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THE DENSITY OF ALGEBRAIC ELEMENTS IN C^*-ALGEBRAS

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Abstract. In this paper we study some problems in C^*-algebra using elementary algebraic method. Our intention is to demonstrate that in many occasions algebraic properties and techniques may be the natural choice.

1. Introduction

In recent years there have been significant developments in the subject of classifying the algebraic elements in C^*-algebra. An element a in a C^*-algebra $\mathcal{A}$ *algebraic element* if there is a complex polynomial $p(x)$ such that $p(a) = 0$. We shall begin by introducing the following terminologies:

Definitions. We say a C^*-algebra $\mathcal{A}$ has

(F.1) $''F''$ property: if every element of $\mathcal{A}$ can be approximated in norm by elements in $\mathcal{A}$ with finite spectrum.

(F.2) $''FS''$ property: if every self-adjoint element of $\mathcal{A}$ can be approximated in norm by self-adjoint elements in $\mathcal{A}$ with finite spectrum.

(F.3) $''FN''$ property: if every normal element of $\mathcal{A}$ can be approximated in norm by normal elements in $\mathcal{A}$ with finite spectrum.

(A.1) $''A''$ property: if every element of $\mathcal{A}$ can be approximated in norm by the algebraic elements in $\mathcal{A}$.

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(A.2) $''AS''$ property: if every self adjoint element of $\mathcal{A}$ can be approximated in norm by self-adjoint algebraic elements in $\mathcal{A}$.

(A.3) $''AN''$ property: if every normal element of $\mathcal{A}$ can be approximated in norm by normal algebraic elements in $\mathcal{A}$.

(I.1) $''I''$ property: if every element of $\mathcal{A}$ can be approximated in norm by invertible elements in $\mathcal{A}$.

(I.2) $''IS''$ property: if every self-adjoint element of $\mathcal{A}$ can be approximated in norm by invertible self-adjoint elements in $\mathcal{A}$.

(I.3) $''IN''$ property: if every normal element of $\mathcal{A}$ can be approximated in norm by invertible normal elements in $\mathcal{A}$.

Here we point out that the IS property (resp. I property) is also known as the property of real rank zero (resp. stable rank one). In fact, in [1], Brown and Pedersen prove that the IS property is equivalent to the FS property.

Let us demonstrate that by considering algebraic techniques, argument for proving some results may turn out to be more straightforward than by considering some classical ones. For example, the Borel functional calculus is considered as a very powerful tool in the study of C^* algebras. For instance, it can be used to show that every self-adjoint element (resp. normal element) in a C^*-algebra can be approximated in norm by real (resp. complex) linear combination of orthogonal projections, provided that $\mathcal{A}$ has the IS property (resp., the IN property). However, we propose that the density of algebraic elements in $\mathcal{A}$ might even be a more appropriate property for the application. Indeed, if a C^*-algebra $\mathcal{A}$ has A property, then every element in $\mathcal{A}$ can be approximated by linear combination of orthogonal idempotents, a property that has little to do with either I or F.

In fact, let us show that the "A" properties might be a better choice then the "I" properties in the following example: We shall begin by showing that $FS \Leftrightarrow AS$ and $FN \Leftrightarrow AN$, while $FS \Leftrightarrow IS$ (by [1]), $FN \Rightarrow IN$ but $IN \nRightarrow FN$. For this purpose, we need

Lemma 1.1. *Let a be a normal element in $\mathcal{A}$. Then a is algebraic if and only if the spectrum of a is finite.*

Proof. Although the conclusion of the lemma is an easy consequence of either the theory of the Gelfand transform on commutative C^*-algebras or the spectral theory for normal elements, we prefer here an algebraic approach. It suffices to show that if a is normal, then a is algebraic if and only if a is in some finite dimensional C^*-subalgebra of $\mathcal{A}$ since in this case $\sigma(a)$ is finite by the Caley-Hamilton theorem. If a is in some finite dimensional C^*-subalgebra of $\mathcal{A}$ then a is of course algebraic. On the other hand, suppose that a is algebraic and $p(a) = 0$ for some polynomial p of degree n. Then $\overline{p}(a^*) = 0$ and the degree of $\overline{p}$ is also n. Hence a is in the (commutative) C^*-subalgebra $C^*(a)$ of $\mathcal{A}$ with dimension at most $n^2 + 1$. ∎

Remark. It is easy to see that the conclusion of Lemma 1.1 will no longer hold if the normality is absent. Indeed, the non-normal element

$$J_\lambda^{(1)} \oplus J_\lambda^{(2)} \oplus \cdots \oplus J_\lambda^{(n)} \oplus \cdots,$$

where $J_\lambda^{(n)}$ is the $n \times n$ Jordan block with λ in the diagonal entries (with norm not exceeding $\sqrt{|\lambda|^2 + 3}$), is not algebraic, but with spectrum $\{\lambda\}$. In general, the A property is stronger than either the F or I property. It is evident that $A \Rightarrow F$. Now suppose that $p(x)$ is a polynomial with $\deg p = n$ such that $p(a) = 0$ for a in some C^*-algebra $\mathcal{A}$. Let $\sigma(a) = \{\lambda_1, \cdots, \lambda_k\}$, $k \le n$ be the spectrum of a. Then either a itself is invertible (i.e., $\lambda_i \neq 0$, $\forall i$) or otherwise (say, $\lambda_1 = 0$) a can be approximated by invertible elements of the form

$$b = (a + \varepsilon)e_0 + a \sum_{i=2}^{k} e_{\lambda_i},\ e_{\lambda_i} = \int_{\Gamma_i} (\lambda - a)^{-1} d\lambda,\ \varepsilon > 0,$$

where $\Gamma_1, \cdots, \Gamma_k$ are mutually disjoint (small) circles around $\lambda_1, \cdots, \lambda_k$, respectively. Note that b is also algebraic since a is. It follows that $A \Rightarrow I$ and $A \Rightarrow IN$ since b is normal if a is.

By Lemma 1.1 $FS \Leftrightarrow AS$ and $FN \Leftrightarrow AN$ follow immediately. On the other hand, beside the fact that $FS \Leftrightarrow IS$, the implication $FN \Rightarrow IN$ follows from a recent result by Friis [3]: A C^*-algebra has the FN property if and only if it is of real rank zero, has the IN property, and that its unitary group is connected (i.e., $K_1(A) = 0$). Finally, since every simple pure infinite C^*-algebra has the IN property (see [6], [7] and [8]), we conclude that $IN \not\Rightarrow FN$ since the Calkin algebra is simple pure infinite but obviously is without the FN property because its K_1 group is $\mathbb{Z}$.

In general, the connection between the properties listed in the beginning of this section is illustrated in the following diagram:

$$\begin{array}{ccccccc}
F & \Leftarrow & A & \overset{\not\Leftarrow}{\Rightarrow} & I & \text{(Stable rank one)} & \\
\not\Uparrow & & \not\Uparrow & & \not\Uparrow\not\Downarrow & & \\
FS & \Leftrightarrow & AS & \Leftrightarrow & IS & \text{(Real rank zero)} & (\textbf{diagram } 1) \\
\not\Downarrow\Uparrow & & \Uparrow\not\Downarrow & & \not\Uparrow & & \\
FN & \Leftrightarrow & AN & \overset{\not\Leftarrow}{\Rightarrow} & IN & &
\end{array}$$

Remark. If a is self-adjoint in $\mathcal{A}$ and $\|a_n - a\| \to 0$ for some normal elements $\{a_n\}$ in $\mathcal{A}$ with finite spectrum, then each a_n is algebraic (by Lemma 1.1) and therefore a can be approximated by $\{\frac{a_n+a_n^*}{2}\}$, a sequence of algebraic self-adjoint elements. So $FN \Rightarrow AS$.

On the other hand, it is well-known that $C[0,2]$ is of stable rank one but not of real rank zero. In fact, the only algebraic elements in $C[0,2]$ are the constant functions. So $C[0,2]$ is not algebraic dense. However, it is evident that $C[0,2]$ has the IN property but has no IS property (since the function $f(x) = x - 1$, for example, can not be approximated by sequences of real-valued invertible continuous functions on $[0,2]$). Hence it follows that $I \not\Rightarrow IS$, $I \not\Rightarrow A$ and $IN \not\Rightarrow IS$. The fact that $FS \not\Rightarrow FN$ again is a consequence of [3], since the Calkin algebra does not have the FN property.

2. Some Results in C^*-Algebra from the Algebraic Point of View

Recalling that a separable C^*-algebra $\mathcal{A}$ is an AF-algebra if for every $\varepsilon > 0$ and for every finite subset $\{a_1, a_2, \cdots, a_n\}$ of $\mathcal{A}$, there exist a finite dimensional C^*algebra $\mathcal{B}$ of $\mathcal{A}$ and elements $b_1, b_2, \cdots, b_n$ in $\mathcal{B}$ such that $\|a_j - b_j\| < \varepsilon$ for each j. Thus the very definition of AF-algebra implies that every AF-algebra is algebraic dense. In this subject, one question is raised by L. G. Brown and G. K Pedersen: Does every AF-algebra have FN? A more convenient way to study the problem is: It is well-known that an element x is normal if its real and imaginary parts commute. Now let $x = a + \imath b$, and suppose that $a_n \to a$, and $b_n \to b$, where a_n, b_n are self-adjoint elements with finite spectrum. Since $\|a_n b_n - b_n a_n\| \to 0$, the question is now equivalent to the question of whether the a_n and b_n can be replaced by c_n and d_n respectively such that c_n and d_n are commuting self-adjoint elements with finite spectrum. H. Lin gave an answer in the positive to this question:

Theorem 2.1. (H. Lin) [5]. *For every $\varepsilon > 0$, there is a $\delta > 0$ such that for any n and any pair of self-adjoint matrices $a, b \in M_n(\mathbb{C})$ such that $\|a\|, \|b\| \leq 1$ and $\| ab - ba \| < \delta$ there is a commuting pair of self-adjoint matrices $a', b' \in M_n(\mathbb{C})$ such that $\| a - a' \| + \| b - b' \| < \varepsilon$.*

Note that Theorem 2.1 may be rephrased as follows: Let $\{x_k\} \subseteq M_{n_k}(\mathbb{C})$ such that $\| x_k \| \leq 1$ and $\| x_k x_k^* - x_k^* x_k \| \to 0$, then there is a sequence of normal matrices $\{y_k\} \subseteq M_{n_k}(\mathbb{C})$ such that $\| x_k - y_k \| \to 0$.

Now let us establish the connection between Lin's theorem and the FN property. Define

$$\mu = \{(a_k) : a_k \in M_{n_k}(\mathbb{C}),\ \sup \| a_k \| < \infty\}$$
$$\wp = \{(a_k) : a_k \in M_{n_k}(\mathbb{C}),\ \lim \| a_k \| = 0\}.$$

Note that $\wp$ is a closed two-sided ideal of μ. So the conclusion of Lin's Theorem is precisely the statement that there is a normal element $y = (y_k) \in \mu$ such that it satisfies $\pi(y) = \pi(x)$, where $\pi : \mu \to \mu/\wp$ is the canonical quotient map. In other words Lin's theorem is equivalently to the fact that every normal element in $\mu/\wp$ lifts to a normal element of μ.

In recent years, the lifting problem for algebraic elements in C^*-algebras have received considerable attention from mathematicians in the area of operator algebra. For example, Olsen and Pedersen shows that if $f(x) = x^n$ then lifting exists, i.e., if I is a closed two sided ideal of $\mathcal{A}, b \in \mathcal{A}/I$ with $b^n = I$ then there is an $a \in \mathcal{A}$ such that $\pi(a) = b$ and $a^n = 0$. On the other hand, Brown and Pedersen show that if $\mathcal{A}$ is of real rank zero then lifting exists for $f(x) = x^2 - x$. At this point we do not know whether the algebraic methods will, in the future, have a fundamental impact on the lifting problems or not. However, some known results can be easily extended by simple algebraic argument. For example:

Proposition 2.2. *Let I be a closed two-sided ideal of $\mathcal{A}$ such that the self-adjoint algebraic elements in $\mathcal{A}/I$ can be lifted to self-adjoint algebraic elements in $\mathcal{A}$, then the normal algebraic elements in $\mathcal{A}/I$ can also be lifted to normal algebraic elements in $\mathcal{A}$.*

Proof. Let $\bar{a} \in \mathcal{A}/I$ be a normal algebraic element and let $\sigma(\bar{a}) = \{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_i \neq \lambda_j$ if $i \neq j$. Define

$$p_1(x) = \begin{cases} x - \lambda_1 + \mu & \text{if } n = 1, \text{ where } \mu > 0 \\ \displaystyle\sum_{i=1}^{n} \mu_i \frac{\prod_{j\neq i}(x - \lambda_j)}{\prod_{j\neq i}(\lambda_i - \lambda_j)} & \text{if } n > 1, \text{ where } \mu_1 < \mu_2 < \cdots < \mu_n \end{cases}.$$

Observe that $p_1(\bar{a})$ is a self-adjoint algebraic element in $\mathcal{A}/I$. So by the hypothesis there is a self-adjoint algebraic element a in $\mathcal{A}$ such that $\pi(a) = p_1(\bar{a})$. Now set

$$p_2(x) = \begin{cases} x - \mu + \lambda_1 & \text{if } n = 1 \\ \displaystyle\sum_{i=1}^{n} \lambda_i \frac{\prod_{j\neq i}(x - \mu_j)}{\prod_{j\neq i}(\mu_i - \mu_j)} & \text{if } n > 1 \end{cases}.$$

We have $p_2 \circ p_1(\bar{a}) = \bar{a}$ and $p_2(a)$ is a normal algebraic element such that $\pi(p_2(a)) = \bar{a}$. ■

Combining Proposition 2.2 and a result in [1] on the lifting property of the self-adjoint elements in C^*-algebras of real rank zero, we immediately obtain

Corollary 2.3. *For every closed ideal I in a real rank zero C^*-algebra $\mathcal{A}$ the normal algebraic elements in $\mathcal{A}/I$ lift to normal algebraic elements in $\mathcal{A}$.*

Combining Proposition 2.2 with a result by Hadwin [2], we have

Corollary 2.4. *Let $\mathcal{A} = \Pi\wp_n$ be the C^*-algebra of all bounded sequences in the cartesian product of a sequence of unital C^*-algebras $\{\wp_n\}$ with coordinatewise multiplication equipped with the supremum norm and $\sum \wp_n$ be the two-sided closed ideal of null sequences in $\Pi\wp_n$, then every normal algebraic elements in $\mathcal{A}/I$ can be lifted.*

Since $AS \Leftrightarrow FS$ and $AN \Leftrightarrow FN$, we would like to, at the end, discuss a topic related to the above discussion: Does the fact that both I and A/I having the FN property implies that $\mathcal{A}$ has the FN property? It turns out that one of the important parts in Friis's characterization for the FN property plays a crucial role here, i.e., a criterion for the IN property: A unital C^*-algebra has the IN property if and only if its normal elements can be approximated by its invertible elements (Corollary 2.3.3, [3]). This means that $I \Rightarrow IN$ and the following is an easy application of this fact:

Theorem 2.5. *Let $\mathcal{A}$ be a unital C^*-algebra of both real rank zero and stable rank one. Suppose that both I and $\mathcal{A}/I$ have the FN property, then $\mathcal{A}$ also has the FN property.*

Proof. First, since $\mathcal{A}$ is unital and is of stable rank one, any element, especially any normal element, in $\mathcal{A}$, can be approximated by invertible elements in $\mathcal{A}$. Therefore $\mathcal{A}$ has the IN property.

Next, since $0 \to I \to \mathcal{A} \to \mathcal{A}/I \to 0$ is short exact, we conclude that $K_1(\mathcal{A}) = 0$, since $K_1(I) = K_1(\mathcal{A}/I) = 0$ and K_1 preserves the exactness. This means that the unitary group of $\mathcal{A}$ is connected. Now the conclusion of the theorem follows immediately from Friis's characterization of the FN property. ■

The above conclusion, however, fails if the FN property is replaced by the weak FN property: A C^*-algebra $\mathcal{A}$ has the weak FN property if and only if whenever x is normal with $\lambda - x \in (InvA)_0$ (the component of 1 in the invertible group), $\forall\lambda \notin \sigma(x)$, there is a sequence of normal elements $x_n \in \mathcal{A}$ with finite spectrum such that $x = \lim_{n\to\infty} x_n$ (see [6]).

Example. The following example is proposed by Loring (see [6]). Let $\mathcal{A}$ be the Calkin algebra. Consider $\mathcal{B} = \mathcal{A} \oplus \mathcal{A}$, and choose an unitary $u \in \mathcal{A}$ such that $[u] \neq 0$ in $K_1(\mathcal{A})$ and a normal element $z \in \mathcal{A}$ with its spectrum being the closed unit disk $\overline{D}$. Then $x = z \oplus u$ is a normal element in $\mathcal{B}$ with $\sigma(x) = D$. So for any

$\lambda > 1, \lambda - x \in (InvB)_0$. But it can be shown [6] that x can not be approximated by normal element with finite spectrum, and hence $\mathcal{B}$ does not have the weak FN property. However, if we take $I = \mathcal{A}$, then

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{A} = \mathcal{A}/I \to 0,$$

and Calkin algebra has weak FN property (see [6]).

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