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ON WEAK TYPE BOUNDS FOR A FRACTIONAL INTEGRAL ASSOCIATED WITH THE BESSEL DIFFERENTIAL OPERATOR

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Abstract. In this study we show that $I_{\Omega,\alpha,v}$ and $M_{\Omega,\alpha,v}$, the fractional integral and maximal operators the generalized shift operator generated by Bessel differential operator respectively, are bounded operators from $L_{1,v}\left(|x|^{\frac{\beta(n+2|v|-\alpha)}{n+2|v|}},\right.$ $\left.\mathbb{R}_n^+\right)$ to $L_{\frac{n+2|v|}{n+2|v|-\alpha},\infty}\left(|x|^{\beta},\mathbb{R}_n^+\right)$ where $0<\alpha<n+2|v|$, $v=(v_1,...,v_n)$, $v_1>0,...,v_n>0, |v|=v_1+...+v_n$and $-1<\beta<0$.

1. Introduction

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x=(x_1,x_2,...,x_n)$, $y=(y_1,y_2,...,y_n)$ are vectors in $\mathbb{R}^n$, $x.y=x_1y_1+...+x_ny_n,\ |x|=(x.x)^{\frac{1}{2}}$,

$$\mathbb{R}_n^+=\{x:\ x=(x_1,...,x_n),\ x_1>0,...,x_n>0\},$$

and

$$B_r^+=\{x: x\in\mathbb{R}_n^+\ |x|\leq r\}.$$

is the sphere centered at origin with radius r.

The Bessel differential operator is defined by

$$B_i=\frac{\partial^2}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i},\ \ i=1,2,...,n,$$

$v=(v_1,...,v_n),\ v_1>0,...,v_n>0,|v|=v_1+...+v_n.$

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$L_{p,v}=L_{p,v}(\mathbb{R}_n^+)$ is defined with respect to the Lebesgue measure $\left(\prod\limits_{i=1}^{n} x_i^{2v_i}\right)dx$ the following

$$L_{p,v}=L_{p,v}(R_n^+)=\left\{f:\ \|f\|_{p,v}=\left(\int\limits_{\mathbb{R}_n^+}|f(x)|^p\left(\prod_{i=1}^{n}x_i^{2v_i}\right)dx\right)^{\frac{1}{p}}<\infty\right\},$$

where $1\leq p<\infty,\ \frac{1}{p}+\frac{1}{p'}=1.\ |B_r^+|=\int\limits_{B_r^+}\left(\prod\limits_{i=1}^{n}x_i^{2v_i}\right)dx=Cr^{n+2|v|}.$

Denote by T_x^y the generalized shift operator acting according to the law

$$\mathrm{T}_x^y f(x)=\mathrm{C}_v\int\limits_0^{\pi}\ldots\int\limits_0^{\pi}f\left(\sqrt{x_1^2+y_1^2-2x_1y_1\cos\alpha_1},\ldots,\sqrt{x_n^2+y_n^2-2x_ny_n\cos\alpha_n}\right)$$
$$\prod_{i=1}^{n}(\sin^{2v_i-1}\alpha_i d\alpha_i)$$

where $x,y\in\mathbb{R}_n^+$, $C_v=\prod\limits_{i=1}^{n}\dfrac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$ [3, 7]. Let f be in $L_{p,v}(\mathbb{R}_n^+),1\leq p<\infty$. Then T_x^yf belongs to $L_{p,v}(\mathbb{R}_n^+)$, and

$$\|T_x^yf\|_{p,v}\leq\|f\|_{p,v}\,.$$

We remark that T_x^y is closely connected with the Bessel differential operator [4].

$$\frac{d^2U}{dx_i^2}+\frac{2v_i}{x_i}\frac{dU}{dx_i}=\frac{d^2U}{dy_i^2}+\frac{2v_i}{y_i}\frac{dU}{dy_i}$$
$$U(x_i,0)=f(x_i)$$
$$U_{y_i}(x_i,0)=0$$

where $x_i>0,y_i>0$, $v_i>0$ and $i=1,2,\ldots,n$.

The convolution operator determined by the T_x^y is defined by

$$(f*\varphi)(x)=\int\limits_{\mathbb{R}_n^+}f(y)T_x^y\varphi(x)(\prod_{i=1}^{n}y_i^{2v_i})dy$$

This convolution known as a B-convolution. We note the following properties of the B-convolution and T^y [4,7].

(a) $f*\varphi=\varphi*f$

(b) $\|f*\varphi\|_{r,v} \le \|f\|_{p,v}\,\|\varphi\|_{q,v}$ $1\le p\,,r\le\infty$, $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$

(c) $T_x^y.1=1,\ T_x^y f(x)=T_y^x f(x)$

(d) If $f(x),g(x)\in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function all $x>0$ and

$$\int\limits_{\mathbb{R}_n^+}|f\,(x)|\,(\prod_{i=1}^{n}x_i^{2v_i})dx<\infty$$

then

$$\int\limits_{\mathbb{R}_n^+} T_x^y f(x)g(y)(\prod_{i=1}^{n}y_i^{2v_i})dy=\int\limits_{\mathbb{R}_n^+} f(y)T_x^y g(x)(\prod_{i=1}^{n}y_i^{2v_i})dy$$

$e.$ $|T_x^y f(x)|\le \sup\limits_{x\ge 0}|f(x)|$.

Suppose that $0<\alpha<n+2\,|v|$, and $\Omega\in L_{p,v}(B_1^+)$ $(p\ge 1)$, where B_1^+ denotes the unit sphere of $\mathbb{R}_n^+$. Moreover, Ω is homogeneous for degree zero. In this work, we define the fractional maximal operator by

$$M_{\Omega,\alpha.v}f(x)=\sup_{r>0}\frac{1}{r^{n+2|v|-\alpha}w(n,v)}\int\limits_{\mathbf{B}_r^+} f(y)T_x^y\Omega(x)(\prod_{i=1}^{n}y_i^{2v_i})dy$$

where

$$w(n,v)=\int\limits_{\mathbf{B}_1^+}(\prod_{i=1}^{n}x_i^{2v_i})dx.$$

and the fractional integral operator by

$$I_{\Omega,\alpha,v}f(x)=\int\limits_{\mathbb{R}_n^+} f(y)T_x^y\left[|x|^{\alpha-n-2|v|}\,\Omega(x)\right](\prod_{i=1}^{n}y_i^{2v_i})dy.$$

The classical fractional integral $I_{\Omega,\alpha}$ is defined by

$$I_{\Omega,\alpha}f(x)=\int\limits_{\mathbb{R}_n^+}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy,\quad 0<\alpha<n.$$

In 1971, B.Muckenhoupt and R.Wheeden [5] studied the weighted norm inequalities for the classical integral $I_{\Omega,\alpha}$ with power weight. In 1993, S.Chanillo, D.Watson and R. Wheeden [1] proved that the classical $I_{\Omega,\alpha}$ fractional integral operator is of weak type $(1,\frac{n}{n-\alpha})$ under the restriction of $s\ge\frac{n}{n-\alpha}$. On the other hand, in

1997, Y.Ding [2] studied the idea in [6] to extend the result about weighted weak type $(1, \frac{n}{n-\alpha})$ for classical $I_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ to power weights. The $I_{\Omega,\alpha,v}$ fractional integral operator generated by the generalized shift operator is the generalized Riesz potential for $\Omega(x-y) = const$. The boundedness of this potential from $L_{p,v}(\mathbb{R}_n^+)$ to $L_{q,v}(\mathbb{R}_n^+)$ was proved for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n+2|v|}$ [7-9]. Moreover, H. Yíldírím and M. Z. Sarikaya studied the boundedness of the derivatives of the this potential [8].

Using the harmonic analysis associated with the Bessel operator (generalized translation operators, generalized convolution, Fourier-Bessel transform, etc.) and the same idea as for the classical case, we defined and study in this paper the fractional integral operator with the Bessel operator. In this paper, we will study the weighted weak type $(1, \frac{n+2|v|}{n+2|v|-\alpha})$ for the $I_{\Omega,\alpha,v}$ fractional integral operator generated by the generalized shift operator. Finaly, we show that the $I_{\Omega,\alpha,v}$ and $M_{\Omega,\alpha,v}$ are bounded operators $L_{1,v}\left(|x|^{\frac{\beta(n+2|v|-\alpha)}{n+2|v|}}, \mathbb{R}_n^+\right)$ to $L_{\frac{n+2|v|}{n+2|v|-\alpha},\infty}\left(|x|^{\beta}, \mathbb{R}_n^+\right)$ where $0<\alpha<n+2|v|$, $v=(v_1,...,v_n)$, $v_1>0,...,v_n>0$, $|v|=v_1+...+v_n$ and $-1<\beta<0$.

2. Main Results

This part is the main part of the article. In this part we prove some main theorems for the fractional integral and maximal operators. Before giving these theorems we will give lemmas with proof for the sake of use the proof of theorems.

Lemma 2.1. *Let $x, y \in \mathbb{R}^+$. In this case there is the following inequality for the generalized shift operator*

$$|x-y|^2 \leq x^2 + y^2 - 2xy\cos\theta \leq (x+y)^2$$

where $\theta \in [0,\pi]$.

Lemma 2.2. *Let $f \in L_{p,v}(\mathbb{R}_n^+)$. There is the following inequality*

$$|T_x^y f(x)|^p \leq T_x^y |f(x)|^p \ , \quad \text{for} \quad \frac{1}{p} + \frac{1}{p'} = 1 \ , \ \ 1 \leq p < \infty.$$

Proof. Let

$$F(\alpha,x,y) = f(\sqrt{x_1^2 + y_1^2 - 2x_1y_1\cos\alpha_1}, ..., \sqrt{x_n^2 + y_n^2 - 2x_ny_n\cos\alpha_n})$$

From Hölder's inequality, we get

$$\begin{aligned} |T_x^y f(x)|^p &= \left| C_v \int_0^\pi ... \int_0^\pi F(\alpha,x,y) \prod_{i=1}^n (\sin^{2v_i-1}\alpha_i d\alpha_i) \right|^p \\ &\leq \left[\left(C_v \int_0^\pi ... \int_0^\pi |F(\alpha,x,y)|^p \prod_{i=1}^n (\sin^{2v_i-1}\alpha_i d\alpha_i) \right) \right] \end{aligned}$$

$$\left[\left(C_v \int_0^{\pi} ... \int_0^{\pi} \prod_{i=1}^{n} (\sin^{2v_i-1} \alpha_i d\alpha_i)\right)^{\frac{1}{p'}}\right]^p$$
$$\leq T_x^y\left(|f(x)|^p\right).$$

Lemma 2.3. *Let* $f \in L_{p,v}(\mathbb{R}_n^+)$, $1 \leq p < \infty$. *Then we have*

$$\|T_x^y f\|_{p,v} \leq \|f\|_{p,v}\,.$$

Proof. From Lemma 2.2 there is the following inequality

$$\begin{aligned}\|T_x^y f\|_{p,v}^p &= \int_{\mathbb{R}_n^+} |T_x^y f(x)|^p \,(\prod_{i=1}^{n} y_i^{2v_i})dy \\ &\leq \int_{\mathbb{R}_n^+} T_x^y\left(|f(x)|^p\right)(\prod_{i=1}^{n} y_i^{2v_i})dy.\end{aligned}$$

If we consider the properties (**c**) and (**d**) of the operator T^y, then we have the following inequality

$$\|T_x^y f\|_{p,v} \leq \left(\int_{\mathbb{R}_n^+} |f(y)|^p \,(\prod_{i=1}^{n} y_i^{2v_i})dy\right)^{\frac{1}{p}} = \|f\|_{p,v}\,.$$

Lemma 2.4. *If* $f, g \in L_{p,v}(\mathbb{R}_n^+)$ *and* f *is a increase function, then*

$$T_x^y\left[fg\right](x) \leq f(x-y)T_x^y g(x).$$

Theorem 2.1. *Let* $0 < \alpha < n + 2\,|v|\,,\ 1 \leq p < \infty$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n+2|v|}$. *If* $f \in L_{p,v}(\mathbb{R}_n^+)$, *then the integral* $I_{\Omega,\alpha,v}f$ *is absolutely convergent almost for everywhere.*

Proof. Let us write $K(x) = |x|^{\alpha-n-2|v|}$ and decompose K as $K_1 + K_\infty$, for arbitrary $\mu > 0$,

$$(1)\qquad K_1(x) = \begin{cases} K(x), & \text{if } |x| \leq \mu \\ 0, & \text{if } |x| > \mu \end{cases}, \quad K_\infty(x) = \begin{cases} 0, & \text{if } |x| \leq \mu \\ K(x), & \text{if } |x| > \mu \end{cases}.$$

Thus we write $K * f$ convolution as follow

$$K * f = (K_1 + K_\infty) * f = (K_1 * f) + (K_\infty * f).$$

Here it is seen that the integral $K_1 * f$ is a convolution of functions K_1 in $L_{1,v}$, and f in $L_{p,v}$. Hence from Minkowsky inequality for integrals, (c), (d), Lemma 2.2, and Lemma 2.3, we have the following inequality

$$\begin{aligned}
&\|K_1 * f\|_{p,v} \\
&\leq \left(\int_{\mathbb{R}_n^+} \left[\int_{|y|\leq\mu} |f(y)T_x^y [K_1(x)\Omega(x)]| \, (\prod_{i=1}^n y_i^{2v_i})dy \right]^p (\prod_{i=1}^n x_i^{2v_i})dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}_n^+} \left[\int_{|y|\leq\mu} |[K_1(y)\Omega(y)] \, T_x^y f(x)| \, (\prod_{i=1}^n y_i^{2v_i})dy \right]^p (\prod_{i=1}^n x_i^{2v_i})dx \right)^{\frac{1}{p}} \\
&\leq \int_{|y|\leq\mu} K_1(y)\Omega(y) \left[\int_{\mathbb{R}_n^+} |T_x^y [f(x)]|^p \, (\prod_{i=1}^n x_i^{2v_i})dx \right]^{\frac{1}{p}} (\prod_{i=1}^n y_i^{2v_i})dy \\
&\leq \int_{|y|\leq\mu} K_1(y)\Omega(y) \left[\int_{\mathbb{R}_n^+} |f(x)|^p \, (\prod_{i=1}^n x_i^{2v_i})dx \right]^{\frac{1}{p}} (\prod_{i=1}^n y_i^{2v_i})dy \\
&= \|K_1\Omega\|_{1,v} \|f\|_{p,v} < \infty.
\end{aligned} \tag{2}$$

Thus $K_1 * f \in L_{p,v}$. Similarly, the integral $K_\infty * f$ is a convolution of function f in $L_{p,v}$ and any function in $L_{p',v}$ dual space. As a consequence, since $(\alpha - n - 2|v|)p' \leq -n - 2|v|$ is equivalent to $q < \infty$, we have the following inequality

$$\|K_\infty * f\|_{\infty,v} \leq \|K_\infty\Omega\|_{p',v} \|f\|_{p,v} .$$

Since $\|f\|_{p,v} < \infty$, we only need to show that $\|K_\infty\Omega\|_{p',v} < \infty$. In fact, using $(\alpha - n - 2|v|)p' \leq -n - 2|v|$ and spherical coordinates we have

$$\begin{aligned}
\|K_\infty\Omega\|_{p',v} &= \left(\int_{|x|>\mu} |x|^{(\alpha-n-2|v|)p'} |\Omega(x)|^{p'} \, (\prod_{i=1}^n x_i^{2v_i})dx \right)^{\frac{1}{p'}} \\
&= C \|\Omega(\theta)\|_{p',v} \left(r^{(\alpha-n-2|v|)p'+n+2|v|} \, |_\mu^\infty \right)^{\frac{1}{p'}} < \infty.
\end{aligned}$$

where $\|\Omega(\theta)\|_{p',v} = \int_{B_1^+} |\Omega(\theta)| \prod_{i=1}^n (\sin^{2v_i-1}\theta_i d\theta_i)$. Therefore $\|K_\infty * f\|_{\infty,v}$ is absolutely convergent. This finishes the proof of Theorem 2.1.

Theorem 2.2. *Let $1 \leq p < q < \infty$ and $f \in L_{p,v}$. Then the inequality*

$$\|I_{\Omega,\alpha,v}f\|_{q,v} \leq C_\alpha(p,q,v) \|f\|_{p,v} , \ 0 < \alpha < n + 2|v|$$

has been hold if and only if

$$\alpha = (n + 2|v|)(\frac{1}{p} - \frac{1}{q}).$$

Therefore, in the case of $p = 1$, *the integral* $I_{\Omega,\alpha,v}$ *is of weak type* $(1, q)$. *That is, if* $f \in L_{1,v}$, *for arbitrary positive* λ,

$$\mathrm{mes}\{x : \ |I_{\Omega,\alpha,v}f| > \lambda\} \ \leq \ \left(\frac{C_q \, \|f\|_{1,v}}{\lambda}\right)^q$$

where

$$\mathrm{mes}E = \int\limits_E (\prod_{i=1}^n x_i^{2v_i})dx, \ E \subset \mathbb{R}_n^+.$$

Proof. We assume that the following inequality holds

$$\|I_{\Omega,\alpha,v}f\|_{q,v} \ \leq \ C_\alpha(p, q, v)\, \|f\|_{p,v}\, .$$

Let us show that

$$\alpha = (n + 2\,|v|)(\frac{1}{p} - \frac{1}{q}).$$

Let γ_ϱ be the dilation by the factor ϱ, $\varrho > 0$, that is $(\gamma_\varrho f)(x) = f(\varrho x)$. Then we have the following equalities

$$\textbf{I.} \quad \gamma_{\varrho^{-1}}[I_{\Omega,\alpha,v}\gamma_\varrho f](x) = \varrho^{-\alpha} I_{\Omega,\alpha,v}f(x)$$

$$\textbf{II.} \quad \|\gamma_\varrho f\|_{p,v} = \varrho^{\frac{-n-2|v|}{p}} \|f\|_{p,v}$$

$$\textbf{III.} \quad \left\|\gamma_{\varrho^{-1}} I_{\Omega,\alpha,v}f\right\| = \varrho^{\frac{n+2|v|}{q}} \|I_{\Omega,\alpha,v}f\|_{q,v}$$

[7]. From **I, II, III,** and our assumption we get

$$\begin{aligned}\|\varrho^{-\alpha} I_{\Omega,\alpha,v}f\|_{q,v} &= \left\|\gamma_{\varrho^{-1}}[I_{\Omega,\alpha,v}\gamma_\varrho f]\right\|_{q,v} \\ &= \varrho^{\frac{n+2|v|}{q}} \|I_{\Omega,\alpha,v}\gamma_\varrho f\|_{q,v} \\ &\leq C_\alpha(p, q, v)\varrho^{\frac{n+2|v|}{q}} \|\gamma_\varrho f\|_{p,v}\, .\end{aligned}$$

Thus we obtain $\|I_{\Omega,\alpha,v}f\|_{q,v} \leq C_\alpha(p, q, v)\varrho^\alpha \varrho^{\frac{n+2|v|}{q}} \varrho^{\frac{-n-2|v|}{p}} \|f\|_{p,v}$. This result necessitate

$$\alpha = (n + 2\,|v|)(\frac{1}{p} - \frac{1}{q}).$$

Now let us assume that $\alpha = (n + 2\,|v|)(\frac{1}{p} - \frac{1}{q})$ where $1 < p < q < \infty$. We show that inequality

$$\|I_{\Omega,\alpha,v}f\|_{q,v} \ \leq \ C_\alpha(p, q, v)\, \|f\|_{p,v}$$

is valid for $f \in L_{p,v}$. Substituting μ with $\mu^* = \mu \|f\|_{p,v}^{\frac{q}{n+2|v|}} > 0$ in (1), then we obtain

$$
\begin{aligned}
\|K_1\Omega\|_{1,v} &= \int\limits_{|x|<\mu} |x|^{\alpha-n-2|v|} |\Omega(x)| (\prod_{i=1}^{n} x_i^{2v_i})dx = C_1(\mu^*)^{\alpha} \\
\|K_\infty\Omega\|_{p',v} &= \left(\int\limits_{|x|\geq\mu} |x|^{(\alpha-n-2|v|)p'} |\Omega(x)|^{p'} (\prod_{i=1}^{n} x_i^{2v_i})dx \right)^{\frac{1}{p'}} \\
&= C_2(\mu^*)^{-\frac{n+2|v|}{q}}
\end{aligned} \tag{3}
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. On the other hand, let us show that the mapping $f \to K * f$ is of type (p, q). That is, for $f \in L_{p,v}(\mathbb{R}_n^+)$ and arbitrary positive λ,

$$
mes\{x : |K * f| > \lambda\} \leq \left(\frac{C(p,q,v) \|f\|_{p,v}}{\lambda} \right)^q
$$

where $1 \leq p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n+2|v|}$.

Now for any positive λ we write

$$
mes\{x : |K * f| > 2\lambda\} < mes\{x : |K_1 * f| > \lambda\} + mes\{x : |K_\infty * f| > \lambda\}.
$$

Then we have

$$
\begin{aligned}
\|K_\infty * f\|_{\infty,v} &\leq \|K_\infty\Omega\|_{p',v} \|f\|_{p,v} \\
&= C_2(\mu^*)^{-\frac{n+2|v|}{q}} \|f\|_{p,v} \\
&= C_2(\mu)^{-\frac{n+2|v|}{q}}.
\end{aligned}
$$

Here if we take $C_2(\mu)^{-\frac{n+2|v|}{q}} = \lambda$, then $\|K_\infty * f\|_{\infty,v} < \lambda$. Therefore, we obtain $mes\{x : |K_\infty * f| > \lambda\} = 0$. On the other hand, we have the following inequality from (2)

$$
\begin{aligned}
\text{mes}\{x : |K * f| > 2\lambda\} &\leq \frac{\|K_1 * f\|_{p,v}^p}{\lambda^p} \\
&< \frac{\|K_1\Omega\|_{1,v}^p \|f\|_{p,v}^p}{\lambda^p}.
\end{aligned}
$$

Since $C_2(\mu)^{-\frac{n+2|v|}{q}} = \lambda$, there is equality $\mu^* = C_3(\lambda)^{-\frac{q}{n+2|v|}} \|f\|^{-\frac{q}{n+2|v|}}$. From

$\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n+2|v|}$ and (3) we have

$$\begin{aligned}\operatorname{mes}\{x:\ |K*f|>2\lambda\} &\leq \left(C_4(\lambda)^{-\frac{q\alpha}{n+2|v|}}\|f\|_{p,v}\,\lambda^{-1}\,\|f\|_{p,v}^{-\frac{q\alpha}{n+2|v|}}\right)^q \\ &= \left(C_4(\lambda)^{-(1+\frac{q\alpha}{n+2|v|})}\,\|f\|_{p,v}^{1+\frac{q\alpha}{n+2|v|}}\right)^q \\ &= \left(\frac{C(p,q,v)\,\|f\|_{p,v}}{\lambda}\right)^q.\end{aligned}$$

This means that the mapping $f\to K*f$ is of type (p,q). If $p=1$ and $\frac{1}{q}=1-\frac{\alpha}{n+2|v|}$, then

$$\operatorname{mes}\{x:\ |K*f|>\lambda\}\ \leq\ \left(\frac{C(p,q,v)\,\|f\|_{1,v}}{\lambda}\right)^q.$$

That is, the mapping $f\to K*f$ is of weak type $(1,q)$.

According to Marcinkiewcz interpolation theorem, a strong type is obtained from simultaneous two weak types. As a consequence of this theorem, we have the following inequality

$$\|I_{\Omega,\alpha,v}f\|_{q,v}\leq C_\alpha(p,q,v)\,\|f\|_{p,v}\,.$$

This is also proof of second part of the theorem. Here, using with weak types $(p_0,q_0)=(1,q_0)$ and (p,q_1), we obtain

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n+2\,|v|}$$

from the following equalities

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}=1-\theta+\frac{\theta}{p_1},\qquad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$$
$$\frac{1}{q_0}=1-\frac{\alpha}{n+2\,|v|},\qquad \frac{1}{q_1}=\frac{1}{p_1}-\frac{\alpha}{n+2\,|v|}$$

Therefore, the integral $I_{\Omega,\alpha,v}f$ is of strong type (p,q).

Lemma 2.5. *Let $q>1$ and Λ be a sublinear operator satisfying for each $a>0$ the estimate*

$$\left|\left\{\tfrac{a}{2}\leq |x|\leq a:\ \left|\Lambda(f\chi_{\{|x|>2a\}})(x)\right|>\lambda\right\}\right|$$
(4)
$$\leq C\left(\frac{1}{\lambda}\int\limits_{|y|>2a}|f(y)|\left(\frac{a}{|y|}\right)^{\frac{1}{q}}(\prod_{i=1}^n y_i^{2v_i})dy\right)^q.$$

Then, if Λ *is of weak type* $(1,q)$*, it is also of weak type* $\left(L_{1,v}(|x|^{\frac{\beta}{q}}), L_{q,\infty}(|x|^{\beta})\right)$ *for* $-1<\beta<0$.

Proof. Given f, we now define, for each $k\in\mathbb{Z}$, $f_{k,0}=f\chi_{\{|x|\leq 2^{k+1}\}}$ and $f_{k,1}=f-f_{k,0}$. Then we can write, as usual,

$$\begin{aligned}|\Lambda f(x)| &\leq \sum_k |\Lambda f_{k,0}|\,\chi_{A_k}+\sum_k |\Lambda f_{k,1}|\,\chi_{A_k}\\ &= \Lambda_0 f(x)+\Lambda_1 f(x)\end{aligned}$$

where $A_k=\{x\in\mathbb{R}_n^+:\ 2^{k-1}\leq |x|<2^k\}$ for each $k\in\mathbb{Z}$.

If we call $w_\beta(x)=|x|^\beta$, we have

$$\begin{aligned}w_\beta\{x:\ \Lambda_0 f>\lambda\} &= \int\limits_{\{x:\,\Lambda_0 f>\lambda\}} w_\beta(x)(\prod_{i=1}^n x_i^{2v_i})dx\\
&\leq C\sum_k w_\beta(2^k)\,|\{x\in A_k:\ \Lambda_0 f_{k,0}>\lambda\}|\\
&\leq \frac{C}{\lambda^q}\sum_k w_\beta(2^k)\left(\int\limits_{|y|\leq 2^{k+1}} |f(y)|\,(\prod_{i=1}^n y_i^{2v_i})dy\right)^{\frac{1}{q}}\\
&= \frac{C}{\lambda^q}\sum_k w_\beta(2^k)\left(\sum_{j\leq k+1}\int\limits_{A_j} |f(y)|\,(\prod_{i=1}^n y_i^{2v_i})dy\right)^{\frac{1}{q}}\\
&\leq \frac{C}{\lambda^q}\left\{\sum_j\left[\left(\sum_{k\geq j-1}\int\limits_{A_j} |f(y)|\,(\prod_{i=1}^n y_i^{2v_i})dy\right)^q w_\beta(2^k)\right]^{\frac{1}{q}}\right\}^q\\
&= \frac{C}{\lambda^q}\left\{\sum_j\int\limits_{A_j} |f(y)|\,(\prod_{i=1}^n y_i^{2v_i})dy\left[\sum_{k\geq j-1} w_\beta(2^k)\right]^{\frac{1}{q}}\right\}^q\\
&\leq \frac{C}{\lambda^q}\left\{\sum_j\int\limits_{A_j} |f(y)|\,w_\beta(2^j)^{\frac{1}{q}}(\prod_{i=1}^n y_i^{2v_i})dy\right\}^q\\
&\leq C\left\{\frac{1}{\lambda}\int\limits_{\mathbb{R}_n^+} |f(y)|\,w_\beta(y)^{\frac{1}{q}}(\prod_{i=1}^n y_i^{2v_i})dy\right\}^q\end{aligned}$$

Here we have used that Λ is a weak type $(1,q)$ bounded operator and $\beta<0$. In order to estimate Λ_1, we make use of (4):

$$
\begin{aligned}
w_\beta\{x: \Lambda_1 f > \lambda\} &= \int\limits_{\{x:\, \Lambda_1 f > \lambda\}} w_\beta(x)(\prod_{i=1}^{n} x_i^{2v_i})dx \\
&\leq C \sum_k w_\beta(2^k) \left|\{x \in A_k : \Lambda_1 f_{k,0} > \lambda\}\right| \\
&\leq \frac{C}{\lambda^q} \sum_k w_\beta(2^k) \left(\int\limits_{|y|>2^{k+1}} |f(y)| \, (\frac{2^k}{|y|})^{\frac{1}{q}} (\prod_{i=1}^{n} y_i^{2v_i}) dy \right)^q \\
&= \frac{C}{\lambda^q} \sum_k w_\beta(2^k) 2^k \left(\sum_{j \geq k} \int\limits_{A_j} |f(y)| \, (\frac{1}{|y|})^{\frac{1}{q}} (\prod_{i=1}^{n} y_i^{2v_i}) dy \right)^{\frac{1}{q}} \\
&\leq \frac{C}{\lambda^q} \left\{ \sum_j \int\limits_{A_j} |f(y)| \, (\frac{1}{|y|})^{\frac{1}{q}} (\prod_{i=1}^{n} y_i^{2v_i}) dy \left[\sum_{k \geq j-1} w_\beta(2^k) 2^k \right]^{\frac{1}{q}} \right\}^q \\
&\leq \frac{C}{\lambda^q} \left\{ \sum_j \int\limits_{A_j} |f(y)| \, (\frac{1}{|y|})^{\frac{1}{q}} (\prod_{i=1}^{n} y_i^{2v_i}) dy w_\beta(2^j)^{\frac{1}{q}} 2^{\frac{j}{q}} \right\}^q \\
&\leq C \left\{ \frac{1}{\lambda} \int\limits_{\mathbb{R}_n^+} |f(y)| \, w_\beta(y)^{\frac{1}{q}} (\prod_{i=1}^{n} y_i^{2v_i}) dy \right\}^q .
\end{aligned}
$$

where we have used that $\beta > -1$. This finishes the proof of Lemma 2.5.

Lemma 2.6. *Let* $0 < \alpha < n + 2|v|$, $\Omega \in L_s(B_1^+)$ *and* $s \geq 1$. *Then there is a* $C > 0$ *depending only on* n, α *and* v *such that*

$$M_{\Omega,\alpha,v} f(x) \;\; \leq \;\; C I_{|\Omega|,\alpha,v}(|f|)(x). \tag{5}$$

Proof. Fix $r > 0$, then we have from (c) and (d) of the operator T^y

$$
\begin{aligned}
I_{|\Omega|,\alpha,v}(|f|)(x) &= \int\limits_{|y|<r} |f(y)| \, T_x^y \left[|x|^{\alpha-n-2|v|} \, |\Omega(x)| \right] (\prod_{i=1}^{n} y_i^{2v_i}) dy \\
&= \int\limits_{|y|<r} |y|^{\alpha-n-2|v|} \, |\Omega(y)| \, T_x^y \, |f(x)| \, (\prod_{i=1}^{n} y_i^{2v_i}) dy \\
&\geq \frac{1}{r^{\alpha-n-2|v|}} \int\limits_{|y|<r} |\Omega(y)| \, T_x^y \, |f(x)| \, (\prod_{i=1}^{n} y_i^{2v_i}) dy \\
&= \frac{1}{r^{\alpha-n-2|v|}} \int\limits_{|y|<r} |f(y)| \, T_x^y \, |\Omega(x)| \, (\prod_{i=1}^{n} y_i^{2v_i}) dy
\end{aligned} \tag{6}
$$

Taking the supremum for $r > 0$ on two sides of (6), we get

$$I_{|\Omega|,\alpha,v}(|f|)(x) \geq \sup_{r>0} \frac{1}{r^{\alpha-n-2|v|}} \int\limits_{|y|<r} |f(y)| \, T_x^y \, |\Omega(x)| \, (\prod_{i=1}^{n} y_i^{2v_i}) dy.$$

This is (5).

Theorem 2.3. *Let* $0 < \alpha < n + 2\,|v|\,, -1 < \beta < 0,\ \frac{n+2|v|}{n+2|v|-\alpha} \leq s \leq \infty$ *and* $f \in L_s(B_1^+)$*. Then* $I_{\Omega,\alpha,v}$ *is a bounded operator from* $L_{1,v}\left(|x|^{\frac{\beta(n+2|v|-\alpha)}{n+2|v|}}, \mathbb{R}_n^+\right)$ *to* $L_{\frac{n+2|v|}{n+2|v|-\alpha},\infty}\left(|x|^{\beta}, \mathbb{R}_n^+\right)$*. That is, for any* $\lambda > 0$ *and any* $f \in L_{1,v}\left(|x|^{\frac{\beta(n+2|v|-\alpha)}{n+2|v|}},\right.$ $\left.\mathbb{R}_n^+\right)$,

$$\int\limits_{\{x:\ |I_{\Omega,\alpha,v} f| > \lambda\}} |x|^{\beta} \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx$$
$$\leq C \left(\frac{1}{\lambda} \int\limits_{\mathbb{R}_n^+} |f(x)| \, |x|^{\frac{\beta(n+2|v|-\alpha)}{n+2|v|}} \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx\right)^{\frac{n+2|v|}{n+2|v|-\alpha}}$$

where C *is independent of* λ *and* f.

Proof. Since $I_{\Omega,\alpha,v}$ is weak type $(1, \frac{n+2|v|}{n+2|v|-\alpha})$ from Theorem 2.2, by Lemma 2.3 we only need to show that $I_{\Omega,\alpha,v}$ satisfies (4) for $q = \frac{n+2|v|}{n+2|v|-\alpha}$ and any $a > 0$. In fact, from Lemma 2.1, 2.2, 2.3 and 2.4

$$\left|\left\{\frac{a}{2} \leq |x| \leq a : \ \left|\Lambda(f\chi_{\{|x|>2a\}})(x) > \lambda\right|\right\}\right|$$
$$< \frac{1}{\lambda^q} \int\limits_{|x|\leq a} \left|I_{\Omega,\alpha,v}(f\chi_{\{|x|>2a\}})(x)\right|^q (\prod_{i=1}^{n} x_i^{2v_i}) dx$$
$$= \frac{1}{\lambda^q} \int\limits_{|x|\leq a} \left[\int\limits_{|y|>2a} |f(y)| \, T_x^y \left[|x|^{\alpha-n-2|v|} \, |\Omega(x)|\right] (\prod_{i=1}^{n} y_i^{2v_i}) dy\right]^q (\prod_{i=1}^{n} x_i^{2v_i}) dx$$
$$\leq \frac{1}{\lambda^q} \left\{\int\limits_{|y|>2a} |f(y)| \left[\int\limits_{|x|\leq a} \left(T_x^y \left[|x|^{\alpha-n-2|v|} \, |\Omega(x)|\right]\right)^q (\prod_{i=1}^{n} x_i^{2v_i}) dx\right]^{\frac{1}{q}} (\prod_{i=1}^{n} y_i^{2v_i}) dy\right\}^q$$
$$\leq \frac{C}{\lambda^q} \left\{\int\limits_{|y|>2a} |f(y)| \left[\int\limits_{|x|\leq a} \frac{1}{|x-y|^{(n+2|v|-\alpha)q}} [T_x^y \, |\Omega(x)|]^q \, (\prod_{i=1}^{n} x_i^{2v_i}) dx\right]^{\frac{1}{q}} (\prod_{i=1}^{n} y_i^{2v_i}) dy\right\}^q$$

$$\leq \frac{C}{\lambda^q}\left\{\int\limits_{|y|>2a}|f(y)|\,\frac{1}{|y|^{n+2|v|-\alpha}}\left[\int\limits_{|x|\leq a}|\Omega(x)|^q\,(\prod_{i=1}^n x_i^{2v_i})dx\right]^{\frac{1}{q}}(\prod_{i=1}^n y_i^{2v_i})dy\right\}^q$$

$$\leq \frac{C}{\lambda^q}\left\{\int\limits_{|y|>2a}|f(y)|\,\frac{1}{|y|^{n+2|v|-\alpha}}\left[\int\limits_{B_1^+}\int\limits_{2|y|-a}^{2|y|+a}|\Omega(\theta)|^q\,r^{n+2|v|-1}drd\theta\right]^{\frac{1}{q}}(\prod_{i=1}^n y_i^{2v_i})dy\right\}^q$$

$$\leq \frac{C}{\lambda^q}\left\{\int\limits_{|y|>2a}|f(y)|\,\frac{1}{|y|^{n+2|v|-\alpha}}\left\|\Omega(\theta)\right\|_{q,v}\left((a\,|y|)^{n+2|v|-1}\right)^{\frac{1}{q}}(\prod_{i=1}^n y_i^{2v_i})dy\right\}^q$$

$$= C\left\|\Omega(\theta)\right\|_{q,v}^q\left\{\frac{1}{\lambda^q}\int\limits_{|y|>2a}|f(y)|\left(\frac{a}{|y|}\right)^{\frac{1}{q}}(\prod_{i=1}^n y_i^{2v_i})dy\right\}^q.$$

Thus, the conclusion of Theorem 2.3 immediately follows from Lemma 2.5.

Theorem 2.4. *Let* $0<\alpha<n+2\,|v|\,, -1<\beta<0,\ \frac{n+2|v|}{n+2|v|-\alpha}\leq s\leq\infty$ *and* $f\in L_s(B_1^+)$. *Then* $M_{\Omega,\alpha,v}$ *is a bounded operator from* $L_{1,v}\left(|x|^{\frac{\beta(n+2|v|-\alpha)}{n+2|v|}},\mathbb{R}_n^+\right)$ *to* $L_{\frac{n+2|v|}{n+2|v|-\alpha},\infty}\left(|x|^{\beta},\mathbb{R}_n^+\right)$. *That is, for any* $\lambda>0$ *and any* $f\in L_{1,v}\Big(|x|^{\frac{\beta(n+2|v|-\alpha)}{n+2|v|}}$, $\mathbb{R}_n^+\Big)$,

$$\int\limits_{\{x:\,\left|M_{\Omega,\alpha,v}f\right|>\lambda\}}|x|^{\beta}\,(\prod_{i=1}^n x_i^{2v_i})dx$$

$$\leq C\left(\frac{1}{\lambda}\int\limits_{\mathbb{R}_n^+}|f(x)|\,|x|^{\frac{\beta(n+2|v|-\alpha)}{n+2|v|}}\,(\prod_{i=1}^n x_i^{2v_i})dx\right)^{\frac{n+2|v|}{n+2|v|-\alpha}}$$

where C *is independent of* λ *and* f.

It is easy to see that proof of Theorem 2.4 is a direct consequence of Theorem 2.3 and Lemma 2.6.

References

1. S. Chanillo. D. Watson and R. L. Wheeden, Some integral and maximal operator related to starlike sets, *Studia Math.*, **107** (1993), 223-255. MR 94g:42027.

2. Y. Ding, Weak type bounds for a class of rough operators with power weights, *Proce. Amer. Math. Soc.*, **125** (1997), 2939-2942.

3. B. M. Levitan, *Generalized Translation Operators and Some of Their Applications*, Moscova 1962 (Translation 1964).

4. B. M. Levitan, Expansion in Fourier Series and Integrals with Bessel Functions, *PUspeki, Mat., Nauka (N.S) 6*, **42(2)** (1952), 102-143, (in Russian).

5. B. Murckenhoupt and R. L. Wheeden, Weighted norm inequalities for singular and fractional integrals, *Trans. Amer. Math. Soc.*, **161** (1971), 249-258.

6. F. Soria and A. Weiss, A remark on singular integrals and power weights, *Indiana Unv. Math. Jour.*, **43** (1994), 187-204.

7. H. Yíldírím, *Riesz Potentials Generated by a Generalized shift operator.* Ankara Uni. Graduate school of Natural and Applieds Sciences Department of Math. Ph.D. thesis 1995.

8. **[**8] H. Yíldírím and M. Z. Saríkaya, On the generalized Riesz type potentials, *Jour. of Inst. of Math. and Comp. Sci.*, **14(3)** (2001), 217-224.

9. H. Yíldírím and O. Akin, Riesz potentials generated by a generalised shift operator, *Bull. Calcutta Math. Soc.*, **90(2)** (1998), 157-162.

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