r^{n-1} = \{(x_1, \ldots, x_{n-1}, 0) : \|(x_1, \ldots, x_{n-1})\| < r\}$ and $B_r^{n-1} \times \{t\}$ is a translation of B_r^{n-1} by $(0, \ldots, 0, t)$. Thus $\int_{B_r^{n-1}\times\{t\}} f(x+y)d\mu_r(x) = 0$ for all $t \in [0, s]$ will imply that $\int_{C_{r,s}} f(x+y)d\mu_{r,s}(x) = 0$. Then consider the comparable statement for the rotation $\gamma C_{r,s}$ of the cylinder. Clearly for $\gamma \in SO(n)$,

$$\gamma C_{r,s} = \cup_{t\in[0,s]} \gamma \left(B_r^{n-1} \times \{t\}\right).$$

Note there exists a translation, $\tau_{t,\gamma}$, such that $\tau_{t,\gamma}\left(\gamma B_r^{n-1}\right) = \gamma\left(B_r^{n-1} \times \{t\}\right)$. Then observe that

$$\int_{\gamma B_r^{n-1}} f(\tau_{t,\gamma} x)d\mu_{\gamma,r}(x) = 0 \qquad \text{for all } t \in [0, s]$$

is equivalent to

$$\int_{\gamma\left(B_r^{n-1}\times\{t\}\right)} f(x)d\mu_{\gamma,r}(x) = 0 \qquad \text{for all } t \in [0, s].$$

This collection of integrals for all $t \in [0, s]$ implies the vanishing of the integral

$$\int_{\gamma C_{r,s}} f(x)d\mu_{\gamma,r,s}(x) = 0.$$

To conclude the proof, we note the result of Brown, Schreiber, and Taylor, [10] applies, since $C_{r,s}$ has points on its boundary which are corners. It follows that $C_{r,s}$ possesses the Pompeiu property. Thus integral conditions (1) imply integral conditions (2), and by the Pomepiu property for $C_{r,s}$, this implies that $f \equiv 0$. Thus B_r^{n-1} also possesses the Pomepiu property. The proof is complete. Note this proof illustrates the point that it is generally easier for sets of higher codimension to possess the Pompeiu property than it is for sets with the same dimension as the ambient space. The relevant integral conditions will contain more information.

Note the obvious modification of this proof to the local setting produces an alternative proof of Theorem 2.2 also.

We now note that the principle of these proofs does not depend on B_r^{n-1} being a ball, but rather is applicable for any set S of the form $S \subset \mathbf{R}^{n-1} \times \{0\}$ and such that the boundary of S is homeomorphic to S^{n-2}. Let C_S^r denote the cylinder $S \times [0, r]$. Note that C_S^r has a corner and thus possesses the Pompeiu property by the above result of Brown, Schreiber, and Taylor. We then have the following theorem, which essentially states that every higher codimension set, together with the full complement of rotations, possesses the Pompeiu property.

Theorem 3.1. *Consider $S \times \{0\} \subset \mathbf{R}^{n-1} \times \{0\}$ such that S has boundary homeomorphic to S^{n-2}. Let $S \times [0, r] = C_S^r$ be a cylinder based on the set S. Then we conclude S possesses the Pompeiu property. In particular, let $f \in C(\mathbf{R}^n)$, let μ represent area measure on S, and assume vanishing of the integrals*

$$\int_{\gamma S} f(y + x) d\mu(x) = 0 \quad \textit{for all } y \in \mathbf{R}^n \textit{ and } \gamma \in SO(n).$$

We may then conclude $f \equiv 0$.

Proof. The idea for this proof is identical to that of the alternative proof of Theorem 2.1 given above in this section. We first observe the cylinder C_S^r can be sliced into translations of S_{n-1}. The same is true of γC_S^r being sliced into translations of γS_{n-1}. It thus follows that vanishing of integrals

$$\int_{\gamma S} f(y + x) d\mu_\gamma(x) = 0 \quad \text{for all } y \in \mathbf{R}^n \text{ and all } \gamma \in SO(n)$$

implies vanishing of the integrals

$$\int_{\gamma C_S^r} f(y + x) d\mu_{\gamma,r}(x) = 0 \quad \text{for all } y \in \mathbf{R}^n \text{ and all } \gamma \in SO(n).$$

Since there are points on the boundary of C_S^r which occur at corners, C_S^r possesses the Pompeiu property by the result of [10]. Thus $f \equiv 0$. Thus the set S possesses the Pompeiu property as well. This completes the proof of this theorem.

Note that each of these results for the Pompeiu problem has a parallel statement regarding analytic functions in the context of the Morera problem. We discussed this relation briefly in the Introduction. It becomes important when we want integral conditions that characterize analyticity of functions. In the next two sections, the focus shifts to the Heisenberg group. One of the primary motivations for working with the Heisenberg group is to study the Morera problem in this setting. The Heisenberg group $\mathbf{H}^n$ is identified with a manifold in $\mathbf{C}^{n+1}$, and in the context of the Heisenberg group, the Morera problem uses integral conditions on the surface of the manifold to characterize analytic extension to one side of the manifold. Further details follow in the upcoming material.

4. Importance of Local Results and Role in the Heisenberg Group Setting

Local versions of the Pompeiu problem are in general more powerful because they can be extended to global results, but also allow us to make conclusions based solely on vanishing of integrals within a local region. Furthermore the above approach to the local Pompeiu problem, developed by Quinto, brings to bear the well developed and interesting theory of microlocal analysis unto our considerations of the Pompeiu problem. One important source of our interest in the higher codimension version of the Pompeiu problem relates to the Heisenberg group. We see below, and see also [2, 3, 6], that the natural expression of the Pompeiu problem on $\mathbf{H}^n$ considers integrals over a ball, or other set S inside of $\mathbf{C}^n \times \{0\}$, already of codimension 1 in $\mathbf{H}^n$. The codimension of a sphere would then be 2. One of the next important questions for the Pompeiu problem on the Heisenberg group is the local version of this question. The authors of this paper are currently working on this local problem for the Heisenberg group. The local problem for the Heisenberg group is a problem of considerable importance. It is highly significant first because the Heisenberg group may be used locally to model strongly pseudoconvex hypersurfaces. Furthermore the solution for the local problem may allow us to remove the growth conditions which are required in the results of [2, 3, 6]. Note that because of the higher codimension of the set in this local problem, the local results of [7] for higher codimension sets in Euclidean space are an important first step in this direction.

We now give a brief description of the Heisenberg group $\mathbf{H}^n$ to prepare for the Heisenberg results of the next section. Let $\Omega_{n+1} = \{(\mathbf{z}, z_{n+1}) : \operatorname{Im} z_{n+1} > \|\mathbf{z}\|^2\}$ be the Siegel upper-half space in $\mathbf{C}^{n+1}$. Then its boundary $\partial\Omega_{n+1} = \{(\mathbf{z}, z_{n+1}) : \operatorname{Im} z_{n+1} = \|\mathbf{z}\|^2\}$ is identified with the Heisenberg group. To make this identification we first give Heisenberg coordinates $\{[\mathbf{z}, t] : \mathbf{z} \in \mathbf{C}^n, t \in \mathbf{R}\}$. Letting $[\mathbf{z}, t] \to (\mathbf{z}, t + i\|\mathbf{z}\|^2)$ gives a bijection onto $\partial\Omega_{n+1}$. The Heisenberg group then follows the group law

$$[\mathbf{z}, t] \cdot [\mathbf{w}, s] = [\mathbf{z} + \mathbf{w}, t + s + 2\operatorname{Im} \mathbf{z} \cdot \bar{\mathbf{w}}].$$

We note this group law carries over to a group action on $\partial\Omega_{n+1}$. The group law can be used to define integral conditions for the Pompeiu problem on $\mathbf{H}^n$ as follows.

$$\int_{\|\mathbf{z}\|<r} L_{\mathbf{g}} f(\mathbf{z}, 0)\, d\mu_r(\mathbf{z}) = 0 \qquad \text{for all } \mathbf{g} \in \mathbf{H}^n,$$

where $L_{\mathbf{g}}$ is left-translation by $\mathbf{g} = [\mathbf{w}, s]$.

The left-invariant vector fields on $\mathbf{H}^n$ are generated by the following left-invariant vector fields

$$Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \quad \text{for } j = 1, \ldots, n,$$

$$\bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - i z_j \frac{\partial}{\partial t}, \quad \text{for } j = 1, \ldots, n,$$

and the extra direction $T = \frac{\partial}{\partial t}$. Note that T is generated by the Lie bracket of Z_j and $\bar{Z}_j$. The Heisenberg group may then be given a CR structure by observing CR functions for $\mathbf{H}^n$ are those functions f on $\mathbf{H}^n$ which are annihilated by $\bar{Z}_j f \equiv 0$ for all $j = 1, \ldots, n$. For $\mathbf{H}^n$, CR functions extend analytically to the entire upper-half space $\partial\Omega_{n+1}$. If $\bar{Z}_j f = 0$ for $j = 1, \ldots, n$, then there exists F defined on $\bar{\Omega}_{n+1}$ and analytic on Ω_{n+1} such that $F|_{\partial\Omega_{n+1}} = f$.

Consider subbundles $H^{1,0}(M)$ and $H^{0,1}(M)$ of holomorphic and anti-holomorphic subbundles of the complexified vector bundle $T^{\mathbf{C}}(M)$, where $M = \mathbf{C}^n$. $H^{1,0}(M)$ is generated by $\{Z_1, \ldots, Z_n\}$ and $H^{0,1}(M)$ is generated by $\{\bar{Z}_1, \ldots, \bar{Z}_n\}$. Observe that $H^{1,0}(M) \cap H^{0,1}(M) = \{0\}$, and furthermore $H^{1,0}$ and $H^{0,1}$ are each involutive. Observe furthermore that $T^{\mathbf{C}}(M) = H^{1,0}(M) \oplus H^{0,1}(M) \oplus \mathbf{C}T$, where T is the additional direction $T = \frac{\partial}{\partial t}$. $H^{1,0}(\mathbf{H}^n)$ is called a CR structure for $\mathbf{H}^n$, and $\mathbf{H}^n$ is a CR manifold.

Although the Heisenberg group is an example of a CR manifold, it is important to keep in mind this is a highly specialized example. In general CR manifolds may not have group structure or a highly developed harmonic analysis. That these are available for $\mathbf{H}^n$ makes the Pompeiu and Morera problems very managable in this setting. We pause briefly to discuss the nature of the Morera problem for $\mathbf{H}^n$. The integral conditions are given by

$$\int_{\|\mathbf{z}\|=r} L_{\mathbf{g}} f(\mathbf{z}, 0) \omega_j(\mathbf{z}) = 0 \qquad \text{for all } \mathbf{g} \in \mathbf{H}^n \text{ and all } j = 1, \ldots, n,$$

where $\omega_j(\mathbf{z}) = dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{j-1} \wedge d\bar{z}_{j+1} \wedge \cdots \wedge d\bar{z}_n$. This is a set of integrals of f along the surface of $\partial\Omega_{n+1}$. For the Morera problem in the Heisenberg setting, we seek to prove f is a CR function. Note this would give analytic extension to the upper-half space Ω_{n+1}. Thus analytic extension from $\mathbf{H}^n$ can be characterized by integrals on the surface of $\mathbf{H}^n$.

The following orthonormal basis of exponential Laguerre functions is integrally related to harmonic analysis on $\mathbf{H}^n$. Consider

$$\mathcal{W}^{\lambda}_{\mathbf{k},\mathbf{k}}(\mathbf{z}) = e^{-2\pi|\lambda||\mathbf{z}|^2} \prod_{j=1}^{n} L^{(0)}_{k_j}(4\pi|\lambda||z_j|^2) \qquad \text{for } (\mathbf{k}, \lambda) \in (\mathbf{Z}_+)^n \times \mathbf{R}*$$

The set $\{\mathcal{W}^{\lambda}_{\mathbf{k},\mathbf{k}}\}_{\mathbf{k} \in (\mathbf{Z}_+)^n}$ forms an orthonormal basis for $L^2_{\mathbf{O}}(\mathbf{C}^n)$, where

$$L^2_{\mathbf{O}}(\mathbf{C}^n) = \{f \in L^2(\mathbf{C}^n) : f(\sigma\mathbf{z}) = f(\mathbf{z}) \text{ for all } \sigma \in \mathbf{T}^n \text{ and all } \mathbf{z} \in \mathbf{C}^n\}.$$

We also form the following

$$\psi^{\lambda}_{\mathbf{k}}(\mathbf{z}, t) = e^{2\pi i \lambda t} \mathcal{W}^{\lambda}_{\mathbf{k},\mathbf{k}}(\mathbf{z}) \qquad \text{for } (\mathbf{k}, \lambda) \in (\mathbf{Z}_+)^n \times \mathbf{R}*,$$

which are bounded spherical functions on $\mathbf{H}^n$. The complete set of bounded spherical functions on $\mathbf{H}^n$ is given by

$$\psi_{\mathbf{k}}^{\lambda}(\mathbf{z},t) = e^{2\pi i\lambda t}\mathcal{W}_{\mathbf{k},\mathbf{k}}^{\lambda}(\mathbf{z}) \qquad \text{for } (\mathbf{k},\lambda) \in (\mathbf{Z}_{+})^n \times \mathbf{R}*,$$

and

$$\mathcal{J}_{\rho}(\mathbf{z}) = \prod_{j=1}^{n} J_0(\rho_j|z_j|) \qquad \text{for } \rho \in (\mathbf{R}_{+})^n.$$

These functions will be needed in some of the results in the next section on the Heisenberg group. Here we will not go into all the methods of harmonic analysis on $\mathbf{H}^n$ which have been used. The interested reader may read more in references such as [6].

5. Higher Codimension for the Heisenberg Group Setting

In this section we extend the higher codimension results for the Pompeiu problem to the setting of the Heisenberg group. We then further consider the Morera side of this problem, which addresses results of [2, 1] in the higher codimension setting. For convenience we have begun with the setting of $\mathbf{H}^2$. The results presented here come from the paper [15]. The Morera problem is in particular interesting because it allows us to characterize CR functions on $\mathbf{H}^2$ by integrals over translations of circles inside of $\mathbf{H}^2$.

For higher codimension in $\mathbf{H}^2$ we consider complex disks B_r^1 and B_r^2 in $\mathbf{C}^2$ given by

$$B_r^1 = \{(z_1, 0) \in \mathbf{C}^2 : \|z_1\|^2 \leq r\},$$

and

$$B_r^2 = \{(0, z_2) \in \mathbf{C}^2 : \|z_2\|^2 \leq r\},$$

which are each of real codimension 3 in $\mathbf{H}^2$. The integral conditions of the Pompeiu problem are then given by

$$\int_{B_r^1} L_{\mathbf{g}} f(z_1, 0, 0) d\sigma_r^1(z_1) = 0 \qquad \text{for all } \mathbf{g} \in \mathbf{H}^2$$

or similarly for B_r^2

$$\int_{B_r^2} L_{\mathbf{g}} f(0, z_2, 0) d\sigma_r^2(z_2) = 0 \qquad \text{for all } \mathbf{g} \in \mathbf{H}^2$$

Here σ_r^1 is area measure on B_r^1 and σ_r^2 is area measure on B_r^2.

The easiest Pompeiu type results to prove for the Heisenberg group are for functions in the class $L^2(\mathbf{H}^n)$. However, these results often extend to the space of functions L^p for $1 \leq p < \infty$. Then at the level of L^∞ we find a qualitative difference in the nature of the results. In general this is the level at which theorems of two radii are required. Here we present first the L^p results for the Pompeiu problem in $\mathbf{H}^2$ for complex disks in the case of $1 \leq p < \infty$. We begin with the case when rotations are not included.

Theorem 5.1. *Let $f \in C \cap L^p(\mathbf{H}^2)$ for $1 \leq p < \infty$ and $r > 0$. Suppose f satisfies the following integral conditions*

$$\int_{B_r^1} L_{\mathbf{g}} f(z_1, 0, 0) d\sigma_r(z_1) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^2.$$

We may then conclude that $f \equiv 0$. We note that the theorem is also valid for integral conditions where B_r^1 is replaced by B_r^2.

We then observe that at the level of L^∞ the Pompeiu property no longer holds (without rotations), but a theorem of two radii is instead required. First we define the functions $\Psi_k^{(n)}$ from an integral of $L_k^{(n)}$. Let

$$\Psi_k^{(n)}(R) = \int_0^R e^{-t/2} L_k^{(n)}(t) t^{n-1} dt.$$

We now have the following theorem.

Theorem 5.2. *Let $f \in C \cap L^\infty(\mathbf{H}^2)$, and let $r_1, r_2 > 0$. Suppose f satisfies the following integral conditions*

$$\int_{B_{r_1}^1} L_{\mathbf{g}} f(z_1, 0, 0) d\sigma_{r_1}(z_1) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^2$$

and

$$\int_{B_{r_2}^1} L_{\mathbf{g}} f(z_1, 0, 0) d\sigma_{r_2}(z_1) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^2$$

Furthermore assume r_1 and r_2 satisfy the following conditions:

(1) $r_1/r_2 \notin \mathcal{Q}(J_1(x))$.

(2) $(r_1/r_2)^2 \notin \mathcal{Q}(\Psi_k^{(0)}(x))$ *for all* $k \in \mathbf{Z}_+$.

Then $f \equiv 0$. However, if r_1 and r_2 do not satisfy conditions 1. and 2., then there exists some $f \in C \cap L^\infty(\mathbf{H}^2)$ such that $f \not\equiv 0$, but f satisfies the integral conditions.

Note that the theorem is also valid when the integral conditions for $B_{r_1}^1$ and $B_{r_2}^1$ are replaced by $B_{r_1}^2$ and $B_{r_2}^2$. The same is also true for $B_{r_1}^1$ and $B_{r_2}^2$.

However when also considering the full set of rotations of the complex disk inside of $\mathbf{C}^2$, we show that the Pompeiu property is restored.

Theorem 5.3. *Let $f \in C \cap L^\infty(\mathbf{H}^2)$, and let $r_1, r_2 > 0$. Suppose f satisfies the following integral conditions*

$$\int_{\gamma B^1_{r_1}} L_{\mathbf{g}} f(\gamma(z_1, 0), 0) d\sigma_{r_1,\gamma}(\gamma(z_1, 0)) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^2 \textit{ and for all } \gamma \in SU(2).$$

We may then conclude that $f \equiv 0$.

Note that these results are consistent with what we have observed in Euclidean space. Without a sufficient set of rotations, balls of higher codimension will require theorems of two radii to preserve the Pompeiu property. However, when rotations are included, balls of higher codimension now have the Pompeiu property.

The Morera side of the problem is particularly interesteing. Here we consider vanishing of integrals along $\mathbf{H}^2$ in order to characterize CR functions. Thus analytic extension to Ω_3 is characterized by vanishing of integrals along the surface of $\partial\Omega_3$, or $\mathbf{H}^2$. The integral conditions would be of the form

$$\int_{S^1_r} L_{\mathbf{g}} f(\mathbf{z}, 0, 0) dz_1 = 0 \quad \text{for all } \mathbf{g} \in \mathbf{H}^2, \tag{3}$$

and

$$\int_{S^2_r} L_{\mathbf{g}} f(0, \mathbf{z}, 0) dz_2 = 0 \quad \text{for all } \mathbf{g} \in \mathbf{H}^2.$$

As $S^1_r = \partial B^1_r$ in $\mathbf{C} \times \{0\}$ and $S^2_r = \partial B^2_r$ in $\{0\} \times \mathbf{C}$, these integrals are similar to the standard integrals for the Morera problem. The relation to Theorem 5.1 is also clear. Regarding the integral conditions in (3), note also that the spheres S^1_r and S^2_r are boundaries for many other surfaces of complex dimension 1 and complex codimension 1 inside of $\mathbf{C} \times \mathbf{C}$. The authors are currently investigating these aspects of the Morera problem. The Morera problem in tis setting is also interesting because of the relation to analytic disks. See the issues raised in this context in Section 7.

6. Pompeiu Problem for Higher CR Codimension in Products of Heisenberg Groups

In the last section on the Heisenberg group we noticed that the complex ball is already of real codimension 1, and we considered complex disks in $\mathbf{H}^2$, which are of a higher codimension (real codimension 3). We noticed how such a complex disk, of higher codimension in the ambient space, allows for additional rotations and thus recovers the Pompeiu property.

In this section we consider products of Heisenberg groups. In this setting we explore another side of higher codimension within CR manifolds. In particular, the product of Heisenberg groups yields a CR manifold whose CR codimension is 2 rather than 1. As a consequence, there are two missing directions T_1 and T_2, rather than only one. When considering rotations of the set S, these must be considered within $\mathbf{C}^{m+n}$, rather than the larger group $\mathbf{H}^{m+n}$. It is thus possible to consider balls or spheres of higher codimension which still remain invariant under all rotations in the ambient space. In this case the higher codimension is assocaited to the manifold itself, and it does not offer more space for rotations of the set S. But rather any rotation takes place in the complex space $\mathbf{C}^{m+n}$, wherein the ball B_r has full dimension (zero codimension). The complex space is associated to the complex vector fields $Z_1, \ldots, Z_n, \bar{Z}_1, \ldots, \bar{Z}_n$, whereas the higher codimension comes from having two extra directions T_1 and T_2. Rotation within this space is achieved by the twisting, built into the translation by the group action. Thus it is built into the approach we take to the harmonic analysis.

We demonstrate that in this case, these higher codimension sets, which are still at the highest possible dimension in $\mathbf{C}^{m+n}$, no longer possess the Pompeiu property at the L^∞ level. This result is still consistent with the main themes of the paper, as there can be nothing added by rotation. In the first case, we consider the case of L^p, for $1 \le p < \infty$. We consider integration over the set $B_r = \{(\mathbf{z}_1, \mathbf{z}_2) \times (0,0) : \|(\mathbf{z}_1, \mathbf{z}_2)\| \le r\}$ inside of $\mathbf{C}^n \times \mathbf{C}^m \times (0,0)$.

Theorem 6.1. *Let $f \in C \cap L^p(\mathbf{H}^n \times \mathbf{H}^m)$, with $1 \le p < \infty$, and let $r > 0$. Suppose f satisfies integral conditions*

$$\int_{B_r} L_{\mathbf{g}} f(\mathbf{z}_1, \mathbf{z}_2, 0, 0)\, d\mu_r(\mathbf{z}_1, \mathbf{z}_2) = 0 \qquad \textit{for all } \mathbf{g} \in \mathbf{H}^n \times \mathbf{H}^m.$$

We may then conclude $f \equiv 0$.

We now observe that at the level of L^∞, a theorem of two radii is required. To simplify matters, we consider the product $\mathbf{H}^1 \times \mathbf{H}^1$, *i.e.*, $n = m = 1$. To state this theorem we define $\Upsilon^{1,1}_{k,\lambda,\rho}$ as follows:

$$\Upsilon^{1,1}_{k,\lambda,\rho}(R) = \int_0^{R^2} e^{-2\pi|\lambda|x} L_k^{(0)}(4\pi|\lambda|x)(R^2 - x) J_1(\rho(R^2 - x))\, dx$$

We also denote the function $\Phi_{a,\mathbf{k}}$, described more fully in [13], which is defined as follows.

$$\begin{aligned}\Phi_{a,\mathbf{k}} &= \int_0^{4\pi|\lambda|} e^{-x_1/2} L_{k_1}^{(0)}(x_1) \int_0^{4\pi|\lambda|-x_1} e^{-ax_2/2} L_{k_2}^{(0)}(ax_2) dx_2 dx_1 \\ &= \sum_{j=1}^{2} e^{-2\pi|\lambda_j|} P(4\pi|\lambda_1|) + C_{a,\mathbf{k}}\end{aligned}$$

We then have the following theorem.

Theorem 6.2. *Let $f \in C \cap L^\infty(\mathbf{H}^1 \times \mathbf{H}^1)$, and let $r > 0$. Suppose f satisfies integral conditions*

$$\int_{B_{r_1}} L_{\mathbf{g}} f(\mathbf{z}_1, \mathbf{z}_2, 0, 0) d\mu_{r_1}(\mathbf{z}_1, \mathbf{z}_2) = 0 \qquad \textit{for all } \mathbf{g} \in \mathbf{H}^1 \times \mathbf{H}^1,$$

and

$$\int_{B_{r_2}} L_{\mathbf{g}} f(\mathbf{z}_1, \mathbf{z}_2, 0, 0) d\mu_{r_2}(\mathbf{z}_1, \mathbf{z}_2) = 0 \qquad \textit{for all } \mathbf{g} \in \mathbf{H}^1 \times \mathbf{H}^1.$$

Suppose further that r_1 and r_2 satisfy the following conditions:

(1) $r_1/r_2 \notin \mathcal{Q}(J_2)$.

(2) $(r_1/r_2)^2 \notin \mathcal{Q}(\Phi_{a,k})$ *for all $a \in \mathbf{R}_+$ and all $k \in \mathbf{Z}_+$.*

(3) $(r_1/r_2)^2 \notin \mathcal{Q}(\Upsilon^{1,1}_{k,\lambda,\rho})$ *for all $k \in \mathbf{Z}_+$.*

We may then conclude $f \equiv 0$. However if the radii r_1 and r_2 do not meet any of conditions 1., 2., and 3., then there exists $f \not\equiv 0$, $f \in C \cap L^\infty(\mathbf{H}^1 \times \mathbf{H}^1)$ satisfying the integral conditions.

Note tht Theorem 6.2 is somewhat more complicated than the two radii theorems of the Heisenberg group $\mathbf{H}^n$ in Section 5, and in [3, 1, 6] in the sense that there are three conditions on the radii rather than two. For this reason we give a brief outline of the proof; for additional details, please see [14]. We rely on the fact that the bounded $\mathbf{T}^n \times \mathbf{T}^m$-spherical functions on $\mathbf{H}^n \times \mathbf{H}^m$ are given by the following:

$$\psi^{\lambda_1,\lambda_2}_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{z}, t, \mathbf{w}, s) = \psi^{\lambda_1}_{\mathbf{k}_1}(\mathbf{z}, t) \cdot \psi^{\lambda_2}_{\mathbf{k}_2}(\mathbf{w}, s)$$

for every $(\mathbf{k}_1, \lambda_1, \mathbf{k}_2, \lambda_2) \in (\mathbf{Z}_+)^n \times \mathbf{R}^* \times (\mathbf{Z}_+)^m \times \mathbf{R}^*$,

$$(\psi\mathcal{J})^{\lambda_1}_{k_1,\rho_2}(\mathbf{z}, t, \mathbf{w}) = \psi^{\lambda_1}_{\mathbf{k}_1}(\mathbf{z}, t) \cdot \mathcal{J}_{\rho_2}(\mathbf{w})$$

for every $(\mathbf{k}_1, \lambda_1, \rho_2) \in (\mathbf{Z}_+)^n \times \mathbf{R}^* \times (\mathbf{R}_+)^m$,

$$(\mathcal{J}\psi)^{\lambda_2}_{\rho_1,k_2}(\mathbf{z}, \mathbf{w}, s) = \mathcal{J}_{\rho_1}(\mathbf{z}) \cdot \psi^{\lambda_2}_{\mathbf{k}_2}(\mathbf{w}, s)$$

for every $(\rho_1, \mathbf{k}_2, \lambda_2) \in (\mathbf{R}_+)^n \times (\mathbf{Z}_+)^m \times \mathbf{R}^*$,

and

$$\mathcal{J}_{\rho_1,\rho_2}(\mathbf{z}, \mathbf{w}) = \mathcal{J}_{\rho_1}(\mathbf{z}) \cdot \mathcal{J}_{\rho_2}(\mathbf{w})$$

for every $(\rho_1, \rho_2) \in (\mathbf{R}_+)^n \times (\mathbf{R}_+)^m$.

Here the product of Heisenberg brushes $\mathcal{H}^{(n)} \times \mathcal{H}^{(m)}$ indexes these functions. Define $f \in L^1_{00}(\mathbf{H}^n \times \mathbf{H}^m)$ as follows

$$L^1_{00}(\mathbf{H}^n \times \mathbf{H}^m) = \{f \in L^1(\mathbf{H}^n \times \mathbf{H}^m : f(\sigma_1 \mathbf{z}, t, \sigma_2 \mathbf{w}, s) = f(\mathbf{z}, t, \mathbf{w}, s) \quad \text{for all} \\ (\sigma_1, \sigma_2) \in \mathbf{T}^n \times \mathbf{T}^m \text{ and } (\mathbf{z}, t, \mathbf{w}, s) \in \mathbf{H}^n \times \mathbf{H}^m\}$$

For $f \in L^1_{00}(\mathbf{H}^n \times \mathbf{H}^m)$, we may form the spherical function transform $\widetilde{f}$, defined on $\mathcal{H}^{(n)} \times \mathcal{H}^{(m)}$ as follows.

$$\widetilde{f}(\mathbf{k}_1, \lambda_1, \mathbf{k}_2, \lambda_2) = \int_{\mathbf{H}^n \times \mathbf{H}^m} f(\mathbf{z}, t, \mathbf{w}, s) \overline{\psi^{\lambda_1, \lambda_2}_{\mathbf{k}_1, \mathbf{k}_2}(\mathbf{z}, t, \mathbf{w}, s)} dm(\mathbf{z}, t, \mathbf{w}, s) \\ \text{for } (\mathbf{k}_1, \lambda_1, \mathbf{k}_2, \lambda_2) \in (\mathbf{Z}_+)^n \times \mathbf{R}^* \times (\mathbf{Z}_+)^m \times \mathbf{R}^*,$$

$$\widetilde{f}(\mathbf{k}_1, \lambda_1, 0, \rho_2) = \int_{\mathbf{H}^n \times \mathbf{H}^m} f(\mathbf{z}, t, \mathbf{w}, s) \overline{(\psi \mathcal{J})^{\lambda_1}_{k_1, \rho_2}(\mathbf{z}, t, \mathbf{w})} dm(\mathbf{z}, t, \mathbf{w}, s) \\ \text{for } (\mathbf{k}_1, \lambda_1, \rho_2) \in (\mathbf{Z}_+)^n \times \mathbf{R}^* \times (\mathbf{R}_+)^m,$$

$$\widetilde{f}(0, \rho_1, \mathbf{k}_2, \lambda_2) = \int_{\mathbf{H}^n \times \mathbf{H}^m} f(\mathbf{z}, t, \mathbf{w}, s) \overline{(\mathcal{J} \psi)^{\lambda_2}_{\rho_1, k_2}(\mathbf{z}, \mathbf{w}, s)} dm(\mathbf{z}, t, \mathbf{w}, s) \\ \text{for } (\rho_1, \mathbf{k}_2, \lambda_2) \in (\mathbf{R}_+)^n \times (\mathbf{Z}_+)^m \times \mathbf{R}^*,$$

$$\widetilde{f}(0, \rho_1, 0, \rho_2) = \int_{\mathbf{H}^n \times \mathbf{H}^m} f(\mathbf{z}, t, \mathbf{w}, s) \overline{\mathcal{J}_{\rho_1, \rho_2}(\mathbf{z}, \mathbf{w})} dm(\mathbf{z}, t, \mathbf{w}, s) \\ \text{for } (\rho_1, \rho_2) \in (\mathbf{R}_+)^n \times (\mathbf{R}_+)^m.$$

For $j = 1, 2$ let T_j be the distribution which represents averaging over the $(n+m)$-ball $B_{r_j} \subset \mathbf{C}^n \times \{0\} \times \mathbf{C}^m \times \{0\}$. T_j is defined as follows.

$$\langle \phi, T_j \rangle = \int_{B_{r_j}} \phi(\mathbf{z}, 0, \mathbf{w}, 0) d\sigma_r(\mathbf{z}, \mathbf{w}) \quad \text{for any } \phi \in L^1(\mathbf{H}^n \times \mathbf{H}^m).$$

We apply the spherical function transform to form $\widetilde{T}_1$ and $\widetilde{T}_2$, each defined on $\mathcal{H}^{(n)} \times \mathcal{H}^{(m)}$. We reduce to the case of $n = m = 1$, for the product space $\mathbf{H}^1 \times \mathbf{H}^1$ addressed in the theorem. This reduction makes the evaluation of these transforms cleaner. These are given by, for $j = 1, 2$,

$$\widetilde{T}_j(\mathbf{k}_1, \lambda_1, \mathbf{k}_2, \lambda_2) = c \Phi_{a, \mathbf{k}}(r_j^2),$$

where $a = |\frac{\lambda_1}{\lambda_2}|$.

$$\widetilde{T}_j(\mathbf{k}_1, \lambda_1, 0, \rho_2) = c \Upsilon^{1,1}_{k_1, \lambda_1, \rho_2}(r_j^2),$$

$$\widetilde{T}_j(0,\rho_1,\mathbf{k}_2,\lambda_2) = c\Upsilon^{1,1}_{k_2,\lambda_2,\rho_1}(r_j^2),$$
$$\widetilde{T}_j(0,\rho_1,0,\rho_2) = cJ_2(\sqrt{\rho_1^2+\rho_2^2}r_j).$$

Conditions 1. through 3. then correspond to the conditions such that $\widetilde{T}_1(p)$ and $\widetilde{T}_2(p)$ are not both (simultaneously) 0 for any $p \in \mathcal{H}^{(1)} \times \mathcal{H}^{(1)}$. Under these conditions, we have a Tauberian theorem, comparable to those of [6, 3], which guarantee that $\mathcal{I} = L^1_{00}(\mathbf{H}^n \times \mathbf{H}^m)$, where $\mathcal{I}$ is the closed ideal generated by $\{\Phi * T_1, \Phi * T_2 : \Phi \in L^1_{00}(\mathbf{H}^1 \times \mathbf{H}^1)\}$. It follows that $f * L^1_{00}(\mathbf{H}^1 \times \mathbf{H}^1) = 0$, or $f \equiv 0$. This completes the sketch of the proof. Note that a similar result holds for any $\mathbf{H}^n \times \mathbf{H}^m$, as $n = m = 1$ was only used for ease in evaluating the integrals that determine the transforms. However for the general case it becomes more difficult to evaluate the integrals and to describe conditions 1. to 3. for the set of exceptional radii.

Notice that an individual ball (or sphere) in $\mathbf{H}^n \times \mathbf{H}^m$ will not possess the Pompeiu property, even though it is of a higher codimension. The ball or sphere is not of higher codimension within $\mathbf{C}^n \times \mathbf{C}^m$, and there is no extra room for rotation within this space. Thus we have no result comparable to Theorem 5.3.

The Morera side of the problem is also interesting in this setting, but we defer treatment of this problem to later work.

The key point to this section is to observe that the codimension plays a different role than it did in Section 5. Here the codimension is built into the manifold itself, *i.e.*, a CR codimension higher than 1, rather than yielding extra space for rotation of the set S. However, it may be possible, at some other point, to consider additional rotations of the plane $\mathbf{C} \times \{0\} \times \mathbf{C} \times \{0\}$ within the space of $\mathbf{H}^1 \times \mathbf{H}^1$. We suspect that such rotations may help recover the Pompeiu property without the use of two radii, but this remains for future investigations.

7. Morera Problem of Higher Codimension and Analytic Disks

The issue of analytic extension provides an important motivation for considering the Morera problem on the Heisenberg group. This problem allows characterization of functions which extend analytically to Ω_{n+1} based on integrals over the surface $\partial\Omega_{n+1}$, or $\mathbf{H}^n$. Let us note that the issue of analytic extension for CR functions is usually proven using the method of analytic disks.

We first define analytic disks as follows.

Definition 7.1. An analytic disk A is a mapping $A : \overline{D} \to \mathbf{C}^{n+1}$ which is continuous on the closed disk $\overline{D} = \{\zeta \in \mathbf{C} : |\zeta| \leq 1\}$ and analytic on the open unit disk D. Let $M \subset \mathbf{C}^{n+1}$ be a manifold. An analytic disk A is said to be attached to the manifold M when the circle ∂D is mapped into M, *i.e.*, $A(\partial D) \subset M$.

The approach taken to holomorphic extension from a submanifold $M \subset \mathbf{C}^n$ using analytic disks is to consider the collection of analytic disks attached to M. Each such analytic disk A is attached to M via $A(\partial D) \subset M$. The focus of our interest is the image of the interior of this disk, $A(D) \subset \mathbf{C}^n$. When considering the collection $\mathcal{A}_M = \{A : A(\partial D) \subset M\}$, the region of interest is that region filled in by the interiors, $\cup_{A \in \mathcal{A}_M} A(D)$. What subset of $\mathbf{C}^n$ is filled in by the interiors of the analytic disks attached to M? A key point in holomorphic extension to a wedge $\mathcal{W}$ is to fill out $\mathcal{W}$ by images of analytic disks attached to M.

Using the theory of analytic disks allows for a characterization of CR functions based on the geometry of the CR manifold M. To give the result we must first give the following definition of minimality for a CR manifold.

Definition 7.2. Let $M \subset \mathbf{C}^n$ be a CR submanifold at $p_0 \in M$. We say that M is minimal at p_0 if there is no (germ of a) real submanifold $S \subset M$ through p_0 such that every $T_p^c M$ is tangent to S at every $p \in S$.

We now observe that minimality that minimality at a point is the precise condition to characterize analytic extension for all CR functions to a wedge centered at that point.

Theorem 7.3. *Let M be a generic submanifold of $\mathbf{C}^{n+d}$ of codimension d, and $p_0 \in M$. If M is minimal at p_0, then for every open neighborhood U of p_0 in M there exists a wedge $\mathcal{W}$ with edge M centered at p_0 such that every continuous CR function in U extends holomorphically to the wedge $\mathcal{W}$. Conversely, if M is not minimal at p_0, then there exists a continuous CR function defined in a neighborhood of p_0 in M which does not extend holomorphically to any wedge of edge M centered at p_0.*

The sufficiency of minimality for this result was proved in [21], and the necessity was proved in [5].

Beyond the theory of analytic extension for generic CR manifolds, there is more that can be stated in the specific case of quadric CR manifolds. We state this result, which follows from direct computation with analytic disks. These cases are most relevant for this paper, as both the Heisenberg group $\mathbf{H}^n$ and the product of Heisenberg groups $\mathbf{H}^n \times \mathbf{H}^m$, the cases of CR manifolds considered in this paper, are cases of quadric CR manifolds. A quadric submanifold M is one of the form given by

$$\{(\mathbf{x} + i\mathbf{y}, \mathbf{w}) \in \mathbf{C}^d \times \mathbf{C}^{n-d} : \mathbf{y} = q(\mathbf{w}, \bar{\mathbf{w}})\}$$

where $q : \mathbf{C}^{n-d} \times \mathbf{C}^{n-d} \to \mathbf{C}^d$ is a quadratic form. The case of quadrics is a special case in which the analytic disks can be explicitly described. First we must define Γ_0, the convex hull of the quadratic form q. Let

$$\Gamma_0 = \left\{ \sum_{j=1}^{N} t_j q(\alpha_j, \bar{\alpha}_j); N \geq 1, 0 \leq t_j \leq 1, \sum_{j=1}^{N} t_j = 1 \text{ and } \alpha_j \in \mathbf{C}^{n-d} \right\}.$$

We now give the result, as stated in Boggess, [9]

Theorem 7.4. *Suppose M is a quadric submanifold of $\mathbf{C}^n$. If the interior or $\{\Gamma_0\}$ is nonempty, then for each CR function f that is of class C^1 on M, there is a function F that is holomorphic on $\Omega = M +$ interior $\{\Gamma_0\}$ and continuous on $\Omega \cap M$ with $F|_M = f$.*

We observe that for the Heisenberg group $\mathbf{H}^n$, this result corresponds to holomorphic extension of CR functions to Ω_{n+1}, the Siegel upper half space in $\mathbf{C}^{n+1}$. Perhaps even more interesting is the case of a quadric CR hypersurface $M \subset \mathbf{C}^{n+1}$ which is defined by a quadratic form with eigenvalues of both signs, such as $q(z, \bar{z}) = |z_1|^2 - |z_2|^2$. In this case CR functions extend holomorphically to all of the space $\mathbf{C}^3$.

Beyond their use in issues of analytic extension, we are also interested in analytic disks based on their relation to vanishing of integrals along the boundaries of disks on the surface of a manifold M. Tumanov has demonstrated a connection between analytic disks, holomorphic extension of CR functions, and the Morera problem in [22]. We first define the holomorphic extension property with respect to a given analytic disk A.

Definition 7.5. Let f be a continuous function on a manifold M, and let A be an analytic disk attached to M. We say that f has the holomorphic extension property for the analytic disk A if

(4) $$f \circ A|_{\partial D}, \text{ extends holomorphically inside } D.$$

He proves the following result.

Theorem 7.6. *Let f be a continuous function on a C^2 smooth generic manifold $M \subset \mathbf{C}^n$. Suppose M has the holomorphic extension property (4) for all analytic discs attached to M. Then f is a CR function on M.*

In order to more fully investigate this connection between analytic disks, holomorphic extension of CR function, and the Morera problem, we really need to consider the Morera problem of higher codimension, where integrating over circles (boundaries of disks) and their translations and rotations. We seek to establish a direct connection between Morera results, such as those found earlier in this paper or those in [1] and [2], and CR extension results using analytic disks.

Note that the current work for the Morera problem in $\mathbf{H}^n$ over circles in $\mathbf{C}$ with real codimension $2n-1$ in $\mathbf{C}^n$, considers vanishing of integrals over translations and rotations of circles (boundaries of disks). In particular, the Morera type integrals addressed in formula (3) of Section 5 correspond to vanishing of integrals over the boundary of a disk inside of the CR manifold $\mathbf{H}^2$. We would like to establish a result regarding CR functions and analytic extension based on these integral conditions in (3) of Section 5. Moreover, a direct comparison between the Morera results and the analytic disk results may allow us to use both relevant methods, the Fourier transform in one case and the theory of analytic disks in the other. Deeper issues we also plan to investigate include the Morera problem for other CR manifolds (starting with other quadrics), analytic extension for cases of specific functions f, and what happens in the case of non-minimality. The most intriguing aspect of this problem is the relation of the Morera problem to geometric properties of the manifold.

References

1. M. Agranovsky, C. Berenstein and D. C. Chang, Morera theorem for holomorphic H^p spaces in the Heisenberg group, *J. Reine Angew. Math.*, **443** (1993), 49-89.
2. M. Agranovsky, C. Berenstein, D. C. Chang and D. Pascuas, A Morera type theorem for L^2 functions on the Heisenberg group, *J. d'Analyse Math.*, **57** (1991), 282-296.
3. M. Agranovsky, C. Berenstein, D. C. Chang and D. Pascuas, Injectivity of the Pompeiu transform in the Heisenberg group, *J. d'Analyse Math.*, **63** (1994), 131-173.
4. M. S. Baouendi, P. Ebenfelt and L. P. Rothschild, *Real Submanifolds in Complex Space and their Mappings,* Princeton Mathematical Studies #47, Princeton University Press, Princeton, NJ, 1999.
5. M. S. Baouendi and L. P. Rothschild, Cauchy Riemann functions on manifolds of higher codimension in complex space, *Invent. Math.*, **101** (1990), 45-56.
6. C. Berenstein, D. C. Chang and J. Tie, *Laguerre calculus and its applications on the Heisenberg group*, AMS/IP Studies in Advanced Mathematics #22, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2001.
7. C. Berenstein and W. Eby, *Local results for the Pompeiu problem in Euclidean space for spheres and balls of higher codimension,* preprint.
8. C. Berenstein and R. Gay, A local version of the two-circles theorem, *Israel J. Math.*, **55** (1986), 267-288.
9. A. Boggess, *CR Manifolds and the Tangential Cauchy-Riemann Complex,* Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1991.
10. L. Brown, B. Schreiber, and B. A. Taylor, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier* (*Grenoble*), **23** (1973), 125-154.

11. D. C. Chang, On the boundary of Fourier and complex analysis: the Pompeiu problem, *Taiwanese J. Math.*, **6**, (2002), 1-37.

12. D.C. Chang and W. Eby, Tauberian theorem for $\mathbf{m}$-spherical transforms on the Heisenberg group, *Mathematische Nachrichten*, **280**, #8, (2007), 815–837.

13. D. C. Chang and W. Eby, Pompeiu problem for complex ellipsoids on the Heisenberg group, *Bull. Inst. Math. Academia Sinica* (*New Series*), **2** (2007), 731-768.

14. D. C. Chang and W. Eby, Pompeiu problem on products of Heisenberg groups, preprint.

15. D. C. Chang and W. Eby, Pompeiu problem in $\mathbf{H}^2$ for complex disks and spheres, preprint.

16. E. Grinberg and E. T. Quinto, Morera theorems for complex manifolds, *J. Funct. Anal.*, **178** (2000), 1-22.

17. V. Guillemin and S. Sternberg, *Geometric Asymptotics*, American Mathematical Society, Providence, RI, 1977.

18. L. Hörmander, Fourier Integral Operators, I, *Acta Math.*, **127** (1971), 79-183.

19. E. T. Quinto, The dependence of the generalized Radon transform on defining measures, *Trans. Amer. Math. Soc.*, **257** (1980), 331-346.

20. E. T. Quinto, Pompeiu transforms on geodesic spheres in real analytic manifolds, *Israel J. Math.*, **84** (1993), 353-363.

21. A. Tumanov, Extension of CR-functions into a wedge from a manifold of finite type, (Russian), *Mat. Sb.* (*N.S.*), **136 (178)** (1988), 128-139; *English translation in Math. USSR-Sb.*, **64** (1989), 129-140.

22. A. Tumanov, Thin disks and a Morera theorem for CR functions, *Math. Zeit.*, **226** (1997), 327-334.

23. S. A. Williams, A partial solution of the Pompeiu problem, *Math. Ann.*, **223** (1976), 183-190.

24. L. Zalcman, Analyticity and the Pompeiu problem, *Arch. Rat. Mech. Anal.*, **47** (1972), 237-254.

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