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ON THE SOLVABILITY OF THE FIRST MIXED PROBLEM FOR STRONGLY HYPERBOLIC SYSTEM IN INFINITE NONSMOOTH CYLINDERS

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Abstract. The purpose of this paper is to establish some new results on the unique solvability of solution of the first mixed problem for strongly hyperbolic systems in infinite cylinders with nonsmooth base.

1. Introduction

The boundary value problems for elliptic equation in domains with smooth boundary have been well studied by S. Agmons, A. Douglis and L. Nirenberg in [2]. The authors considered the normal solvability of the problems with Sapiro-Lopatinsky's condition. Besides, they also studied the smoothness of solutions which depend on coefficients of the right hand part of the equation and the boundary of considered domains.

The general elliptic boundary valued problems in nonsmooth domains were considered by V. A. Kondratiev [4], where author established important results on the unique existence of solutions and asymptotic expansion of solutions for the problems in the weight Sobolev spaces.

In this paper we consider the first mixed problem for strongly hyperbolic system in infinite cylinders with the nonsmooth base. We establish some results on the unique existence of generalized solution of the problem by using the method of approximate Galerkin. It is noted that the problem for the case of finite cylinders has been studied in the work [3], where the author investigated the solvability of this problem and the asymptotic expansion of generalized solutions in a neighbourhood of the conical point.

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The rest of the paper consists of two sections. Section 2 is devoted to some notations and formulation of the problem. In section 3 we will establish the uniqueness and existence of the solution in Sobolev spaces.

2. Formulation of the Problem

Let Ω be a bounded domain in R^n with the boundary $\partial\Omega$. For each T, $0 < T < \infty$, we put

$$\Omega_T = \Omega \times (0,T),\ S_T = \partial\Omega \times (0,T).$$

We use notation:

$$\Omega_\infty = \Omega \times (0,\infty),\ S_\infty = \partial\Omega \times (0,\infty),$$

$u = (u_1, u_2, ..., u_s), u_{t^k} = (u_{1t^k}, u_{2t^k}, ..., u_{st^k}),\ u_{jt^k} = \partial^k u_j/\partial t^k,$

$$D^\alpha u = (D^\alpha u_1, D^\alpha u_2, ..., D^\alpha u_s), D^\alpha u_j = \frac{\partial^{|\alpha|} u_j}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Throughout the paper we will use the following functional spaces:

(1) $H^m(\Omega)$ is the space of complex vector functions $u = u(x)$, $x \in \Omega$ having generalized derivatives $D^\alpha u \in L^2(\Omega)$ with the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha u|^2 dx \right)^{1/2}.$$

(2) $H^{m,k}(\Omega_T)$ is the space of complex vector functions $u = u(x,t)$, $(x,t) \in \Omega_T$ having generalized derivatives $D^\alpha u \in L^2(\Omega_T)$, $|\alpha| \le m$ and $u_{t^l} \in L^2(\Omega_T)$, $1 \le l \le k$ with the norm

$$\|u\|^2_{H^{m,k}(\Omega_T)} = \int_{\Omega_T} \left(\sum_{|\alpha| \le m} |D^\alpha u|^2 + \sum_{l=1}^{k} |u_{t^l}|^2 \right) dxdt.$$

In particular,

$$\|u\|^2_{H^{m,0}(\Omega_T)} = \sum_{|\alpha| \le m} \int_{\Omega_T} |D^\alpha u|^2 dxdt.$$

(3) $\overset{\circ}{H}{}^{m,k}(\Omega_T)$ is the closure in $H^{m,k}(\Omega_T)$ of the set of all infinitely differentiable complex vector functions on Ω_T which vanish near S_T.

(4) $H^{m,k}(e^{-\gamma t};\Omega_\infty)$ is the space of complex vector functions $u = u(x,t)$ on Ω_∞ which have generalized derivatives $D^\alpha u$, $|\alpha| \le m$, u_{t^l}, $1 \le l \le k$ such that

$$\|u\|^2_{H^{m,k}(e^{-\gamma t},\Omega_\infty)} = \int\limits_{\Omega_\infty} \left(\sum_{|\alpha|\le m} |D^\alpha u|^2 + \sum_{l=1}^{k} |u_{t^l}|^2 \right) e^{-2\gamma t} dxdt < \infty,$$

where γ is a positive constant.

(5) $\overset{\circ}{H}{}^{m,k}(e^{-\gamma t},\Omega_\infty)$ is the closure in $H^{m,k}(e^{-\gamma t},\Omega_\infty)$ of the set containing all infinitely differentiable complex vector functions on Ω_∞ which vanish near S_∞.

(6) $L^\infty(0,\infty;L^2(\Omega))$ is the space of measurable complex valued functions $u : (0,\infty) \to L^2(\Omega); t \mapsto u(\cdot,t)$ satisfying

$$\|u\|_{L^\infty(0,\infty;L^2(\Omega))} = \operatorname*{ess\,sup}_{\mathrm{t}>0} \|\mathrm{u}(\cdot,\mathrm{t})\|_{\mathrm{L}^2(\Omega)} < \infty.$$

We introduce the differential operator

$$L(x,t,D) := \sum_{|\alpha|,|\beta|\le m} D^\alpha(a_{\alpha\beta}(x,t)D^\beta) + \sum_{|\alpha|=1}^{m} a_\alpha(x,t)D^\alpha + a(x,t), \tag{2.1}$$

where $a_{\alpha\beta} = a_{\alpha\beta}(x,t), a_\alpha = a_\alpha(x,t), a = a(x,t)$ are $s \times s$ matrices of measurable complex functions which are bounded in $\overline{\Omega}_\infty$; $a_{\alpha\beta} = (-1)^{|\alpha|+|\beta|}a^*_{\beta\alpha}$, $a^*_{\beta\alpha}$ are complex conjugate transportation matrices of $a_{\alpha\beta}$ with continuous elements in $\overline{\Omega}_\infty$.

Let f belong to $L^\infty(0,\infty;L^2(\Omega))$. We consider the following in the cylinder Ω_∞:

$$(-1)^{m-1}L(x,t,D)u - u_{tt} = f, \tag{2.2}$$

with the initial condition

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0, \tag{2.3}$$

and boundary conditions

$$\frac{\partial^j u}{\partial \nu_j}|_{S_\infty} = 0; j = 0, ..., (m-1), \tag{2.4}$$

where $\frac{\partial^j u}{\partial \nu_j}$ is derivative with respect to the outer unit normal of S_∞.

Definition 2.1. The differential operator L is said to be strongly hyperbolic in Ω_∞ if there exists a constant h_0 such that

$$\sum_{|\alpha|,|\beta|=m} a_{\alpha\beta}(x,t)\xi^\alpha\xi^\beta\eta\overline{\eta} \geq h_0|\xi|^{2m}|\eta|^2, \tag{2.5}$$

for al $\xi \in R^n\backslash\{0\}$, $\eta \in C^s\backslash\{0\}$ and $(x,t) \in \overline{\Omega}_\infty$.

Definition 2.2. A function $u(x,t)$ is called a generalized solution of the problem (2.2)-(2.4) in the space $H^{m,1}(e^{-\gamma t};\Omega_\infty)$, if and only if $u(x,t)$ belongs to $\overset{0}{H}{}^{m,1}(e^{-\gamma t};\Omega_\infty)$, $u(x,0)=0$, and for each $T>0$ the equality

$$\begin{aligned}(-1)^{m-1}\int\limits_{\Omega_T}\left(\sum_{|\alpha|,|\beta|=1}^{m}(-1)^{|\alpha|}a_{\alpha\beta}D^\beta u\overline{D^\alpha}\varphi+\sum_{|\alpha|=1}^{m}a_\alpha D^\alpha u\overline{\varphi}+au\overline{\varphi}\right)dxdt \\ +\int\limits_{\Omega_T}u_t\overline{\varphi}_t dxdt=\int\limits_{\Omega_T}f\overline{\varphi}dxdt.\end{aligned} \tag{2.6}$$

holds for all $\varphi \in \overset{\circ}{H}{}^{m,1}(\Omega_T)$ satisfying $\varphi(\cdot,T)=0$.

3. Solvability of the Problem

In this section we establish theorems on the uniqueness and existence of a generalized solution of problem (2.2)-(2.4). First we prove the following lemma which is a generalization of the Garding inequality (see [8]).

Lemma 3.1. *Assume that L is strongly hyperbolic in $\overline{\Omega}_\infty$ and $a_{\alpha\beta}(x,t)$ are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0,\infty)$ whenever $|\alpha|=|\beta|=m$. Then there exist constants $\mu_0>0$ and $\lambda_0 \geq 0$ such that*

$$(-1)^m B(u,u)(t) \geq \mu_0\|u(\cdot,t)\|^2_{H^m(\Omega)} - \lambda_0\|u(\cdot,t)\|^2_{L^2(\Omega)} \tag{3.1}$$

for all $u \in \overset{\circ}{H}{}^{m,1}(e^{-\gamma t};\Omega_\infty)$, where B is defined by

$$B(u,u)(t)=\sum_{|\alpha|,|\beta|=1}^{m}(-1)^{|\alpha|}\int\limits_\Omega a_{\alpha\beta}D^\beta u\overline{D^\alpha u}dx. \tag{3.2}$$

Proof. We use the notation $\beta_{ls}=(-1)^{|l|+m}a_{ls}$ for $|l|+|s|<2m$ and rewrite

$$(-1)^m B(u,u)(t)=\sum_{|\alpha|=|\beta|=m}\int\limits_\Omega a_{\alpha\beta}D^\beta u\overline{D^\alpha u}dx+\sum_{|l|+|s|<2m}\int\limits_\Omega b_{ls}D^s u\overline{D^l u}dx.$$

Since Ω is bounded, there exists a cube $\Pi \subset R^n$ such that $\overline{\Omega} \subset \Pi$. We now consider the following cases.

Case 1. $b_{ls} = 0$ for all l, s satisfying $|l| + |s| < 2m$ and $a_{\alpha\beta}(x,t) = a_{\alpha\beta}(t)$. For each $u \in \overset{\circ}{H}{}^m(\Omega)$, using the Fourier expansion, we have

$$u(x,t) = (2\pi)^{-n/2} \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

where

$$c_k = c_k(t) = (2\pi)^{-n/2} \int\limits_{\Omega} u(x,t) e^{-ikx} dx.$$

Hence

$$D^\alpha u = (2\pi)^{-n/2} \sum_{k=-\infty}^{\infty} i^{|\alpha|} k^\alpha c_k e^{ikx},$$

$$a_{\alpha\beta} D^\beta u = (2\pi)^{-n/2} \sum_{k=-\infty}^{\infty} i^{|\beta|} k^\beta a_{\alpha\beta} c_k e^{ikx}.$$

Using Parseval equality, the strongly hyperbolic condition of L and Friedrichs inequality we obtain

$$\begin{aligned}(-1)^m B(u,u)(t) &= (2\pi)^{-n/2} \sum_{k=-\infty}^{\infty} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} k^\alpha k^\beta c_k \overline{c_k} \\ &\geq h_0 \sum_{k=-\infty}^{\infty} |k|^{2m} |c_k|^2 \\ &\geq \mu_0 \|u(\cdot,t)\|^2_{H^m(\Omega)},\end{aligned}$$

where $\mu_0 = \frac{h_0 C}{(mes\Omega)^{2/m}}$, C is a constant depending on n only.

Case 2. We put

$$G := \{x \in \Omega : u(x,t) \neq 0\}$$

and assume that $\text{Diam}G < \delta_0$ for some $\delta_0 > 0$. Let $(x^0, t) \in G$. than there exists a positive constant C_1 which is independent of (x^0, t) such that

$$C_1 \|u(\cdot,t)\|^2_{H^m(\Omega)} \leq \sum_{|\alpha|=|\beta|=m} \int\limits_{\Omega} a_{\alpha\beta}(x^0,t) D^\beta u \overline{D^\alpha u} dx. \tag{3.3}$$

On the other hand we have

$$\begin{aligned}
&\sum_{|\alpha|=|\beta|=m}\int_\Omega a_{\alpha\beta}(x^0,t)D^\beta u\overline{D^\alpha u}dx\\
&=\sum_{|\alpha|=|\beta|=m}\int_\Omega a_{\alpha\beta}(x,t)D^\beta u\overline{D^\alpha u}dx+\sum_{|l|+|s|<2m}\int_\Omega b_{ls}(x,t)D^s u\overline{D^l u}dx\\
&\quad+\sum_{|\alpha|=|\beta|=m}\int_\Omega [a_{\alpha\beta}(x^0,t)-a_{\alpha\beta}(x,t)]D^\beta u\overline{D^\alpha u}dx\\
&\quad-\sum_{|l|+|s|<2m}\int_\Omega b_{ls}(x,t)D^s u\overline{D^l u}dx\\
&\le (-1)^m B(u,u)(t)+C_2\|u(\cdot,t)\|_{H^m(\Omega)}\|u(\cdot,t)\|_{H^{m-1}(\Omega)}\\
&\quad+C_3\max_{x\in G}\left\{\sum_{|\alpha|=|\beta|=m}|a_{\alpha\beta}(x^0,t)-a_{\alpha\beta}(x,t)|\right\}\|u(\cdot,t)\|^2_{H^m(\Omega)},
\end{aligned}$$

where C_2 and C_3 are positive constants. Since $a_{\alpha\beta}(x,t)$ is continuous in $x\in\overline{\Omega}$ uniformly with respect to $t\in(0,\infty)$, for $\delta_0>0$ sufficiently small, we have

$$\max_{x\in G}\left\{\sum_{|\alpha|=|\beta|=m}|a_{\alpha\beta}(x^0,t)-a_{\alpha\beta}(x,t)|\right\}<C_1/2C_3.$$

Combining this with (3.3) we get

$$\frac{C_1}{2}\|u(\cdot,t)\|^2_{H^m(\Omega)}\le(-1)^m B(u,u)(t)+C_2\|u(\cdot,t)\|_{H^m(\Omega)}\|u(\cdot,t)\|_{H^{m-1}(\Omega)}.$$

By the interpolation inequality (see [8]) we obtain the desired inequality.

Case 3. Consider the general case. We choose a partition of unity in $\overline{\Omega}$,

$$\sum_{1\le h\le N}\psi_h^2=1,\psi_h\in C^\infty(\overline{\Omega}),\mathrm{diam(supp)}\psi<\delta_0,$$

where δ_0 is chosen such that

$$\left\{\sum_{|\alpha|=|\beta|=m}|a_{\alpha\beta}(x^1,t)-a_{\alpha\beta}(x^2,t)|\right\}<C_1/2C_3$$

for $|x^1-x^2|<\delta_0$.

Using the partition of unity above we have

$$(-1)^m B(u,u)(t) = \sum_{|\alpha|=|\beta|=m} \int_\Omega \sum_{h=1}^N \psi_h^2 a_{\alpha\beta} D^\beta u \overline{D^\alpha u} dx$$
$$+ \sum_{|l|+|s|<2m} \int_\Omega b_{ls} D^s u \overline{D^l u} dx$$
$$= \sum_{h=1}^N \sum_{|\alpha|=|\beta|=m} \int_\Omega a_{\alpha\beta} \psi_h^2 D^\beta (\psi_h u) \overline{D^\alpha (\psi_h u)} dx + \sum_{|l|+|s|<2m} \int_\Omega b_{ls} D^s u \overline{D^l u} dx$$
$$= \sum_{h=1}^N \sum_{|\alpha|=|\beta|=m} \int_\Omega a_{\alpha\beta} D^\beta (\psi_h u) \overline{D^\alpha (\psi_h u)} dx + O(\|u\|_{H^m(\Omega)} \|u\|_{H^{m-1}(\Omega)}).$$

From the second case we have

$$\|\psi_h u(\cdot,t)\|^2_{H^m(\Omega)} \le C_4 \sum_{|\alpha|=|\beta|=m} \int_\Omega a_{\alpha\beta} D^\beta (\psi_h u) \overline{D^\alpha (\psi_h u)} dx + C_5 \|\psi_h u(\cdot,t)\|^2_{L^2(\Omega)},$$

where C_4, C_5 are positive constants. Therefore we have

$$(-1)^m B(u,u)(t) \ge \sum_{h=1}^N \left(C_6 \|\psi_h u(\cdot,t)\|^2_{H^m(\Omega)} - C_7 \|\psi_h u\|^2_{L^2(\Omega)} \right)$$
$$- C_8 \|u\|_{H^m(\Omega)} \|u\|_{H^{m-1}(\Omega)}$$
$$= C_6 \sum_{h=1}^N \left(\sum_{|\alpha|\le m} \int_\Omega (D^\alpha \psi_h u)^2 dx \right) - C_7 \sum_{h=1}^N \int_\Omega (\psi_h u)^2 dx - C_8 \|u\|_{H^m(\Omega)} \|u\|_{H^{m-1}(\Omega)}$$
$$= C_6 \sum_{|\alpha|\le m} \left(\sum_{h=1}^N \int_\Omega (D^\alpha \psi_h u)^2 dx \right) - C_7 \sum_{h=1}^N \int_\Omega (\psi_h u)^2 dx - C_8 \|u\|_{H^m(\Omega)} \|u\|_{H^{m-1}(\Omega)}$$
$$\ge C_6 \sum_{|\alpha|\le m} \frac{1}{N} \int_\Omega \left(\sum_{h=1}^N D^\alpha \psi_h u)^2 dx \right) - C_7 \int_\Omega |u|^2 dx - C_8 \|u\|_{H^m(\Omega)} \|u\|_{H^{m-1}(\Omega)}$$
$$= \frac{C_6}{N} \|u(\cdot,t)\|^2_{H^m(\Omega)} - C_7 \|u\|^2_{L^2(\Omega)} - C_8 \|u\|_{H^m(\Omega)} \|u\|_{H^{m-1}(\Omega)}$$

where $C_6, C_7, C_8 - const > 0$.
Using arguments as in the proof at the end of Case 2, we obtain the conclusion of the lemma. The proof is complete. ■

Theorem 3.2. (Uniqueness of genrealized solution). *Assume that the coefficients of operator L satisfy conditions of lemma 3.1 and there exists a positive constant* μ

such that $|\partial a_{\alpha\beta}|, |\partial a_{\alpha}|, |\partial t|, |a| \leq \mu$ *for* $1 \leq |\alpha|, |\beta| \leq m$ *and* $(x,t) \in \overline{\Omega}_{\infty}$. *Then the problem (2.2)-(2.4) has at most generalized solution in* $H^{m,1}(e^{-\gamma t};\Omega_{\infty})$ *for all* $\gamma > 0$.

Proof. Suppose that there exists $\gamma > 0$ such that the problem (2.2) -(2.4) has two generalized u_1 and u_2 belong to $H^{m,1}(e^{-\gamma t};\Omega_{\infty})$. Putting $u = u_1 - u_2$, we see that $u \in \overset{\circ}{H}{}^{m,1}(e^{-\gamma t};\Omega_{\infty})$ and $u(x,0) = 0$. Moreover, u satisfies (2.6) for any $T > 0$.

Define a function φ by

$$\varphi(x,t) = \begin{cases} 0 & \text{if } b \leq t < T, \\ \int\limits_b^t u(x,\tau)d\tau & \text{if } 0 < t \leqslant b. \end{cases}$$

Applying φ to (2.6) and noting that $\varphi_t = u$, we obtain

$$\begin{aligned}(3.4)\qquad & (-1)^{m-1}\int\limits_{\Omega_b}\Big(\sum_{|\alpha|,|\beta|=1}^{m} a_{\alpha\beta}(D^{\beta}\varphi_t)\overline{D^{\alpha}\varphi} \\ & + \sum_{|\alpha|=1}^{m} a_{\alpha}(D^{\alpha}\varphi_t)\overline{\varphi} + a\varphi_t\overline{\varphi}\Big)dxdt + \int\limits_{\Omega_b}\varphi_{tt}\overline{\varphi_t}dxdt = 0.\end{aligned}$$

Put

$$B_1(u,u)(t) = B(u,u)(t) + 2Re\sum_{|\alpha|=1}^{m}\int\limits_{\Omega} a_{\alpha}(D^{\alpha}u)\overline{u}dx.$$

From lemma 3.1 and Cauchy's inequality we have

$$(-1)^m B_1(u,u)(t) \geq \mu_1\|u\|^2_{H^m(\Omega)} - \lambda_1\|u\|^2_{L_2(\Omega)},$$

where μ_1 and λ_1 are constants, $\mu_1 > 0$. We denote by I the $s \times s$ unit matrix and define a_{00}, a_1 by $a_{00} := (-1)^m\lambda_1 I$ and $a_1 := a - a_{00}$.

From equality (3.4) we have

$$(-1)^{m-1}\int\limits_{\Omega_b}\left(\sum_{|\alpha|,|\beta|=0}^{m} a_{\alpha\beta}(D^{\beta}\varphi_t)\overline{D^{\alpha}\varphi} + \sum_{|\alpha|=1}^{m} a_{\alpha}(D^{\alpha}\varphi_t)\overline{\varphi} + a_1\varphi_t\overline{\varphi}\right)dxdt$$
$$+\int\limits_{\Omega_b}\varphi_{tt}\overline{\varphi_t}dxdt = 0.$$

Since $a_{\alpha\beta} = (-1)^{|\alpha|+|\beta|}a^*_{\beta\alpha}$, adding this equality to its complex conjugate we receive the following equality.

$$\int\limits_{\Omega_b} \frac{\partial}{\partial t}(\varphi_t\overline{\varphi}_t)dxdt + \mathrm{Re}\int\limits_{\Omega_b}\left(\sum_{|\alpha|,|\beta|=0}^{m}(-1)^{|\alpha|+m-1}\frac{\partial}{\partial t}(a_{\alpha\beta}D^\beta\varphi\overline{D^\alpha\varphi})\right)dxdt$$

$$(3.5)\qquad -\mathrm{Re}\int\limits_{\Omega_b}\left(\sum_{|\alpha|,|\beta|=0}^{m}(-1)^{|\alpha|+m-1}\frac{\partial a_{\alpha\beta}}{\partial t}D^\beta\varphi\overline{D^\alpha\varphi})\right)dxdt$$

$$(-1)^{m-1}2\mathrm{Re}\int\limits_{\Omega_b}\left(\sum_{|\alpha|=1}^{m}a_\alpha(D^\alpha\varphi_t)\overline{\varphi}+a_1\varphi_t\overline{\varphi}\right)dxdt = 0.$$

Since

$$\int\limits_{\Omega_b} a_\alpha(D^\alpha\varphi_t)\overline{\varphi}dxdt = \int\limits_{\Omega_b}\frac{\partial}{\partial t}(a_\alpha(D^\alpha\varphi)\overline{\varphi})dxdt$$
$$-\int\limits_{\Omega_b}\frac{\partial a_\alpha}{\partial t}(D^\alpha\varphi)\overline{\varphi}dxdt - \int\limits_{\Omega_b}a_\alpha(D^\alpha\varphi)\overline{\varphi_t}dxdt$$

and

$$B_1(\varphi,\varphi)(0)+(-1)^m\lambda_1\|\varphi(\cdot,0)\|^2_{L^2(\Omega)}$$
$$(3.6)\qquad = \sum_{|\alpha|,|\beta|=0}^{m}(-1)^{|\alpha|}\int\limits_{\Omega}a_{\alpha\beta}D^\beta\varphi\overline{D^\alpha\varphi}|_{t=0}dx + 2\mathrm{Re}\sum_{|\alpha|=1}^{m}\int\limits_{\Omega}a_\alpha(D^\alpha\varphi)\overline{\varphi}|_{t=0}dx,$$

(3.5) implies

$$\int\limits_{\Omega}|\varphi_t(x,t)|^2dx+(-1)^mB_1(\varphi,\varphi)(0)+\lambda_1\|\varphi(\cdot,0)\|^2_{L^2(\Omega)}$$
$$+(-1)^m2\mathrm{Re}\int\limits_{\Omega_b}a_1\varphi_t\overline{\varphi}dxdt$$
$$(3.7)\qquad -\mathrm{Re}\sum_{|\alpha|,|\beta|=0}^{m}(-1)^{|\alpha|+m-1}\int\limits_{\Omega_b}\frac{\partial a_{\alpha\beta}}{\partial t}D^\beta\varphi\overline{D^\alpha\varphi}dxdt+$$
$$+(-1)^m2\mathrm{Re}\sum_{|\alpha|=1}^{m}\int\limits_{\Omega_b}(\frac{\partial a_\alpha}{\partial t}(D^\alpha\varphi)\overline{\varphi}+a_\alpha D^\alpha\varphi)\overline{\varphi_t})dxdt = 0.$$

Put $v_\alpha(x,t) = \int\limits_t^0 D^\alpha u(x,\tau)d\tau$ for $0<\tau<b$. It is clear that

$$D^\alpha\varphi(x,t) = v_\alpha(x,b)-v_\alpha(x,t).$$

Therefore we have

$$(-1)^m B_1(\varphi,\varphi)(0) + \lambda_1 \|\varphi(x,0)\|^2_{L^2(\Omega)} \geq \mu_1 \sum_{|\alpha|=0}^{m} \int_{\Omega} |v_\alpha(x,b)|^2 dx.$$

From (3.7) we obtain

$$\begin{aligned}&\int_{\Omega} |\varphi_t(x,b)|^2 dx + \mu_1 \sum_{|\alpha|=0}^{m} \int_{\Omega} |v_\alpha(x,b)|^2 dx \\ &\leq C_1 \Big(\sum_{|\alpha|=0}^{m} \int_{\Omega_b} |v_\alpha(x,t)|^2 dxdt + \int_{\Omega_b} |\varphi_t(x,t)|^2 dxdt \Big) \\ &+ C_2 \int_0^b \Big(\sum_{|\alpha|=0}^{m} \int_{\Omega} |v_\alpha(x,b)|^2 dx \Big) dt,\end{aligned}$$

where C_1 and C_2 are positive constants. This is equivalent to

$$\begin{aligned}&\int_{\Omega} (|\varphi_t(x,b)|^2 + (\mu_1 - bC_2) \sum_{|\alpha|=0}^{m} |v_\alpha(x,b)|^2) dx \\ &\leq C_1 \int_{\Omega_b} (|\varphi_t(x,t)|^2 + \sum_{|\alpha|=0}^{m} |v_\alpha(x,t)|^2) dxdt.\end{aligned}$$

Putting $y(t) = \int_{\Omega} (|\varphi_t(x,t)|^2 + \sum_{|\alpha|=0}^{m} |v_\alpha(x,t)|^2) dx$, we obtain $y(b) \leq C \int_0^b y(t)dt$ for almost $b \in (0, \frac{\mu_1}{2C_2}] \subset (0, \frac{\mu_1}{C_2})$. Hence from the Gronwall-Bellman inequality we have $y(b) = 0$ for a.e. $b \in (0, \frac{\mu_1}{2C_2}]$. Consequently, $u(x,b) = 0$ for a.e. $b \in [0, \frac{\mu_1}{2C_2}]$. Using the similar arguments as the above we can prove that $u(x,b) = 0$ for a.e. $b \in [\frac{\mu_1}{2C_2}, \frac{\mu_1}{C_2}]$. By the same procedure, after some steps we obtain $u(x,b) = 0$ for a.e. $b \in [0,T]$. Since T is arbitrary, we obtain $u_1 = u_2$. The proof of the theorem is complete. ∎

Theorem 3.3. (The existence of generalized solution). *Suppose that* $f \in L^\infty(0,\infty; L^2(\Omega))$ *and hypothesis of theorem 3.2 are satisfied.*
Then there exists $\gamma_0 > 0$ *such that for all* $\gamma > \gamma_0$, *problem (2.2)-(2.4) has at least a generalized solution in* $H^{m,1}(e^{-\gamma t}, \Omega_\infty)$. *Moreover*

$$\|u\|_{H^{m,1}(e^{-\gamma t},\Omega_\infty)} \leqslant C\|f\|_{L^\infty(0,\infty;L^2(\Omega))},$$

where C *is a positive constant which is independent of* u *and* f.

Proof. We shall use the Galerkin's approximate method to prove the existence of generalized solutions.

Let $\{\psi_k\} \subset C_0^\infty(\Omega)$ be an orthogonal system in $L^2(\Omega)$ such that its linear closure in $H^m(\Omega)$ coincides with $\overset{\circ}{H}{}^m(\Omega)$. For each natural number N, we consider the function

$$u^N(x,t) = \sum_{k=1}^{N} c_k^N(t)\psi_k(x),$$

where $c_k^N(t)$ are the solution of the system of ordinary differential equations of second order:

$$\begin{aligned} &\int_\Omega \big(u_{tt}^N\overline{\psi_l} + \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} a_{\alpha\beta} D^\beta u^N \overline{D^\alpha \psi_l}\big)dx + a_0 u^N)\overline{\psi_l}dx \\ &+(-1)^m \int_\Omega \Big(\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha u^N + a_0 u^N\Big)\overline{\psi_l}dx \\ &= -\int_\Omega f\overline{\psi_l}dx,\ l=1,2,\ldots,N \end{aligned} \tag{3.8}$$

with the initial conditions

$$c_k^N(0) = \frac{d}{dt}c_k^N(0) = 0. \tag{3.9}$$

Here we use the notations $a_{00} = (-1)^m\lambda_0 I$, $a_0 = a - a_{00}$ and λ_0 is the constant as in Lemma 3.1.

Since (3.8) is a linear system with initial condition (3.9), it has unique solution c_k^N. Moreover, $\frac{d^2c_k^N}{dt^2} \in L^2(0,T)$. Multiplying (3.8) by $\frac{dc_k^N}{dt}$, taking the sum with respect to l and integrating the obtained equality with respect to t on $(0,T]$, we get

$$\begin{aligned} &\int_{\Omega_T} \big(u_{tt}^N\overline{u_t^N} + \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} a_{\alpha\beta} D^\beta u^N \overline{D^\alpha u_t^N}\big)dxdt + \\ &+(-1)^m \int_{\Omega_T} \Big(\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha u^N\overline{u_t^N} + a_0 u^N\overline{u_t^N}\Big)dxdt = -\int_{\Omega_T} f\overline{u_t^N}dxdt. \end{aligned}$$

Adding this equality to its complex conjugate and integrating by part we obtain

$$\begin{aligned} &\int_\Omega |u_t^N(x,t)|^2dx + \int_\Omega \Big(\sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} a_{\alpha\beta}(x,t) D^\beta u^N \overline{D^\alpha u^N}\Big)dx \\ &+(-1)^m 2\mathrm{Re}\int_{\Omega_T} \Big(\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha u^N\overline{u_t^N} + a_0 u^N\overline{u_t^N}\Big)dxdt \\ &-\mathrm{Re}\int_{\Omega_T}\Big(\sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} \frac{\partial a_{\alpha\beta}}{\partial t} D^\beta u^N \overline{D^\alpha u^N}\Big)dxdt = -2\mathrm{Re}\int_{\Omega_T} f\overline{u_t^N}dxdt. \end{aligned} \tag{3.10}$$

By Lemma 3.1 we have

$$\int_\Omega \Big(\sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} a_{\alpha\beta}(x,t) D^\beta u^N \overline{D^\alpha u^N} \Big) dx$$
$$= (-1)^m B(u^N, u^N)(t) + \lambda_0 \|u^N(\cdot,t)\|^2_{L^2(\Omega)}$$
$$\geq \mu_0 \|u^N(\cdot,t)\|^2_{H^m(\Omega)}.$$

Combing this with (3.10) and using the Causchy inequality yields

$$\begin{aligned}
&\|u_t^N\|^2_{L^2(\Omega)} + \mu_0 \|u^N\|^2_{H^m(\Omega)} \\
&\leq 2\mu \int_{\Omega_T} \big(\sum_{|\alpha|=1}^{m} |D^\alpha u^N||u_t^N| + |u^N||u_t^N| \big) dxdt \\
&\quad + \mu \int_{\Omega_T} \big(\sum_{|\alpha|,|\beta|=0}^{m} |D^\beta u^N||D^\alpha u^N| \big) dxdt + 2 \int_{\Omega_T} |f||u_t^N| dxdt \\
&= 2\mu \int_{\Omega_T} \sum_{\alpha=0}^{m} \frac{1}{\sqrt{\epsilon}} |D^\alpha u^N| \sqrt{\epsilon} |u_t^N| dxdt \\
&\quad + \mu \int_{\Omega_T} \big(\sum_{|\alpha|,|\beta|=0}^{m} |D^\beta u^N||D^\alpha u^N| \big) dxdt + 2 \int_{\Omega_T} \frac{1}{\sqrt{\epsilon_1}} |f| \sqrt{\epsilon_1} |u_t^N| dxdt \\
&\leq \mu \int_{\Omega_T} \big(\sum_{|\alpha|=0}^{m} (\frac{1}{\epsilon} |D^\alpha u^N|^2 + \epsilon |u_t^N|^2 \big) dxdt \\
&\quad + \mu \int_0^T \|u^N\|^2_{H^m(\Omega)} dt + \epsilon_1 \int_{\Omega_T} |u_t^N|^2 dxdt + \frac{1}{\epsilon_1} \int_{\Omega_T} |f|^2 dxdt \\
&= \mu \int_0^T [(\frac{1}{\epsilon} + 1) \|u^N\|^2_{H^m(\Omega)} + (\epsilon m^* + \frac{\epsilon_1}{\mu}) \|u_t^N\|^2_{L^2(\Omega)}] dt \\
&\quad + \frac{1}{\epsilon_1} \int_{\Omega_T} |f|^2 dxdt,
\end{aligned} \tag{3.11}$$

where $m^* = \sum_{|\alpha|=0}^{m} 1$, $\epsilon > 0$ and $\epsilon_1 > 0$. This implies that

$$\|u_t^N\|^2_{L^2(\Omega)} + \mu_0\|u^N\|^2_{H^m(\Omega)}$$

$$\leq (\epsilon m^*\mu + \epsilon_1)\int_0^T \big(\|u_t^N\|^2_{L^2(\Omega)} + \frac{\mu(1+\epsilon)}{\epsilon(\mu\epsilon m^* + \epsilon_1)}\|u^N\|^2_{H^m(\Omega)}\big)dt \tag{3.12}$$

$$+ \frac{T}{\epsilon_1}\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

Choosing ϵ such that $\frac{\mu(1+\epsilon)}{\epsilon(\mu\epsilon m^*+\epsilon_1)} = \mu_0$, we get

$$\epsilon = \frac{\mu - \epsilon_1\mu_0 + \sqrt{(\epsilon_1\mu_0 - \mu)^2 + 4\mu^2\mu_0 m^*}}{2\mu\mu_0 m^*}.$$

Put

$$J_N(t) = \|u_t^N(x,t)\|^2_{L^2(\Omega)} + \mu_0\|u^N(x,t)\|^2_{H^m(\Omega)}.$$

From (3.12) we have

$$J_N(t) \leq (\epsilon m^*\mu + \epsilon_1)\int_0^T J_N(t)dt + \frac{T}{\epsilon_1}\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

From this and the Gronwal- Bellman inequality, we obtain

$$\|u_t^N(x,t)\|^2_{L^2(\Omega)} + \mu_0\|u^N(x,t)\|^2_{H^m(\Omega)}$$

$$\leq \frac{C}{\epsilon_1}\exp((\epsilon m^*\mu + \epsilon_1)T)\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))} \tag{3.13}$$

If $\mu_0 \geq 1$ then (3.13) implies

$$\|u_t^N(x,t)\|^2_{L^2(\Omega)} + \|u^N(x,t)\|^2_{H^m(\Omega)} \leq \frac{C}{\epsilon_1}\exp((\epsilon m^*\mu + \epsilon_1)T)\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

If $0 < \mu_0 < 1$ then we have

$$\|u_t^N(x,t)\|^2_{L^2(\Omega)} + \|u^N(x,t)\|^2_{H^m(\Omega)} \leq \frac{C}{\mu_0\epsilon_1}\exp((\epsilon m^*\mu + \epsilon_1)T)\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

Thus there exist a constant $\overline{C} > 0$ such that

$$\|u_t^N(x,t)\|^2_{L^2(\Omega)} + \|u^N(x,t)\|^2_{H^m(\Omega)} \leq \overline{C}\exp((\epsilon m^*\mu + \epsilon_1)T)\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

Put $\gamma(\epsilon_1) = \frac{\epsilon m^*\mu+\epsilon_1}{2}$. Multiplying both sides of the later inequality by $e^{-\gamma(\epsilon_1)T}$ and integrating with respect to $t \in (0,\infty)$, we have

$$\|u^N\|^2_{H^{m,1}(e^{-\gamma(\epsilon_1)T},\Omega_\infty)} \leq C\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))},$$

where C depends on μ_0 and ϵ_1. Since

$$\gamma'(\epsilon_1) = \frac{\sqrt{(\epsilon_1\mu_0 - \mu)^2 + 4\mu^2\mu_0 m^*} + \epsilon_1\mu_0 - \mu}{4\sqrt{(\epsilon_1\mu_0 - \mu)^2 + 4\mu^2\mu_0 m^*}} > 0,$$

we have $\inf\limits_{\epsilon>0} \gamma(\epsilon_1) = \gamma_0 := \frac{\mu(1+\sqrt{1+4\mu_0 m^*})}{4\mu_0}$. Hence for all $\gamma > \gamma_0$, there exists $\epsilon_1 > 0$ such that $\gamma > \gamma(\epsilon_1) > \gamma_0$. Consequently,

$$\|u^N\|^2_{H^{m,1}(e^{-\gamma t},\Omega_\infty)} \leq C\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}, \tag{3.14}$$

where C depends on γ and μ_0. Thus we have shown that $\{u^N\}$ is bounded in $\overset{0}{H}{}^{m,1}(e^{-\gamma t}, \Omega_\infty)$. By extracting further subsequence, we can assume that $u^N \rightharpoonup u$ weakly. Since $u^N \in \overset{\circ}{H}{}^{m,1}(e^{-\gamma t}; \Omega_\infty)$ and $u(x, 0) = 0$ on Ω, $u \in \overset{\circ}{H}{}^{m,1}(e^{-\gamma t}; \Omega_\infty)$.

It remains to show that u is a generalized solution of the problem. In fact, we define the set M_N by

$$M_N = \{\varphi = \sum_{l=1}^{N} d_l\psi_l : d_l \in H^1(0, T), d_l(T) = 0\}.$$

Taking any $\varphi \in M_N$, we have $\varphi = \sum\limits_{l=1}^{N} d_l\psi_l$. By multiplying (3.8) by d_l, taking the sum with respect to l and integrating in $t \in (0, T)$, we obtain

$$\begin{aligned}&\int\limits_{\Omega_T} u^N_{tt}\overline{\varphi}dxdt + (-1)^m \int\limits_{\Omega_T} \Big(\sum_{|\alpha|,|\beta|=1}^{m} (-1)^{|\alpha|} a_{\alpha\beta} D^\beta u^N \overline{D^\alpha\varphi} \\ &+ \sum_{|\alpha|=1}^{m} a_\alpha D^\alpha u^N \overline{\varphi} + a u^N\overline{\varphi}\Big)dxdt = -\int\limits_{\Omega_T} f\overline{\varphi}dxdt.\end{aligned} \tag{3.15}$$

Integrating by part with respective to t in the first term of (3.15) yields

$$\begin{aligned}&(-1)^{m-1}\int\limits_{\Omega_T}\Big(\sum_{|\alpha|,|\beta|=1}^{m} (-1)^{|\alpha|} a_{\alpha\beta} D^\beta u^N \overline{D^\alpha\varphi} + \sum_{|\alpha|=1}^{m} a_\alpha D^\alpha u^N \overline{\varphi} + a u^N\overline{\varphi})dxdt \\ &+ \int\limits_{\Omega_T} u^N_t \overline{\varphi}_t)dxdt = \int\limits_{\Omega_T} f\overline{\varphi}dxdt.\end{aligned}$$

By letting $N \to \infty$, we obtain

$$\begin{aligned}&(-1)^{m-1}\int\limits_{\Omega_T}\Big(\sum_{|\alpha|,|\beta|=1}^{m} (-1)^{|\alpha|} a_{\alpha\beta} D^\beta u \overline{D^\alpha\varphi} + \sum_{|\alpha|=1}^{m} a_\alpha D^\alpha u \overline{\varphi} + a u\overline{\varphi})dxdt \\ &+ \int\limits_{\Omega_T} u_t \overline{\varphi}_t)dxdt = \int\limits_{\Omega_T} f\overline{\varphi}dxdt.\end{aligned} \tag{3.16}$$

It is noted that the set $M := \bigcup_{N=1}^{\infty} M_N$ is dense in the space of functions $\varphi \in \overset{\circ}{H}{}^{m,1}(\Omega_T)$, $\varphi(x,T)=0$. Therefore (3.16) is valid for all $\varphi \in \overset{\circ}{H}{}^{m,1}(\Omega_T)$ satisfying $\varphi(x,T)=0$. This implies that u is a generalized solution of (2.2)-(2.4). Moreover, from (3.14) we have

$$\|u\|^2_{H^{m,1}(e^{-\gamma t},\Omega_\infty)} \le \liminf_{N\to\infty} \|u^N\|^2_{H^{m,1}(e^{-\gamma t},\Omega_\infty)} \le C\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

The proof is complete. ■

4. Applications

In this section we will apply the previous results to the study of generalized solution existence of elasticity problems.

Let us assume that u and f be real vector functions which defined on $\Omega \subset R^n$. Consider the differential operators:

$$L_s(x,t,D)(u) = \sum_{j,h,k=1}^{n} \frac{\partial}{\partial x_h}\left(a^{jk}_{sh}(x,t)\frac{\partial u^j}{\partial x_k}\right),\ s=1,...,n \tag{4.1}$$

where a^{jh}_{sh} are continuous real functions on $\overline{\Omega}$ satisfying condition

$$a^{jk}_{sh} = a^{jk}_{hs} = a^{sh}_{jk}. \tag{4.2}$$

Put

$$e_{sh} = \frac{1}{2}\left(\frac{\partial u_s}{\partial x_h} + \frac{\partial u_h}{\partial x_s}\right) \tag{4.3}$$

and

$$W(x,t,e) = \sum_{s,h,j,k=1}^{n} a^{jk}_{sh} e_{sh} e_{jk}. \tag{4.4}$$

Assume that $W(x,t,e)$ is an elastic potential which is a positive definite quadratic form with respect to e_{sh}, $1 \le s \le h \le n$ for every $(x,t) \in \overline{\Omega}_\infty$.

According to [4], for any $u(x,t)$ satisfying $u(\cdot,t) \in H^1(\Omega)$ for $t \in (0,\infty)$ we have

$$\sum_{s,h=1}^{n} \int_\Omega \left(\frac{\partial u_s}{\partial x_h} + \frac{\partial u_h}{\partial x_s}\right)^2 dx \ge C\|u(x,t)\|^2_{H^1(\Omega)},$$

where C is a positive constant which is independent of u and t. From this inequality we obtain

$$\sum_{s,h,j,k=1}^{n} \int_{\Omega} a_{sh}^{jk} \frac{\partial u_s}{\partial x_h} \frac{\partial u_j}{\partial x_k} \geq C\|u(x,t)\|_{H^1(\Omega)}^2, \tag{4.5}$$

where C is a positive constant which is independent of u and t. It follows that the elastic potential satisfies the inequality (3.1) with $\lambda_0 = 0$.

We now consider the following problem in Ω_∞.

$$L_s(x,t,D)u - u_{tt} = f(x,t), s = 1,2,...,n \tag{4.6}$$

with the initial conditions

$$u\,|_{t=0} = u_t\,|_{t=0} = 0 \tag{4.7}$$

and boundary condition

$$u\,|_{S_\infty} = 0. \tag{4.8}$$

From Theorem 3.1, Theorem 3.2 and (4.5) we obtain the following result.

Theorem 4.1. *Assume that $W(x,t,e)$ is a positive definite quadratic form with respect to the variable e_{sh}, $1 \leq s \leq h \leq n$, for all $(x,t) \in \overline{\Omega}_\infty$. Assume further the following conditions:*

(*i*) $|\partial a_{sh}^{jk}/\partial t| \leq \mu_2, \ (x,t) \in \overline{\Omega}_\infty$
(*ii*) $f \in L^\infty(0,\infty; L^2(\Omega))$.

Then there exists $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$ problem (4.6)-(4.8) has a unique generalized solution $u(x,t)$ in the space $H^{1,1}(e^{-\gamma t}, \Omega_\infty)$. Moreover

$$\|u\|_{H^{1,1}(e^{-\gamma t},\Omega_\infty)} \leq C\|f\|_{L^\infty(0,\infty;L^2(\Omega))}$$

where C is independent of u and f.

Now we study the differential operator of the Lame's type

$$\mu\Delta + (\lambda+\mu)\mathrm{grad}(\mathrm{div})$$

where Δ is the Laplace operator, $\mu > 0$, λ and $\mu + \lambda > 0$. Consider the following problem in Ω_∞.

$$\mu\Delta u + (\lambda+\mu)\mathrm{grad}(\mathrm{div} u) - u_{tt} = f(x,t) \tag{4.9}$$

$$u\,|_{t=0} = u_t\,|_{t=0} = 0 \tag{4.10}$$

$$u\,|_{S_\infty} = 0. \tag{4.11}$$

Put $a_{ss}^{ss} = \lambda + \mu$, $a_{sh}^{hs} = a_{hs}^{sh} = \mu$ with $s \neq h$, $a_j j^{ss} = 0$ with $s \neq j$ and $a_{jk}^{hs} = 0$ with $jk \neq sh$. Hence we have

$$\begin{aligned} 2W(x,t,e) &= \sum_{s,h,j,k=1}^{n} a_{sh}^{jk} e_{sh} e_{jk} \\ &= (2\mu + \lambda)\sum_{s=1}^{n} e_{ss}^2 + \lambda \sum_{h\neq s}^{1,n} e_{ss}e_{hh} + \mu \sum_{h\neq s}^{1,n} e_{hs}^2 \\ &= 2\mu \sum_{s=1}^{n} e_{ss}^2 + \lambda(\sum_{s=1}^{n} e_{ss})^2 + \mu \sum_{h\neq s}^{1,n} e_{hs}^2. \end{aligned} \tag{4.12}$$

Since $\mu > 0$ and $\mu + \lambda > 0$ we can find $\epsilon > 0$ such that $\mu - \epsilon > 0$ and $\mu + \lambda - \epsilon > 0$. Hence (4.12) implies

$$\begin{aligned} 2W(x,t,e) &= 2\epsilon \sum_{s=1}^{n} e_{ss}^2 + 2(\mu - \epsilon)\sum_{s=1}^{n} e_{ss}^2 + \lambda(\sum_{s=1}^{n} e_{ss})^2 + \mu \sum_{h\neq s}^{1,n} e_{hs}^2 \\ &\geq 2\epsilon \sum_{s=1}^{n} e_{ss}^2 + \mu \sum_{h\neq s}^{1,n} e_{hs}^2 + \frac{2}{n}(\mu - \epsilon)(\sum_{s=1}^{n} e_{ss})^2 + \lambda(\sum_{s=1}^{n} e_{ss})^2 \\ &= 2\epsilon \sum_{s=1}^{n} e_{ss}^2 + \mu \sum_{h\neq s}^{1,n} e_{hs}^2 + (\frac{2}{n}(\mu - \epsilon) + \lambda)(\sum_{s=1}^{n} e_{ss})^2. \end{aligned} \tag{4.13}$$

This implies that b

$$W(x,t,e) \geq \mu_0 \sum_{s,h=1}^{n} e_{sh}^2 \tag{4.14}$$

where μ_0 is a positive constant. Therefore system (4.9) satisfies conditions of Theorem 3.1 and Theorem 3.2, where we can choose $\mu_2 = 0$. From Theorem 3.1 and Theorem 3.2 we obtain

Theorem 4.2. *Let f be a function which belongs to $L^\infty(0,\infty;L^2(\Omega))$. Then for each $\epsilon > 0$ problem (4.9)-(4.11) has a unique generalized solution $u = u(x,t)$ in $H^{1,1}(e^{-\epsilon t},\Omega_\infty)$. Moreover,*

$$\|u\|_{H^{1,1}(e^{-\epsilon t},\Omega_\infty)} \leq C\|f\|_{L^\infty(0,\infty;L^2(\Omega))},$$

where C is a positive constant which is independent of u and t.

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