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SUBOPTIMALITY CONDITIONS FOR MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS

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Dedicated to Professor Phan Quoc Khanh

Abstract. In this paper we study mathematical programs with equilibrium constraints (MPECs) described by generalized equations in the extended form

$$0 \in G(x, y) + Q(x, y),$$

where both mappings G and Q are set-valued. Such models arise, in particular, from certain optimization-related problems governed by variational inequalities and first-order optimality conditions in nondifferentiable programming. We establish new *weak* and *strong suboptimality* conditions for the general MPEC problems under consideration in finite-dimensional and infinite-dimensional spaces that do *not assume the existence of optimal solutions*. This issue is particularly important for infinite-dimensional optimization problems, where the existence of optimal solutions requires quite restrictive assumptions. Our techniques are mainly based on modern tools of variational analysis and generalized differentiation revolving around the fundamental *extremal principle* in variational analysis and its analytic counterpart known as the *subdifferential variational principle.*

1. INTRODUCTION

This paper concerns the study of a broad and important class of parametric optimization problems unified under the name of *Mathematical Programs with Equilibrium Constraints* (MPECs) that can be generally described as follows:

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$$
\left\{
\begin{array}{ll}
\text{minimize} & \varphi_0(x,y)\\
\text{subject to} & \varphi_i(x,y)\le 0,\ i=1,\ldots,m,\\
 & \varphi_i(x,y)=0,\ i=m+1,\ldots,m+r,\\
 & (x,y)\in\Omega,\\
 & y\in S(x),
\end{array}
\right.
\tag{1.1}
$$

where $y \in Y$ stands for the decision variable while $x \in X$ signifies the parameter, which is also included into the optimization process. The most characteristic feature of problems (1.1) is that, together with more conventional functional constraints of the equality and inequality types defined by (extended-)real-valued functions φ_i as well as geometric constraints given by sets Ω, they contain parameterized constraints in the form $y \in S(x)$ described by set-valued mappings $S\colon X \rightrightarrows Y$. The latter constraints often arise as solution maps to lower-level parametric optimization problems (as in bilevel programming), or sets of Lagrange multipliers/Karush-Kuhn-Tucker vectors in first-order optimality conditions, or solution sets to various complementarity problems and variational/hemivariational/quasivariational inequalities, etc. In general, constraints of this type describe certain *equilibria*; that's where the name comes from. In numerous publications (see, e.g., books [15, 17, 22] and the references therein) the reader can find more examples, discussions, and various qualitative and numerical results for particular classes of MPECs written in form (1.1) with underlying specifications of equilibrium constraint mappings S.

It has been well recognized that a convenient model for describing equilibrium constraints in MPECs is provided by Robinson's framework of *generalized equations*

$$S(x) = \big\{ y \in Y \big|\ 0 \in g(x,y) + Q(y) \big\} \tag{1.2}$$

originally introduced in [24] for the case when the set-valued "field" mapping $Q\colon Y \rightrightarrows Y^*$ is parameter-independent and is given as the normal cone mapping $Q(y) = N(y;\Theta)$ to a *convex* set $\Theta \subset Y$, while the "base" parameter-dependent mapping $g\colon X \times Y \to Y^*$ is single-valued. This particularly covers the classical *variational inequalities* and *complementarity* problems. Other important equilibrium models (e.g., *quasivariational inequalities*) admit adequate descriptions in somewhat more general framework of type (1.2) with *parameter-dependent fields* $Q = Q(x,y)$; see, e.g., [12, 19] and the references therein.

However, there are broad classes of MPECs (1.1) whose equilibrium constraints cannot be described in form (1.2) while require the *extended generalized equation* framework

$$S(x) = \big\{ y \in Y \big|\ 0 \in G(x,y) + Q(x,y) \big\}, \tag{1.3}$$

where *both* base and field mappings are *set-valued*. Let us mention two particular classes of equilibrium constraints that can be written in the extended form (1.3) while not in the previously developed forms of generalized equations.

• Consider the so-called *set-valued/generalized variational inequalities* defined by:

$$\text{find } y \in \Theta \ \text{ such that there is } \ y^* \in G(x,y) \quad \text{with } \ \langle y^*, u-y\rangle \ge 0 \ \text{ for all } \ u \in \Theta, \tag{1.4}$$

where $G\colon X \times Y \rightrightarrows Y^*$; we refer the reader to the handbook [28] for the theory and applications of (1.4) and related problems. It is easy to see that model (1.4) can be written in form (1.3) with $Q(y) = N(y;\Theta)$. The classical case of parameterized variational inequalities corresponds to (1.4) with a single-valued mapping $G = g\colon X \times Y \to Y^*$.

• Consider a parametric problem of *nonsmooth constrained optimization* in the form:

$$\text{minimize } \ \varphi(x,y) \ \text{ subject to } \ y \in \Theta,$$

where $\varphi\colon X \times Y \to \overline{I\!R} := (-\infty, \infty]$ is a lower semicontinuous function and where $\Theta \subset Y$ is a closed set. Then general *first-order necessary optimality conditions* for this problem can be written as

$$0 \in \partial_y \varphi(x,y) + N(y;\Theta) \tag{1.5}$$

via appropriate *subdifferentials* of φ with respect to y and *normal cones* to Θ, where $\partial_y\varphi$ and $N(\cdot;\Theta)$ reduce to the corresponding constructions of convex analysis if $\varphi(x,\cdot)$ and Θ are convex; in the latter case condition (1.5) is known to be *necessary and sufficient* for optimality. We refer the reader to [14] for more discussions and various results on model (1.5), which is obviously a particular case of (1.3) with $G(x,y) = \partial_y\varphi(x,y)$ and $Q(y) = N(y;\Theta)$. Furthermore, MPEC problems with equilibrium constraints of type (1.5) relate to the so-called *optimistic version* of *nondifferentiable bilevel programming*; see [4] for more details and recent results in this direction.

As usual in optimization theory, the mainstream of studying various classes of MPECs consists of deriving *necessary optimality conditions* associated with appropriate notions of *stationarity*; see particularly [1, 2, 7, 15, 17, 19, 22, 29] and the references therein. However, as it is pointed by Young [30], any theory ofnecessary optimality conditions is “naïve” unless the *existence of optimal solutions* is guaranteed. The latter issue is far from being trivial for important classes of MPECs and related problems, especially in infinite-dimensional spaces, imposing rather restrictive requirements on the initial data; see, e.g., [2, 12, 15, 17, 28] for various results and discussions.

On the other hand, there is an *alternative route* in optimization theory and applications, which allows us to avoid difficulties with justifying the existence of optimal solutions while providing an efficient approach to the study of qualitative aspects of optimization and the development of numerical algorithms. This approach is based on deriving *suboptimality conditions* that give "almost necessary conditions" (up to an arbitrary $\varepsilon > 0$) for "almost optimal (suboptimal) solutions", which *automatically exist*.

The first systematic results of this type go back probably to Ekeland's seminal paper [5] being among the strongest motivations to develop his now classical *variational principle*. Based on this principle, it is shown in [5] that, given any $\varepsilon > 0$, there is an ε-minimizer $\overline{x}$ to a *smooth* function $\varphi\colon X \to I\!R$ on a Banach space X that satisfies an *ε-counterpart* of the *Fermat stationary rule*:

$$\big\|\nabla\varphi(\overline{x})\big\| \le \varepsilon. \tag{1.6}$$

Suboptimality conditions of type (1.6) and their appropriate (more involved) extensions and analogs have been further developed for and applied to various kinds of constrained optimization-related problems in [8, 9, 10, 14, 17, 21, 26, 27] and their references, although this direction in optimization theory is somehow *underestimated* and *not sufficiently explored*. In particular, we are not familiar with *any* suboptimality conditions for MPECs.

The primary goal of this paper is to derive *suboptimality conditions for general MPECs* given in the form

$$\begin{cases} \text{minimize} & \varphi_0(x,y) \\ \text{subject to} & \varphi_i(x,y) \le 0,\ i = 1,\dots,m, \\ & \varphi_i(x,y) = 0,\ i = m+1,\dots,m+r, \\ & (x,y) \in \Omega, \\ & 0 \in G(x,y) + Q(x,y), \end{cases} \tag{1.7}$$

which corresponds to (1.1) with the equilibrium constraints $y \in S(x)$ defined by solution maps to the extended generalized equations (1.3). Following the terminology of [17, 21], where suboptimality conditions are derived for mathematical programs with *no* equilibrium constraints, we distinguish between the *two generally independent forms* of suboptimality conditions: weak and strong. The *weak form* of suboptimality conditions holds under very mild assumptions on the initial data involving however *weak* neighborhoods* from the corresponding dual spaces in their formulations. The *strong form* of suboptimality conditions imposes more requirements while provides stronger results with the replacement of weak* neighborhoods by *small dual balls*, i.e., it establishes the underlying estimates in suboptimality conditions in the *norm topology* versus the *weak* topology* of dual spaces. Furthermore, strong suboptimality conditions are expressed via *limiting* normals/subgradients/coderivatives of the initial data instead of *Fréchet-like* constructions in the weak form. The limiting constructions are *robust* and enjoy comprehensive rules of *full calculus* in contrast to the Fréchet ones; see Section 2 for more discussions and references.

Our approach to deriving suboptimality conditions is based on *extremal/ variational principles* of variational analysis whose versions needed in the paper are recalled in Section 2. We significantly modify the scheme developed in [17, 21] to be able to apply it to establishing suboptimality conditions for the MPECs under consideration. As a by-product of the new scheme, we also improve the results obtained in [17, 21] for mathematical programs with *no* equilibrium (just functional) constraints.

The rest of the paper is organized as follows. Section 2 contains some preliminaries from variational analysis and generalized differentiation needed for the formulation and justification of the main results. Section 3 is devoted to deriving weak suboptimality conditions for MPECs (1.7), while in Section 4 we present strong suboptimality conditions for the general MPECs under consideration and discuss some of their specifications.

The notation used throughout the paper is basically standard; see [16, 17, 25]. Recall that $I\!N := \{1, 2, \ldots\}$ and that $I\!B$ and $I\!B^*$ stands, respectively, for the closed unit ball of the space X in question and of its topological dual X^*. Given a nonempty set $\Omega \subset X$, we denote by $\delta(x; \Omega)$ the *indicator function* of Ω equal to 0 if $x \in \Omega$ and ∞ otherwise.

2. Tools of Variational Analysis

For the reader's convenience, we briefly overview in this section some underlying constructions and principles of variational analysis and generalized differentiation widely used in the sequel. We mainly follow the recent book by Mordukhovich [16], where the reader can find all the details and commentaries. The main framework of our study is the Asplund space setting. Thus we assume, unless otherwise stated, that all the spaces under consideration are *Asplund*, i.e., such Banach spaces whose separable subspaces have separable duals. The class of Asplund spaces is sufficiently large particularly including every reflexive Banach space and every Banach space with a separable dual; see, e.g., [16, 23] for more details, discussions, and references. The definitions and properties presented below are adjusted to the case of Asplund spaces; see [16] for their modifications and analogs in more general Banach space settings.

Given a nonempty subset Ω of an Asplund space X, we define the *prenormal/Fréchet normal cone* to Ω at $\overline{x} \in \Omega$ by

$$\widehat{N}(\overline{x}; \Omega) := \Big\{ x^* \in X^* \Big| \limsup_{x \xrightarrow{\Omega} \overline{x}} \frac{\langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \le 0 \Big\}, \tag{2.1}$$

where the symbol $x \xrightarrow{\Omega} \overline{x}$ means that $x \to \overline{x}$ with $x \in \Omega$. While the set $\widehat{N}(\overline{x};\Omega) \subset X^*$ is always convex, it may be empty at boundary points $\overline{x} \in \Omega$ and does *not* possess satisfactory pointwise calculus rules while enjoying the so-called *fuzzy calculus*; see [3, 16] for more details and references. The situation is dramatically improved when we consider the following "sequential robust regularization" of (2.1) known as the *basic/limiting/Mordukhovich normal cone* to Ω at $\overline{x} \in \Omega$:

$$N(\overline{x};\Omega) := \Big\{x^* \in X^* \Big|\ \exists \text{ sequences } x_k \xrightarrow{\Omega} \overline{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in \widehat{N}(x_k;\Omega) \text{ for all } k \in I\!N\Big\}, \tag{2.2}$$

where $x_k^* \xrightarrow{w^*} x^*$ signifies the *sequential* convergence in the *weak** topology of X^*. Despite being nonconvex (actually due to this), the basic normal cone (2.2) and the associated coderivative/subdifferential constructions given below satisfy comprehensive *pointwise* rules of *full calculus*; see [16] for probably the complete account in Asplund spaces and partly in the arbitrary Banach space setting.

Given further a set-valued mapping $F\colon X \rightrightarrows Y$ with the graph

$$\operatorname{gph} F := \big\{(x,y) \in X \times Y \big|\ y \in F(x)\big\},$$

define its (basic, normal) *coderivative* at $(\overline{x},\overline{y}) \in \operatorname{gph} F$ by

$$D^*F(\overline{x},\overline{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in N\big((\overline{x},\overline{y});\operatorname{gph} F\big)\big\}, \tag{2.3}$$

which is a positively homogeneous mapping of y^*; we always omit $\overline{y} = f(\overline{x})$ in (2.3) if $F = f\colon X \to Y$ is single-valued. It easily follows from (2.2) that the coderivative (2.3) admit the *sequential limiting representation*

$$D^*F(\overline{x},\overline{y})(y^*) = \Big\{x^* \in X^* \Big|\ \exists\, (x_k,y_k) \to (\overline{x},\overline{y}) \text{ and } (x_k^*,y_k^*) \xrightarrow{w^*} (x^*,y^*) \text{ as } k \to \infty \text{ with } y_k \in F(x_k) \text{ and } x_k^* \in \widehat{D}^*F(x_k,y_k)(y_k^*),\ k \in I\!N\Big\},$$

where the *Fréchet-type* coderivative $\widehat{D}^*F$ is defined similarly to (2.3) with the replacement of the basic normal cone N by its Fréchet counterpart $\widehat{N}$ from (2.1). If $F = f\colon X \to Y$ is *strictly differentiable* at $\overline{x}$ with the derivative $\nabla f(\overline{x})\colon X \to Y$ (this is automatic when $f \in C^1$ around this point), we have

$$D^*f(\overline{x})(y^*) = \widehat{D}^*f(\overline{x})(y^*) = \big\{\nabla f(\overline{x})^* y^*\big\} \text{ for all } y^* \in Y^*.$$

Considering an extended-real-valued function $\varphi\colon X \to \overline{I\!R}$ finite at $\overline{x}$, define its *Fréchet subdifferential* at $\overline{x}$ (known also as the regular or viscosity subdifferential of φ at $\overline{x}$) by

$$\widehat{\partial}\varphi(\overline{x}) := \Big\{x^* \in X^* \Big|\ \liminf_{x \to \overline{x}} \frac{\varphi(x) - \varphi(\overline{x}) - \langle x^*, x - \overline{x}\rangle}{\|x - \overline{x}\|} \ge 0\Big\} \tag{2.4}$$

and the (basic, limiting, Mordukhovich) *subdifferential* of φ at $\overline{x}$ by:

$$\begin{aligned}\partial\varphi(\overline{x}) := \big\{x^* \in X^* \big|\ \exists \ \text{sequences}\ x_k \xrightarrow{\varphi} \overline{x},\ x_k^* \xrightarrow{w^*} x^* \\ \text{with}\ x_k^* \in \widehat{\partial}\varphi(x_k)\big\},\end{aligned} \tag{2.5}$$

where $x_k \xrightarrow{\varphi} \overline{x}$ stands for $x_k \to \overline{x}$ with $\varphi(x_k) \to \varphi(\overline{x})$. We also need in what follows the *singular subdifferential* construction for φ at $\overline{x}$ defined (sequentially) by

$$\begin{aligned}\partial^\infty\varphi(x) := \big\{x^* \in X^* \big|\ \exists\, x_k \xrightarrow{\varphi} \overline{x},\ \lambda_k \downarrow 0,\ x_k^* \xrightarrow{w^*} x^* \\ \text{with}\ x_k^* \in \lambda_k\widehat{\partial}\varphi(x_k)\big\},\end{aligned} \tag{2.6}$$

which reduces to $\{0\}$ if φ is locally Lipschitzian around $\overline{x}$. If φ is lower semicontinuous around $\overline{x}$, there are the following useful geometric representations

$$\begin{aligned}\widehat{\partial}\varphi(\overline{x}) &= \widehat{D}^*\mathcal{E}_\varphi\big(\overline{x},\varphi(\overline{x})\big)(1),\ \partial\varphi(\overline{x}) = D^*\mathcal{E}_\varphi\big(\overline{x},\varphi(\overline{x})\big)(1),\ \partial^\infty\varphi(\overline{x}) \\ &= D^*\mathcal{E}_\varphi\big(\overline{x},\varphi(\overline{x})\big)(0)\end{aligned} \tag{2.7}$$

of the subdifferentials (2.4)–(2.6) via the corresponding coderivatives of the *epigraphical multifunction* $\mathcal{E}_\varphi\colon X \rightrightarrows I\!R$ associated with φ and defined by

$$\mathcal{E}_\varphi(x) := \big\{\mu \in I\!R \big|\ \mu \ge \varphi(x)\big\}\ \ \text{with}\ \ \operatorname{gph}\mathcal{E}_\varphi = \operatorname{epi}\varphi;$$

see, respectively, Theorem 1.86, Theorem 1.89, and Theorem 2.38 from the book [16].

We conclude this section with formulating two underlying results that play a crucial role in deriving the suboptimality conditions for MPECs obtained in this paper. The first result, known as the *subdifferential variational principle*, is established by Mordukhovich and Wang [21] (see also [16, Theorem 2.28]) as an analytic description of the fundamental *extremal principle* of variational analysis; see [16, Theorem 2.20] and the related material of [16, Chapter 2] with the commentaries and references therein.

Theorem 2.1. (subdifferential variational principle). *Let $\varphi\colon X \to \overline{I\!R}$ be a lower semicontinuous function bounded from below on X. Then for every $\epsilon > 0$, $\nu > 0$, and $x_0 \in X$ satisfying $\varphi(x_0) < \inf_X \varphi + \epsilon$ there are $\overline{x} \in X$ and $x^* \in \widehat{\partial}\varphi(\overline{x})$ such that*

$$\|\overline{x} - x_0\| \le \nu, \quad \varphi(\overline{x}) \le \inf_X \varphi + \epsilon, \ \text{ and } \ \|x^*\| \le \frac{\epsilon}{\nu}.$$

The next result, known as the *weak fuzzy sum rule*, is established by Fabian [6] as a consequence of the Borwein-Preiss smooth variational principle (see [3, 16]) by the method of separable reduction. It also follows from the extremal principle; see [16, Corollary 2.29] and [17, Lemma 5.27] and the discussions therein.

Theorem 2.2. (weak fuzzy sum rule). *Let $\varphi_i\colon X \to \overline{I\!R}$, $i = 1,\ldots,n$, be lower semicontinuous functions on X. Then for every $\overline{x} \in X$, $\eta > 0$, $x^* \in \widehat{\partial}\big(\varphi_1 + \ldots + \varphi_n\big)(\overline{x})$ and for any weak* neighborhood V of the origin in X^* there are $x_i \in \overline{x} + \eta I\!B$ and $x_i^* \in \widehat{\partial}\varphi_i(x_i)$ such that $|\varphi_i(x_i) - \varphi_i(\overline{x})| \le \eta$ for all $i = 1,\ldots,n$ and*

$$x^* \in \sum_{i=1}^{n} x_i^* + V.$$

3. Weak Suboptimality Conditions

In this section we derive *weak suboptimality conditions* for MPECs (1.7) under very general assumptions on the initial data. We begin with the following lemma giving weak suboptimality conditions for mathematical programs with *only the geometric constraint*:

$$\text{minimize } \varphi_0(x) \text{ subject to } x \in \Xi \subset X \tag{3.1}$$

where $\varphi_0\colon X \to \overline{I\!R}$ with $\inf_\Xi \varphi_0 > -\infty$. We say that $x \in \Xi$ is an *ε-optimal solution* to the constrained problem (3.1) if

$$\varphi_0(x) < \inf_{\Xi} \varphi_0 + \varepsilon.$$

Obviously for any $\varepsilon > 0$ the set of ε-optimal solutions to (3.1) is *nonempty*. The result below is actually a specification of [17, Theorem 5.29] for problems with no functional constraints. Nevertheless, for completeness and the reader's convenience we present a simplified proof of this lemma in the case needed in what follows.

Lemma 3.1. (weak suboptimality conditions for problems with geometric constraints). *Suppose that φ_0 is lower semicontinuous on the set of ε-optimal solutions to problem* (3.1) *for $\varepsilon > 0$ sufficiently small and suppose that the constraint set Ξ is locally closed. Then given an arbitrary weak* neighborhood U of the origin $0 \in X^*$, there exists $\overline{\varepsilon} > 0$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ and every ε^2-optimal solution $\overline{x}$ to* (3.1) *there are $(x_0, x_\Xi, x_0^*, x_\Xi^*) \in X \times X \times X^* \times X^*$ satisfying the relationships*

$$\|x_0 - \overline{x}\| \le \varepsilon \text{ with } |\varphi_0(x_0) - \varphi_0(\overline{x})| \le \varepsilon,\ \|x_\Xi - \overline{x}\| \le \varepsilon \text{ with } x_\Xi \in \Xi, \tag{3.2}$$

$$x_0^* \in \widehat{\partial}\varphi_0(x_0), \quad x_\Xi^* \in \widehat{N}(x_\Xi;\Xi), \quad 0 \in x_0^* + x_\Xi^* + U. \tag{3.3}$$

Proof. For any $v \in X$ and $\gamma > 0$ we consider a family of weak* neighborhoods of the origin in X^* defined by

$$U(v;\gamma) := \Big\{x^* \in X^* \Big| \, \big|\langle x^*, v\rangle\big| < \gamma\Big\};$$

this family forms a *base* of the weak* topology on X^*. Then picking an arbitrary weak* neighborhood U in the theorem, find $\overline{\gamma} > 0$, $p \in I\!N$, and $v_j \in X$ with $\|v_j\| = 1$ as $1 \le j \le p$ satisfying the inclusion

$$\bigcap_{j=1}^{p} U(v_j; 2\overline{\gamma}) \subset U. \tag{3.4}$$

and show that the conclusions of the theorem hold for every ε such that

$$0 < \varepsilon < \overline{\varepsilon} := \min\big\{\overline{\gamma}, 1\big\}.$$

To proceed, take any $\overline{x} \in \Xi$ with $\varphi_0(\overline{x}) < \inf_\Xi \varphi_0 + \varepsilon^2$ and find $\eta \in (0,\varepsilon)$ such that $\varphi_0(\overline{x}) < \inf_\Xi \varphi_0 + (\varepsilon-\eta)^2$. Observe that for the function

$$\varphi(x) := \varphi_0(x) + \delta(x;\Xi), \quad x \in X, \tag{3.5}$$

we have $\varphi(\overline{x}) < \inf_X \varphi + (\varepsilon-\eta)^2$. Applying now the *subdifferential variational principle* from Theorem 2.1 to the above function φ with the parameters

$$\epsilon := (\varepsilon-\eta)^2 \;\text{ and }\; \nu := \varepsilon - \eta$$

and taking into account the structure of φ in (3.5), we get $u \in \Xi$ and $u^* \in \widehat{\partial}\varphi(u)$ satisfying the relationships

$$\begin{aligned} &\|u-\overline{x}\| \le \varepsilon-\eta, \quad \|u^*\| \le \varepsilon - \eta < \overline{\gamma}, \\ &\varphi_0(u) \le \inf_X \varphi + (\varepsilon-\eta)^2 < \inf_\Xi \varphi_0 + \varepsilon - \eta, \end{aligned} \tag{3.6}$$

which imply, by the $(\varepsilon-\eta)^2$-optimality of $\overline{x}$ to problem (3.1), that $|\varphi_0(u)-\varphi_0(\overline{x})| \le \varepsilon - \eta$.

Next apply the *weak fuzzy sum rule* from Theorem 2.2 to $u^* \in \widehat{\partial}\varphi(u)$ for the sum of two functions in (3.5) with the weak* neighborhood

$$V := \bigcap_{j=1}^{p} U(v_j;\overline{\gamma})$$

of the origin in X^* and the number $\eta > 0$ chosen above. In this way we find elements $(x_0, x_\Xi, x_0^*, x_\Xi^*) \in X \times X \times X^* \times X^*$ such that

$$
\begin{aligned}
&\|x_0 - u\| \le \eta \ \text{ with } \ |\varphi_0(x_0) - \varphi_0(u)| \le \eta,\\
&\|x_\Xi - u\| \le \eta \ \text{ with } \ x_\Xi \in \Xi,\\
&x_0^* \in \widehat{\partial}\varphi_0(x_0), \quad x_\Xi^* \in \widehat{N}(x_\Xi;\Xi), \ \text{ and } \ u^* \in x_0^* + x_\Xi^* + V.
\end{aligned}
\tag{3.7}
$$

Taking finally into account the relationships in (3.4) and (3.6), the above construction of the weak* neighborhood V, and that $\|u^*\| \le \overline{\gamma}$, we arrive from (3.7) at the desired conclusions (3.2) and (3.3) and thus complete the proof of the lemma. ■

The next theorem provides *weak suboptimality* conditions for the general class of MPECs (1.7), where all the spaces under consideration are Asplund.

Theorem 3.2. (weak suboptimality conditions for MPECs). *Consider MPEC* (1.7) *defined by* $\varphi_i\colon X\times Y \to \overline{I\!R}$ *as* $i = 0,\ldots,m+r$, $\Omega \subset X\times Y$, $G\colon X\times Y \rightrightarrows Z$, *and* $Q\colon X\times Y \rightrightarrows Z$. *Assume that the functions* φ_i *are all finite and lower semicontinuous for* $i = 0,\ldots,m$ *while continuous for* $i = m+1,\ldots,m+r$ *on the set of* ε*-optimal solutions to* (1.7) *for each* $\varepsilon > 0$ *sufficiently small and that the sets* Ω, $\operatorname{gph} G$, *and* $\operatorname{gph} Q$ *are locally closed around the points under consideration. Let* U *be an arbitrary weak** *neighborhood of the origin in* $X^*\times Y^*$, *and let* γ *be an arbitrary positive number. Then we can find a number* $\overline{\varepsilon} > 0$ *such that for every* $\varepsilon \in (0,\overline{\varepsilon})$, *every* ε^2*-optimal solution* $(\overline{x},\overline{y})$ *to* (1.7), *and every* $\overline{z} \in G(\overline{x},\overline{y}) \cap \big(-Q(\overline{x},\overline{y})\big)$ *there are elements*

$$(x_i, y_i, x_\Omega, y_\Omega, x_G, y_G, z_G, x_Q, y_Q, z_Q, \lambda_i, x_i^*, y_i^*, x_\Omega^*, y_\Omega^*, x_G^*, y_G^*, z_G^*, x_Q^*, y_Q^*, z_Q^*)$$

as $i = 0,\ldots,m+r$ *from the corresponding spaces, with* $\lambda_0 = 1$, *satisfying the relationships*

$$\|(x_0,y_0) - (\overline{x},\overline{y})\| \le \varepsilon \ \textit{ with } \ |\varphi_0(x_0,y_0) - \varphi_0(\overline{x},\overline{y})| \le \varepsilon, \tag{3.8a}$$

$$\|(x_i,y_i) - (\overline{x},\overline{y})\| \le \varepsilon, \quad i = 1,\ldots,m, \tag{3.8b}$$

$$
\begin{aligned}
&\|(x_i,y_i) - (\overline{x},\overline{y})\| \le \varepsilon\\
&\quad \textit{with } \ |\varphi_i(x_i,y_i) - \varphi_i(\overline{x},\overline{y})| \le \varepsilon, \ \ i = m+1,\ldots,m+r,
\end{aligned}
\tag{3.8c}
$$

$$\|(x_G,y_G,z_G) - (\overline{x},\overline{y},\overline{z})\| \le \varepsilon \ \textit{ with } \ (x_G,y_G,z_G) \in \operatorname{gph} G, \tag{3.8d}$$

$$\|(x_Q,y_Q,z_Q) - (\overline{x},\overline{y},\overline{z})\| \le \varepsilon \ \textit{ with } \ (x_Q,y_Q,z_Q) \in \operatorname{gph} Q, \tag{3.8e}$$

$$\|(x_\Omega,y_\Omega) - (\overline{x},\overline{y})\| \le \varepsilon \ \textit{ with } \ (x_\Omega,y_\Omega) \in \Omega, \tag{3.8f}$$

(3.8g) $$(x_0^*, y_0^*) \in \widehat{\partial}\varphi_0(x_0, y_0), \quad (x_\Omega^*, y_\Omega^*) \in \widehat{N}\big((x_\Omega, y_\Omega); \Omega\big),$$

(3.8h) $$(x_i^*, y_i^*) \in \lambda_i \widehat{\partial}\varphi_i(x_i, y_i) \ \textit{with} \ \lambda_i \geq 0, \quad i = 1, \ldots, m,$$

(3.8i) $$\begin{aligned}&(x_i^*, y_i^*) \in \lambda_i\big[\widehat{\partial}\varphi_i(x_i, y_i) \cup \widehat{\partial}(-\varphi_i)(x_i, y_i)\big] \ \textit{with} \ \lambda_i \geq 0,\\ &\quad i = m+1, \ldots, m+r,\end{aligned}$$

(3.8j) $$(x_G^*, y_G^*) \in \widehat{D}^*G(x_G, y_G, z_G)(z_G^*), \quad (x_Q^*, y_Q^*) \in \widehat{D}^*Q(x_Q, y_Q, z_Q)(z_Q^*)$$

(3.8k) $$\textit{with} \ z_G^* \in Z^*, \ z_Q^* \in Z^*, \ \textit{and} \ \|z_G^* - z_Q^*\| \leq \gamma,$$

(3.8l) $$0 \in \sum_{i=0}^{m+r}(x_i^*, y_i^*) + (x_G^*, y_G^*) + (x_Q^*, y_Q^*) + (x_\Omega^*, y_\Omega^*) + U.$$

Proof. Our approach to deriving the suboptimality conditions formulated in the theorem employs the following procedure. Construct first a mathematical program of type (3.1) with *only the geometric constraint* given by a *set intersection* in such a way that this problem is *equivalent* to the general MPEC (1.7) under consideration. Applying then the suboptimality conditions from Lemma 3.1 to the designed problem (3.1), we need to express them constructively in terms of the initial data of (1.7). This will be done by using the *weak fuzzy sum rule* from Theorem 2.2 and the efficient descriptions of Fréchet normals to graphs and epigraphs of functions established in [16, Section 2.4] on the base of *variational/extremal principles*. Details follow.

Consider the *product space* $W := X \times Y \times I\!R^{m+r} \times Z$ endowed with the standard *sum norm* on the product. It is well known [23] that W is *Asplund* as a product of Asplund spaces. Define the following subsets of W by

(3.9) $$\begin{aligned}&\Omega_i := \big\{(x, y, a, z) \in W \ \big|\ (x, y, \alpha_i) \in \operatorname{epi}\varphi_i\big\}, \ i = 1, \ldots, m,\\ &\Omega_i := \big\{(x, y, a, z) \in W \ \big|\ (x, y, \alpha_i) \in \operatorname{gph}\varphi_i\big\}, \ i = m+1, \ldots, m+r,\\ &\Omega_G := \big\{(x, y, a, z) \in W \ \big|\ (x, y, z) \in \operatorname{gph}G\big\},\\ &\Omega_Q := \big\{(x, y, a, z) \in W \ \big|\ (x, y, -z) \in \operatorname{gph}Q\big\},\\ &\Omega_\Omega := \Omega \times I\!R^m_- \times \{0\} \times Z \subset W, \quad \Xi := \bigcap_{i=1}^{m+r}\Omega_i \cap \Omega_G \cap \Omega_Q \cap \Omega_\Omega,\end{aligned}$$

where $a = (\alpha_1, \ldots, \alpha_{m+r}) \in I\!R^{m+r}$, where $I\!R^m_-$ stands for the nonpositive orthant of $I\!R^m$, and where $0 \in I\!R^r$. It is easy to see that all these sets are locally closed around the points in question due to the semicontinuity/continuity/closedness assumptions made in the theorem. Observe also that for every feasible solution (x,y) to MPEC (1.7) we have

$$(x,y,a,z)\in\Xi \ \text{ with } a=\big(\varphi_1(x,y),\ldots,\varphi_{m+r}(x,y)\big)$$
$$\text{and any } \ z\in G(x,y)\cap\big(-Q(x,y)\big)$$

by the construction of Ξ in (3.9). Conversely, the inclusion $(x,y,a,z)\in\Xi$ implies that (x,y) is a feasible solution to MPEC (1.7), since

$$\varphi_i(x,y)\le\alpha_i\le 0 \ \text{ for } \ i=1,\ldots,m,\ \varphi_i(x,y)=\alpha_i=0$$
$$\text{for } \ i=m+1,\ldots,m+r, z-z=0\in G(x,y)+Q(x,y), \ \text{ and } \ (x,y)\in\Omega$$

due to the set structures in (3.9). Furthermore, define $\widetilde{\varphi}_0\colon X\times Y\times I\!R^{m+r}\times Z\to\overline{I\!R}$ by

$$\widetilde{\varphi}_0(x,y,a,z):=\varphi_0(x,y) \ \text{ for all } \ (x,y,a,z)\in W \tag{3.10}$$

and construct a mathematical program of type (3.1) with *only the geometric constraint* given by the set Ξ from (3.9) as follows:

$$\text{minimize } \ \widetilde{\varphi}_0(x,y,a,z) \ \text{ subject to } \ (x,y,a,z)\in\Xi. \tag{3.11}$$

Having $(\overline{x},\overline{y})\in X\times Y$, denote $\overline{a}:=\big(\varphi_1(\overline{x},\overline{y}),\ldots,\varphi_{m+r}(\overline{x},\overline{y})\big)\in I\!R^{m+r}$ and pick any element $\overline{z}\in G(\overline{x},\overline{y})\cap\big(-Q(\overline{x},\overline{y})\big)$. By construction we get that $(\overline{x},\overline{y})$ is an ε-optimal solution to MPEC (1.7) *if and only if* $(\overline{x},\overline{y},\overline{a},\overline{z})\in W$ is an ε-optimal solution to (3.11).

Fix an arbitrary number $\gamma>0$ and take an arbitrary weak* neighborhood U of the origin in $X^*\times Y^*$ from the formulation of the theorem. Let us construct a weak* neighborhood V of the origin in W^* by

$$V:=\big(\tfrac{1}{m+r+1}U\big)\times I\!R^{m+r}\times\gamma\big(\tfrac{1}{2}\text{int}\, I\!B^*\big), \tag{3.12}$$

where $I\!B^*$ stands for the closed unit ball in Z^*. Applying now Lemma 3.1 to problem (3.11) with the neighborhood V from (3.12) and taking into account the structures of $\widetilde{\varphi}_0$ and V in (3.10) and (3.12), respectively, we find $\overline{\varepsilon}>0$ such that for every $\varepsilon\in(0,\overline{\varepsilon}/3)$ and every ε^2-optimal solution $(\overline{x},\overline{y},\overline{a},\overline{z})$ to (3.11)–corresponding to the designated ε^2-optimal solution $(\overline{x},\overline{y})$ to MPEC (1.7)—there are elements

$$\big(x_0,y_0,a_0,z_0,x_\Xi,y_\Xi,a_\Xi,z_\Xi,x_0^*,y_0^*,a_0^*,z_0^*,x_\Xi^*,y_\Xi^*,a_\Xi^*,z_\Xi^*\big)\in W\times W\times W^*\times W^*$$

satisfying the following relationships

$$\|(x_0,y_0,a_0,z_0)-(\overline{x},\overline{y},\overline{a},\overline{z})\|\le\varepsilon \ \text{ with } \ |\varphi_0(x_0,y_0)-\varphi_0(\overline{x},\overline{y})|\le\varepsilon, \tag{3.13a}$$

(3.13b) $$\|(x_\Xi, y_\Xi, a_\Xi, z_\Xi) - (\overline{x}, \overline{y}, \overline{a}, \overline{z})\| \le \varepsilon \ \text{ with } \ (x_\Xi, y_\Xi, a_\Xi, z_\Xi) \in \Xi,$$

(3.13c) $$(x_0^*, y_0^*, 0, 0) \in \widehat{\partial}\widetilde{\varphi}_0(x_0, y_0, a_0, z_0) = \widehat{\partial}\varphi_0(x_0, y_0) \times \{0\},$$

(3.13d) $$(x_\Xi^*, y_\Xi^*, a_\Xi^*, z_\Xi^*) \in \widehat{N}\big((x_\Xi, y_\Xi, a_\Xi, z_\Xi); \Xi\big) \ \text{ with } \ \|z_\Xi^*\| \le \gamma/2,$$

(3.13e) $$0 \in (x_0^*, y_0^*) + (x_\Xi^*, y_\Xi^*) + \tfrac{1}{m+r+1}U.$$

It is easy to observe from the intersection structure of the set Ξ in (3.9) that inclusion (3.13d) can be equivalently written as

(3.14) $$(x_\Xi^*, y_\Xi^*, a_\Xi^*, z_\Xi^*) \in \widehat{\partial}\Big(\sum_{i=1}^{m+r} \delta(\cdot;\Omega_i) + \delta(\cdot;\Omega_G) + \delta(\cdot;\Omega_Q) + \delta(\cdot;\Omega_\Omega)\Big)(x_\Xi, y_\Xi, a_\Xi, z_\Xi).$$

Taking the Fréchet subgradient $(x_\Xi^*, y_\Xi^*, a_\Xi^*, z_\Xi^*)$ in (3.14) and applying to it the *weak fuzzy sum rule* from Theorem 2.2 with the neighborhood V defined in (3.12) and with any fixed number $\eta = \varepsilon \in (0, \overline{\varepsilon}/3)$ from above, we find elements

$$(\widetilde{x}_i, \widetilde{y}_i, \widetilde{\alpha}_i, x_G, y_G, z_G, x_Q, y_Q, z_Q, x_\Omega, y_\Omega, \widetilde{x}_i^*, \widetilde{y}_i^*, \widetilde{\lambda}_i, x_G^*, y_G^*, z_G^*, x_Q^*, y_Q^*, z_Q^*, x_\Omega^*, y_\Omega^*)$$

as $i = 1, \ldots, m+r$ satisfying the following relationships, where $\alpha_{\Xi i}$ stands for the i-th component of the vector $a_\Xi \in I\!R^{m+r}$ from (3.13):

(3.15a) $$\|(\widetilde{x}_i, \widetilde{y}_i, \widetilde{\alpha}_i) - (x_\Xi, y_\Xi, \alpha_{\Xi i})\| \le \varepsilon, (\widetilde{x}_i, \widetilde{y}_i, \widetilde{\alpha}_i) \in \text{epi}\,\varphi_i, \ i = 1, \ldots, m,$$

(3.15b) $$\|(\widetilde{x}_i, \widetilde{y}_i, \widetilde{\alpha}_i) - (x_\Xi, y_\Xi, \alpha_{\Xi i})\| \le \varepsilon, (\widetilde{x}_i, \widetilde{y}_i, \widetilde{\alpha}_i) \in \text{gph}\,\varphi_i, \ i = m+1, \ldots, m+r,$$

(3.15c) $$\|(x_G, y_G, w_G) - (x_\Xi, y_\Xi, z_\Xi)\| \le \varepsilon, \|(x_Q, y_Q, w_Q) - (x_\Xi, y_\Xi, z_\Xi)\| \le \varepsilon,$$

(3.15d) $$\|(x_\Omega, y_\Omega) - (x_\Xi, y_\Xi)\| \le \varepsilon \ \text{ with } \ (x_\Omega, y_\Omega) \in \Omega,$$

(3.15e) $$(\widetilde{x}_i^*, \widetilde{y}_i^*, -\widetilde{\lambda}_i) \in \widehat{N}\big((\widetilde{x}_i, \widetilde{y}_i, \widetilde{\alpha}_i); \text{epi}\,\varphi_i\big), \ i = 1, \ldots, m,$$

(3.15f) $$(\widetilde{x}_i^*, \widetilde{y}_i^*, -\widetilde{\lambda}_i) \in \widehat{N}\big((\widetilde{x}_i, \widetilde{y}_i, \widetilde{\alpha}_i); \text{gph}\,\varphi_i\big), \ i = m+1, \ldots, m+r,$$

(3.15g) $$(x_\Omega^*, y_\Omega^*) \in \widehat{N}\big((x_\Omega, y_\Omega); \Omega\big), (x_G^*, y_G^*, -z_G^*) \in \widehat{N}\big((x_G, y_G, z_G); \text{gph}\,G\big),$$

(3.15h) $$(x_Q^*, y_Q^*, z_Q^*) \in \widehat{N}\big((x_Q, y_Q, z_Q); \text{gph}\,(-Q)\big) \ \text{ with } \ \|z_G^* - z_Q^* + z_\Xi^*\| \le \gamma/2,$$

$$\text{(3.15i)} \quad (x^*_\Xi, y^*_\Xi) \in \sum_{i=1}^{m+r} (\widetilde{x}^*_i, \widetilde{y}^*_i) + (x^*_G, y^*_G) + (x^*_Q, y^*_Q) + (x^*_\Omega, y^*_\Omega) + \tfrac{1}{m+r+1} U.$$

Let us further elaborate conditions (3.15). Consider first the relationships for *inequality constraints* and fix $i \in \{1, \ldots, m\}$ in (3.15a) and (3.15e). It is easy to check, due to the definition of Fréchet normals in (2.1) and $\widetilde{\alpha}_i \ge \varphi_i(\widetilde{x}_i, \widetilde{y}_i)$, that (3.15e) implies the inclusion

$$\text{(3.16)} \quad \big(\widetilde{x}^*_i, \widetilde{y}^*_i, -\widetilde{\lambda}_i\big) \in \widehat{N}\big((\widetilde{x}_i, \widetilde{y}_i, \varphi_i(\widetilde{x}_i, \widetilde{y}_i)); \text{epi}\,\varphi_i\big) \quad \text{with} \quad \widetilde{\lambda}_i \ge 0,$$

and thus there are the two possible cases in (3.16): $\widetilde{\lambda}_i > 0$ and $\widetilde{\lambda}_i = 0$.

If $\widetilde{\lambda}_i > 0$ in (3.16), we immediately get $(\widetilde{x}^*_i, \widetilde{y}^*_i) \in \widetilde{\lambda}_i \widehat{\partial} \varphi_i(\widetilde{x}_i, \widetilde{y}_i)$ from (3.16) due to the first relationship in (2.7); hence by (3.13b) and (3.15a) we arrive at conclusions (3.8b) and (3.8h) of the theorem with $(x_i, y_i, \lambda_i, x^*_i, y^*_i) := (\widetilde{x}_i, \widetilde{y}_i, \widetilde{\lambda}_i, \widetilde{x}^*_i, \widetilde{y}^*_i)$.

The other case of $\widetilde{\lambda}_i = 0$ in (3.16) means that $(\widetilde{x}^*_i, \widetilde{y}^*_i)$ is a *horizontal Fréchet normal* to the *epigraph* of φ_i at $(\widetilde{x}_i, \widetilde{y}_i)$. Using [16, Lemma 2.37] on the description of such normals in Asplund spaces, we find $(x_i, y_i, \lambda_i, x^*_i, y^*_i) \in X \times Y \times I\!R \times X^* \times Y^*$ satisfying the relationships

$$\text{(3.17)} \quad \begin{aligned} &\|(x_i, y_i) - (\widetilde{x}_i, \widetilde{y}_i)\| \le \varepsilon, \quad (x^*_i, y^*_i) \in (\widetilde{x}^*_i, \widetilde{y}^*_i) + \tfrac{1}{m+r+1} U, \\ &\lambda_i \ge 0, \ \text{ and } \ (x^*_i, y^*_i) \in \lambda_i \widehat{\partial} \varphi_i(x_i, y_i), \quad i = 1, \ldots, m, \end{aligned}$$

which imply those in (3.8h) in the case under consideration.

Next we elaborate the relationships for *equality constraints* in (3.15) and fix an index $i \in \{m+1, \ldots, m+r\}$ in (3.15b) and (3.15f). Again, explore the two possible cases in (3.15f): $\widetilde{\lambda}_i \ne 0$ and $\widetilde{\lambda}_i = 0$.

If $\widetilde{\lambda}_i \ne 0$, we get from [16, Theorem 1.80] that (3.15f) yields

$$(\widetilde{x}^*_i, \widetilde{y}^*_i) \in \lambda_i \big[\widehat{\partial} \varphi_i(\widetilde{x}_i, \widetilde{y}_i) \cup \widehat{\partial}(-\varphi_i)(\widetilde{x}_i, \widetilde{y}_i)\big] \quad \text{with} \quad \lambda_i := |\widetilde{\lambda}_i|,$$

which justifies (3.8i) with $(x_i, y_i, x^*_i, y^*_i) := (\widetilde{x}_i, \widetilde{y}_i, \widetilde{x}^*_i, \widetilde{y}^*_i)$. If $\widetilde{\lambda}_i = 0$ in (3.15f), this means that $(\widetilde{x}^*_i, \widetilde{y}^*_i)$ is a *horizontal Fréchet normal* to the *graph* of φ_i at $(\widetilde{x}_i, \widetilde{y}_i)$. Using the description of such normals for continuous functions on Asplund spaces from [16, Theorem 2.40(i)], we find elements $(x_i, y_i, \lambda_i, x^*_i, y^*_i) \in X \times Y \times I\!R \times X^* \times Y^*$ satisfying the relationships

$$\text{(3.18)} \quad \begin{aligned} &\|(x_i, y_i) - (\widetilde{x}_i, \widetilde{y}_i)\| \le \varepsilon, \quad (x^*_i, y^*_i) \in (\widetilde{x}^*_i, \widetilde{y}^*_i) + \tfrac{1}{m+r+1} U, \quad \lambda_i \ge 0, \\ &(x^*_i, y^*_i) \in \lambda_i \big[\widehat{\partial} \varphi_i(x_i, y_i) \cup \widehat{\partial}(-\varphi_i)(x_i, y_i)\big], \quad i = m+1, \ldots, m+r, \end{aligned}$$

which imply those in (3.8i) in this case.

Considering finally the inclusions and estimates in the above relationships (3.13) and (3.15) for the cases of *geometric* and *equilibrium constraints* and taking into account the construction of the Fréchet coderivative in Section 2, we easily arrive at the corresponding condition in (3.8d)-(3.8g) and (3.8j) of the theorem from those in (3.13) and (3.15). Estimate (3.8k) for the equilibrium constraints follows from

$$\|z_G^* - z_Q^*\| \le \|(z_G^* - z_Q^* + z_\Xi^*) - z_\Xi^*\| \le \|z_G^* - z_Q^* + z_\Xi^*\| + \|z_\Xi^*\| \le \gamma/2 + \gamma/2 = \gamma$$

due to (3.13d) and (3.15h). Furthermore, the relationships in (3.8a) and (3.8g) for the cost function of (1.7) are implied directly by those in (3.13a) and (3.13c); the cost function is not involved in the conditions of (3.15).

To complete the proof of theorem, it remains to justify the *generalized Euler equation* (3.8l) involving the given weak* neighborhood U of the origin in $X^* \times Y^*$. We get this by combining relationships (3.13e), (3.15i) with those for (x_i^*, y_i^*) in (3.17) and (3.18). ■

Remark 3.3. (qualified suboptimality conditions with no constraint qualifications). As we see from Theorem 3.2, the suboptimality conditions for MPECs obtained therein are in the *qualified/normal form*, which means that $\lambda_0 = 1$ for the multiplier corresponding to the cost function φ_0; see (3.8g) and (3.8l) in comparison with (3.8h) and (3.8i). This is a *new result* even for problems with just *functional* (no equilibrium) constraints derived in [17, Theorem 5.29], which contains conditions in the *non-qualified* (Fritz John) form:

$$(3.19) \qquad (x_0^*, y_0^*) \in \lambda_0 \partial\varphi_0(x_0, y_0), \ \sum_{i=0}^{m+r} \lambda_i = 1, \ \lambda_i \ge 0 \ \text{ for all } \ i = 0, \ldots, m+r$$

instead of the qualified ones with $\lambda_0 = 1$ in the counterpart of Theorem 3.2 for problems with no equilibrium constraints. It is easy to check that Theorem 3.2 implies its non-qualified version with conditions (3.19). Indeed, letting

$$\lambda := 1 + \sum_{i=1}^{m+r} \lambda_i, \quad \widetilde{\lambda}_0 := \frac{1}{\lambda}, \quad \widetilde{\lambda}_i := \frac{\lambda_i}{\lambda} \ \text{ for } \ i = 1, \ldots, m+r,$$
$$\widetilde{x}_i^* := \frac{x_i^*}{\lambda} \ \text{ for } \ i = 0, \ldots, m+r, \quad \widetilde{x}_G^* := \frac{x_G^*}{\lambda}, \ \widetilde{x}_Q^* := \frac{x_Q^*}{\lambda},$$
$$\widetilde{x}_\Omega^* =: \frac{x_\Omega^*}{\lambda}, \ \widetilde{z}_G^* := \frac{z_G^*}{\lambda}, \ \widetilde{z}_Q^* := \frac{z_Q^*}{\lambda}$$

in the suboptimality conditions of Theorem 3.2, we arrive the non-qualified version of this theorem with conditions (3.19).

At the first glance it looks rather surprising that we get *qualified* conditions with *no constraint qualifications*. The key here is that the conditions obtained are *not* *pointwise* but *fuzzy*, i.e., they involve all points from a neighborhood of suboptimal solutions as well as dual elements measured by an arbitrary small number $\varepsilon > 0$. We can see from the proof of Theorem 3.2 that deriving such conditions benefits from the possibility of *limiting subgradient* representations of *horizontal normals* to epigraphs and graphs of functions, which are based on *variational principles*; see [16, Subsection 2.4.2] for more details.

Remark 3.4. (comparison with another approach). It is worth mentioning that the proof of suboptimality conditions in [17, Theorem 5.29] for standard mathematical programs with no equilibrium constraints

$$\text{(3.20)} \quad \begin{aligned} &\text{minimize } \ \varphi_0(x) \quad \text{subject to} \quad x \in \Omega \subset X, \\ &\varphi_i(x) \le 0,\ i = 1, \ldots, m, \ \text{ and } \ \varphi_i(x) = 0,\ i = m+1, \ldots, m+r, \end{aligned}$$

employs a different device in comparison with that of Theorem 3.2 above. The former is based on considering the auxiliary *unconstrained minimization* problem

$$\text{minimize } \ \varphi_0(x) + \delta(x; \Omega) + \sum_{i=1}^{m+r} \delta(x; \Omega_i), \quad x \in X,$$

equivalent to (3.20), where the sets Ω_i are defined by

$$\Omega_i := \big\{x \in X \big|\ \varphi_i(x) \le 0\big\},\ i = 1, \ldots, m;$$
$$\Omega_i := \big\{x \in X \big|\ \varphi_i(x) = 0\big\},\ i = m+1, \ldots, m+r.$$

To adopt this scheme in the case of equilibrium-type constraints given by $0 \in G(x) + Q(x)$, we need to involve the set $(G + Q)^{-1}(0)$, which is essentially *more complicated* to handle and often fails to be *closed* even when both mappings G and Q are assumed to be closed-graph. Observe that the closedness requirements are *necessary* to employ *variational arguments.*

Remark 3.5. (implementation and applications of weak suboptimality conditions). Theorem 3.2 deals with a general MPEC model particularly including problems with equilibrium constraints described by set-valued/generalized variational inequalities, solutions sets to lower-level problems in hierarchical optimization; see, e.g., Section 1 above and [1, 17] for more examples and discussions. To implement suboptimality conditions obtained in this way and to apply them to particular models, we need to calculate the corresponding coderivatives that appear in (3.8). It has been partly done in [1, 17] and the references therein in the case of necessary *optimality* conditions obtained via our *basic coderivative* (2.3), which enjoys comprehensive *pointwise* rules of *full calculus* and has been computed for broad classes of mappings arising in many important applications. The situation is more complicated with the Fréchet-like constructions used in Theorem 3.2, which satisfy a much modest amount of pointwise/exact calculus; see [16] and also [18] for recent results in this direction. However, the latter constructions possess many useful rules of *fuzzy calculus* in Asplund spaces (see, e.g., [3, 11, 16, 20] with more references and discussions), which are appropriate to be employed in the *fuzzy* framework of *suboptimality* conditions.

4. Strong Suboptimality Conditions

In this section we derive new suboptimality conditions for MPECs (1.7) in the *strong form*, which–as discussed in Section 1–is different from the weak form of suboptimality conditions in the following two major aspects:

(a) the strong form provides estimates of dual elements in the *strong/norm* topology instead of the weak* topology of the dual spaces in question as in the weak form;

(b) the strong form uses our *robust limiting* subgradient, normal, and coderivative constructions instead of the Fréchet-like constructions in the weak form.

The strong form undoubtedly offers significant advantages over the weak form-even in finite dimensions, where there is no difference between weak and strong topologies of dual spaces—due to essentially more developed calculus for the limiting constructions and their efficient computation for various classes of sets and mappings important in applications; see [16, 17] and the discussions in Remark 3.5 and Remark 4.4. On the other hand, strong suboptimality conditions require more assumptions in comparison with weak ones: qualification conditions in both finite and infinite dimensions and the so-called SNC properties [16], which are automatic in finite-dimensional spaces.

Recall that a set $\Omega \subset X$ is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if the following implication holds:

$$\big[x_k \xrightarrow{\Omega} \bar{x}, \quad x_k^* \xrightarrow{w^*} 0, \quad x_k^* \in \widehat{N}(x_k; \Omega)\big] \Longrightarrow \|x_k^*\| \to 0 \ \text{ as } \ k \to \infty \tag{4.1}$$

for any sequences involved in (4.1). Further, we say that an extended-real-valued function $\varphi\colon X \to \overline{I\!R}$ is *sequentially normally epi-compact* (SNEC) at $\bar{x}$ with $\varphi(\bar{x}) < \infty$ if its epigraph $\operatorname{epi}\varphi \subset X \times I\!R$ is SNC at $\big(\bar{x}, \varphi(\bar{x})\big)$. Besides the obvious validity of both SNC and SNEC properties in finite-dimensional spaces, Ω is SNC at $\bar{x}$ if it is *compactly epi-Lipschitzian* around this point in the sense of Borwein and Strójwas, while φ is SNEC at $\bar{x}$ if it is *directionally Lipschitzian* around this point in the sense of Rockafellar; in particular, when it is *locally Lipschitzian* around $\bar{x}$–see [16] for more details and references and for other sufficient conditions for the fulfillment of the SNC and SNEC properties. Furthermore, these and related properties of sets, mappings, and functions enjoy well-developed *SNC calculus* ensuring their preservation under various operations. Note that SNC calculus is also based on *variational/extremal principles* of variational analysis; see [16].

To derive strong suboptimality conditions for MPECs (1.7), we start–similarly to Section 3–with such conditions for problem (3.1) involving *only the geometric constraint* given by a closed set $\Xi \subset X$. The following result is a specification and a small modification of [17, Theorem 5.30] for this case, while we present its simplified proof for completeness and the reader's convenience.

Lemma 4.1. (strong suboptimality conditions for problems with geometric constraints). *Let φ_0 be lower semicontinuous on the sets of ε-optimal solutions to problem* (3.1) *for all $\varepsilon > 0$ sufficiently small, and let Ξ be locally closed. Assume also that either φ_0 is SNEC or Ξ is SNC and that the qualification condition*

$$\partial^{\infty}\varphi_0(x) \cap \big(-N(x;\Xi)\big) = \{0\} \tag{4.2}$$

is satisfied on the afore-mentioned set. Then for every $\varepsilon > 0$ sufficiently small and every ε^2-optimal solution $\overline{x}$ to (3.1) *there is an ε^2-optimal solution $\widehat{x}$ to this problem such that*

$$\|\widehat{x} - \overline{x}\| \le \varepsilon \;\; \textit{and} \;\; \|\widehat{x}_0^* + x_{\Xi}^*\| \le \varepsilon \tag{4.3}$$

for some $\widehat{x}_0^* \in \partial\varphi_0(\widehat{x})$ and $\widehat{x}_{\Xi}^* \in N(\widehat{x};\Xi)$.

Proof. Consider the unconstrained problem

$$\text{minimize } \varphi(x) := \varphi_0(x) + \delta(x;\Xi), \quad x \in X, \tag{4.4}$$

equivalent to (3.1) and observe that $\overline{x}$ is an ε^2-optimal solution to (4.4). Applying the *subdifferential variational principle* from Theorem 2.1 with the parameters

$$\epsilon := \varepsilon^2 \;\text{ and }\; \nu := \varepsilon$$

to the function φ in (4.4), we find an ε^2-optimal solution $\widehat{x} \in \Xi$ to (4.4)—and hence to the original constrained problem (3.1)—satisfying conditions (4.3) with a subgradient

$$\widehat{x}^* \in \partial\big[\varphi_0 + \delta(\cdot;\Xi)\big](\widehat{x}). \tag{4.5}$$

Using now the sum rule for the basic subdifferential in (4.5), which holds under the SNC and qualification conditions imposed in the theorem (see [16, Theorem 3.36]), we get

$$\partial\varphi(\widehat{x}) \subset \partial\varphi_0(\widehat{x}) + \partial\delta(\widehat{x};\Xi) = \partial\varphi_0(\widehat{x}) + N(\widehat{x};\Xi),$$

and thus arrive from (4.5) at the second condition in (4.3). ■

Note that we automatically have the suboptimality conditions of the lemma if φ_0 is *locally Lipschitzian* on the sets of ε-optimal solutions to (3.1). Indeed, in this case both the SNEC and qualification condition (4.2) are satisfied.

The next theorem provides *strong suboptimality conditions* in the qualified form for general MPECs (1.7) in Asplund spaces. Denote

$$\lambda := (\lambda_1, \ldots, \lambda_{m+r}) \in I\!R^{m+r} \text{ and } \Lambda := \big\{\lambda \in I\!R^{m+r} \,\big|\, \lambda_i \ge 0 \text{ for all } i = 1, \ldots, m\big\}.$$

Theorem 4.2. (strong suboptimality conditions for MPECs). *Let the sets* Ω, $\operatorname{gph} G$, *and* $\operatorname{gph} Q$ *be locally closed, and let the functions* φ_i *be lower semicontinuous for* $i=0,\ldots,m$ *and continuous for* $i=m+1,\ldots,m+r$ *on the set of* ε*-optimal solutions to MPEC* (1.7) *for all* $\varepsilon > 0$ *sufficiently small. Suppose also that the following two groups of conditions hold on the latter set of* ε*-optimal solutions:*

(a) *The SNC conditions:*

- *either* φ_0 *is SNEC and all but one of the sets* $\operatorname{epi} \varphi_i$ *for* $i = 1, \ldots, m$, $\operatorname{gph} \varphi_i$ *for* $i = m+1, \ldots, m+r$, $\operatorname{gph} G$, $\operatorname{gph} Q$, *and* Ω *are SNC;*
- *or all of the sets* $\operatorname{epi} \varphi_i$ *for* $i = 1, \ldots, m$, $\operatorname{gph} \varphi_i$ *for* $i = m+1, \ldots, m+r$, $\operatorname{gph} G$, $\operatorname{gph} Q$, *and* Ω *are SNC;*

(b) *The qualification condition: the only zero elements*

$$(x_0^*, y_0^*) = \ldots = (x_{m+r}^*, y_{m+r}^*) = (x_G^*, y_G^*) = (x_Q^*, y_Q^*) = (x_\Omega^*, y_\Omega^*) = 0,$$
$$\lambda = 0, \; z^* = 0$$

satisfy the relationships

$$\begin{cases} (x_0^*, y_0^*) \in \partial^\infty \varphi_0(x,y), \quad \lambda \in \Lambda, \\ (x_i^*, y_i^*) \in D^* \mathcal{E}_{\varphi_i}(x,y,\alpha_i)(\lambda_i) \;\; \textit{for} \;\; i = 1, \ldots, m, \\ (x_i^*, y_i^*) \in D^* \varphi_i(x,y)(\lambda_i) \;\; \textit{for} \;\; i = m+1, \ldots, m+r, \\ (x_\Omega^*, y_\Omega^*) \in N\big((x,y); \Omega\big), \\ (x_G^*, y_G^*) \in D^* G(x,y,z)(z^*), \quad (x_Q^*, y_Q^*) \in D^* Q(x,y,-z)(z^*), \quad \textit{and} \\ (x_0^*, y_0^*) + \displaystyle\sum_{i=1}^{m+r} (x_i^*, y_i^*) + (x_G^*, y_G^*) + (x_Q^*, y_Q^*) + (x_\Omega^*, y_\Omega^*) = 0 \end{cases}$$

whenever $\alpha_i \ge \varphi_i(x,y)$ *for* $i = 1, \ldots, m$ *and* $z \in G(x,y) \cap \big(-Q(x,y)\big)$.

Then given any number $\varepsilon > 0$, for every ε^2-optimal solution $(\overline{x}, \overline{y})$ to MPEC (1.7) and every $\overline{z} \in G(\overline{x}, \overline{y}) \cap \big(-Q(\overline{x}, \overline{y})\big)$ there is an ε^2-optimal solution $(\widehat{x}, \widehat{y})$ to this problem and $\widehat{z} \in G(\widehat{x}, \widehat{y}) \cap \big(-Q(\widehat{x}, \widehat{y})\big)$ such that

$$\|(\widehat{x}, \widehat{y}) - (\overline{x}, \overline{y})\| \le \varepsilon, \quad \|\widehat{z} - \overline{z}\| \le \varepsilon, \quad \text{and} \tag{4.6}$$

$$\Big\|(\widehat{x}_0^*, \widehat{y}_0^*) + \sum_{i=1}^{m+r} (\widehat{x}_i^*, \widehat{y}_i^*) + (\widehat{x}_G^*, \widehat{y}_G^*) + (\widehat{x}_Q^*, \widehat{y}_Q^*) + (\widehat{x}_\Omega^*, \widehat{y}_\Omega^*)\Big\| \le \varepsilon \tag{4.7}$$

where the dual elements $(\widehat{x}_0^*, \widehat{y}_0^*, \widehat{x}_i^*, \widehat{y}_i^*, \widehat{x}_G^*, \widehat{y}_G^*, \widehat{x}_Q^*, \widehat{y}_Q^*, \widehat{x}_\Omega^*, \widehat{y}_\Omega^*)$ satisfy the relationships

$$\begin{aligned}
&(\widehat{x}_0^*, \widehat{y}_0^*) \in \partial\varphi_0(\widehat{x}, \widehat{y}), \quad (\widehat{x}_\Omega^*, \widehat{y}_\Omega^*) \in N\big((\widehat{x}, \widehat{y}); \Omega\big),\\
&(\widehat{x}_i^*, \widehat{y}_i^*) \in D^*\mathcal{E}_{\varphi_i}(\widehat{x}, \widehat{y}, \widehat{\alpha}_i)(\widehat{\lambda}_i) \text{ with } \widehat{\alpha}_i \ge \varphi_i(\widehat{x}, \widehat{y}),\ \widehat{\lambda}_i \ge 0,\ i = 1, \ldots, m,\\
&(\widehat{x}_i^*, \widehat{y}_i^*) \in D^*\varphi_i(\widehat{x}, \widehat{y})(\widehat{\lambda}_i) \text{ with } \widehat{\lambda}_i \in I\!R, \quad i = m+1, \ldots, m+r,\\
&(\widehat{x}_G^*, \widehat{y}_G^*) \in D^*G(\widehat{x}, \widehat{y}, \widehat{z})(\widehat{z}_G^*), (\widehat{x}_Q^*, \widehat{y}_Q^*) \in D^*Q(\widehat{x}, \widehat{y}, -\widehat{z})(\widehat{z}_Q^*)\\
&\text{with } \|z_G^* - z_Q^*\| \le \varepsilon.
\end{aligned} \tag{4.8}$$

Proof. We start proceeding similarly to the proof of Theorem 3.2 and consider the optimization problem (3.11) with *only the geometric constraint* equivalent to MPEC (1.7), where the constraint set $\Xi \subset W$ and the cost function $\widetilde{\varphi}_0$ are defined in (3.9) and (3.10), respectively. Taking an ε^2-optimal solution $(\overline{x}, \overline{y})$ to MPEC (1.7) from the formulation of the theorem and picking any $\overline{z} \in G(\overline{x}, \overline{y}) \cap \big(-Q(\overline{x}, \overline{y})\big)$, we conclude similarly to the proof of Theorem 3.2 that $(\overline{x}, \overline{y}, \overline{a}, \overline{z}) \in W$ with $\overline{a} := \big(\varphi_1(\overline{x}, \overline{y}), \ldots, \varphi_{m+r}(\overline{x}, \overline{y})\big)$ is an ε^2-optimal solution to problem (3.11). Applying now the strong suboptimality conditions (4.3) from Lemma 4.1 to the designated ε^2-optimal solution $(\overline{x}, \overline{y}, \overline{a}, \overline{z})$ to problem (3.11) and taking into account the structure of $\widetilde{\varphi}_0$ in (3.10), we find an ε^2-optimal solution $(\widehat{x}, \widehat{y}, \widehat{a}, \widehat{z}) \in W$ to this problem and dual elements

$$(\widehat{x}_0^*, \widehat{y}_0^*) \in \partial\varphi_0(\widehat{x}, \widehat{y}) \ \text{ and } \ (\widehat{x}_\Xi^*, \widehat{y}_\Xi^*, \widehat{a}_\Xi^*, \widehat{z}_\Xi^*) \in N\big((\widehat{x}, \widehat{y}, \widehat{a}, \widehat{z}); \Xi\big) \tag{4.9}$$

satisfying the relationships

$$\|(\widehat{x}, \widehat{y}, \widehat{a}, \widehat{z}) - (\overline{x}, \overline{y}, \overline{a}, \overline{z})\| \le \varepsilon \ \text{ and } \ \|(x_0^*, y_0^*, 0, 0) + (x_\Xi^*, y_\Xi^*, a_\Xi^*, z_\Xi^*)\| \le \varepsilon \tag{4.10}$$

provided that the qualification condition

$$\big(\partial^\infty\varphi_0(x, y), 0, 0\big) \cap \big[-N\big((x, y, a, z); \Xi\big)\big] = \{0\} \tag{4.11}$$

coming from (4.2) holds on the set of ε-optimal solutions to problem (3.11) for all ε sufficiently small and that *either* φ_0 is SNEC, *or* Ξ is SNC on this set.

To proceed further, we need to represent the basic normal cone $N(\cdot;\Xi)$ in (4.9) and (4.11) in terms of the initial data of MPEC (1.7) and also to express the SNC condition for Ξ via requirements imposed on the initial data of (1.7). It can be done by using efficient rules of *generalized differential and SNC calculi* developed in [16], both of which are based on the *extremal principle* of variational analysis.

Indeed, by the *intersection rule* for basic normals from [16, Corollary 3.5] we have for the set intersection Ξ in (3.9) that

$$\begin{aligned} N\big((x,y,a,z);\Xi\big) \subset \ & \sum_{i=1}^{m+r} N\big((x,y,a,z);\Omega_i\big) + N\big((x,y,a,z);\Omega_G\big) \\ & + N\big((x,y,a,z);\Omega_Q\big) + N\big((x,y,a,z);\Omega_\Omega\big) \end{aligned}$$

provided that *all but one* of the set Ω_i, $i=1,\ldots,m+r$, Ω_G, Ω_Q, and Ω are SNC at (x,y,a,z) and the qualification condition

$$\begin{aligned} & \Big[w_i^* \in N(w;\Omega_i),\ i=1,\ldots,m+r, \quad w_G^* \in N(w;\Omega_G), \quad w_Q^* \in N(w;\Omega_Q), \\ & w_\Omega^* \in N(w;\Omega_\Omega), \quad \sum_{i=1}^{m+r} w_i^* + w_G^* + w_Q^* + w_\Omega^* = 0\Big] \\ & \Longrightarrow w_i^* = w_G^* = w_Q^* = w_\Omega^* = 0 \end{aligned}$$

is satisfied for $w=(x,y,a,z)$ from above. By the set structures in (3.9) and the coderivative definition in (2.3) we can easily conclude that the latter qualification condition reduces to the one formulated in part (b) of the theorem. On the other hand, by [16, Corollary 3.81] the intersection set Ξ is SNC at (x,y,a,z) if *all* sets Ω_i, $i=1,\ldots,m+r$, Ω_G, Ω_Q, and Ω are SNC at the point under the validity of the qualification condition (b).

Combining this with (4.9)–(4.11), taking into account the particular structures of the sets in (3.9), and adjusting the corresponding notation, we arrive at the suboptimality conditions (4.6)–(4.8) and thus complete the proof of the theorem. The reader can easily reproduce all the corresponding details. ■

Remark 4.3. (specifications of strong suboptimality conditions under additional assumptions). If for some $i \in \{1,\ldots,m\}$ the function φ_i is *continuous* at the points in question, then without loss of generality we can let $\alpha_i = \varphi_i(x,y)$ in the qualification condition (b) of Theorem 4.2 and $\widehat{\alpha}_i = \varphi_i(\widehat{x},\widehat{y})$ in suboptimality conditions (4.8) for the corresponding inequality constraint. In this case the coderivative terms in (b) reduces to either $\lambda_i\partial\varphi_i(x,y)$ for $\lambda_i > 0$ or $\partial^\infty\varphi_i(x,y)$ for $\lambda_i = 0$, and similarly in (4.8); see the formulas in (2.7) justifying these representations. Furthermore, if all φ_i, $i=1,\ldots,m+r$, are *Lipschitz continuous* around the points in question, then the coderivative conditions corresponding to the inequality constraints in (4.8) can be equivalently replaced by

$$\widehat{\lambda}_i \partial\varphi_i(\widetilde{x},\widetilde{y}) \ \text{ with } \ \widehat{\lambda}_i \geq 0, \quad i = 1,\ldots,m,$$

while the coderivative conditions for the equality constraints can be replaced by

$$|\widehat{\lambda}_i|\big[\partial\varphi_i(\widetilde{x},\widetilde{y}) \cup \partial\big(-\varphi_i\big)(\widetilde{x},\widetilde{y})\big], \quad i = m+1,\ldots,m+r;$$

similarly for the qualification condition (b) in Theorem 4.2. Moreover, the sets $\operatorname{epi}\varphi_i$ for $i = 1,\ldots,m$ and $\operatorname{gph}\varphi_i$ for $i = m+1,\ldots,m+r$ are automatically SNC in this setting. Thus we get back to [17, Theorem 5.30] established for Lipschitzian functional constraints with no constraints of the equilibrium type.

Remark 4.4. (implementation and applications of strong suboptimality conditions). The strong suboptimality conditions obtained in Theorem 4.2 in terms of our basic/limiting normals, subgradients, and coderivatives can be applied to a broad range of problems with specific structures due to full calculus available for them and due to efficient computing these constructions in numerous settings important for applications; see [16, 17] for more results, discussions, and examples. Actually, there is no much difference between implementation and applications of necessary *optimality* conditions for MPECs (see, e.g., [1, 2, 7, 17, 29] and their references) and the *strong suboptimality* conditions established in this paper. We can particularly handle in this way complementarity problems, variational inequalities and their extensions, problems of bilevel programming, etc., by using *second-order subdifferentials* of extended-real-valued functions as in [1, 17].

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