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# WEAK AND STRONG CONVERGENCE FOR SOME OF NONEXPANSIVE MAPPINGS

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**Abstract.** In this paper, we deal with a class of nonexpansive mappings with the property $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, as $n \to \infty$, where $D$ is the Hausdorff metric. We show that nonexpansive mappings with compact domains enjoy this property and give some examples of this kind of mappings with noncompact domains in $l^{\infty}$. Then we prove a nonlinear ergodic theorem, and a convergence theorem of mann's type for this kind of mappings.

## 1. Introduction

The first nonlinear ergodic theorem for nonexpansive mappings with bounded domains in a Hilbert space was established by Baillon [5]: Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself. If the set $F(T)$ of fixed points of $T$ is nonempty, then for each $x \in C$, the Cesaro means $S_n(x) = \frac{1}{n}\sum_{k=0}^{n} T^k x$ converge weakly to some $y \in F(T)$. Bruck [7] extended Baillon's theorem to a uniformly convex Banach space whose norm is Frechet differentiable. Before that, Edeleshtein [9] had obtained a nonlinear strong ergodic theorem for nonexpansive mappings with compact domains in a Banach space. Atsushiba and Takahashi [2] improved the Edelestein's theorem: Let $C$ be a nonempty compact convex subset of a strictly convex Banach space and let $T$ be a nonexpansive mapping of $C$ into itself. Then for each $x \in C$, the Cesaro means $S_n(x) = \frac{1}{n}\sum_{k=0}^{n} T^{k+h} x$ converge strongly to some $y \in F(T)$, uniformly in $h$.

The first purpose of this paper is to prove a nonlinear ergodic theorem for a specific class of nonexpansive mappings from a nonempty closed convex subset of a Bananch space into itself, which extends the Atsushiba and Takahashi's theorem. Our second goal is to prove a strong convergence theorem of mann's type [11] for this specific class of mappings.

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## 2. Preliminaries

Let $E$ be a real Banach space and let $C$ be a nonempty closed convex subset of $E$. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for each $x, y \in C$. We denote by $F_\varepsilon(T)$ the $\varepsilon$-approximate fixed points of $T$; i.e. $F_\varepsilon(T) = \{x \in C : \|x - Tx\| \le \varepsilon\}$. If $C$ is bounded, then $F_\varepsilon(T) \neq \varnothing$ for each $\varepsilon > 0$ (see [6]). A Banach space $E$ is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $E^*$ be the topological dual of $E$. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $x^*(x)$. The open ball of radius $r$ centered at 0 is denoted by $B_r$. For a subset $A$ of $E$, we denote by $\overline{co}A$ and $\bar{A}$ the closed convex hull and the closure of $A$, respectively. The distance from $x$ to $A$ is denoted by $dist(x, A)$. We denote by $\Gamma$ the set of all strictly increasing, continuous convex functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. For each $\gamma \in \Gamma$, a mapping $T : C \to C$ is said to be of type $(\gamma)$, if for every $x, y \in C$ and $\lambda \in [0, 1]$, $\gamma\left(\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\|\right) \le \|x - y\| - \|Tx - Ty\|$. Obviously, if $T$ is of type $(\gamma)$ for some $\gamma \in \Gamma$, then $T$ is nonexpansive and $F(T)$ is a convex set. Moreover if $C$ is also weakly compact, then $F(T) \neq \varnothing$ (see [10]). If $C$ is compact and $E$ is a strictly convex Banach space, then every nonexpansive mapping $T : C \to C$ is of type $(\gamma)$ (see [2, 7]).

## 3. Convergence to the Fixed Point Set

First, we prove a lemma which we need in the following.

**Lemma 3.1.** *Let $E$ be a locally convex space and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ be a decreasing sequence of nonempty compact subsets. Then* $\overline{co}(\bigcap_{i=1}^{\infty} A_i) = \bigcap_{i=1}^{\infty} (\overline{co}A_i)$.

*Proof.* Obviously $\overline{co}(\bigcap_{i=1}^{\infty} A_i) \subseteq \bigcap_{i=1}^{\infty}(\overline{co}A_i)$. Let $a \in \bigcap_{i=1}^{\infty}(\overline{co}A_i)$ and $a \notin \overline{co}(\bigcap_{i=1}^{\infty} A_i)$. Since $\bigcap_{i=1}^{\infty} A_i \neq \varnothing$, there exist $\varphi \in E^*$, $r \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\varphi(a) < r - \varepsilon \quad \text{and} \quad r + \varepsilon < \varphi(x) \tag{1}$$

for every $x \in \overline{co}(\bigcap_{i=1}^{\infty} A_i)$. Let $H_r = \{x \in E;\ \varphi(x) \le r\}$ and $A_i^* := A_i \bigcap H_r$ for every $i \in \mathbb{N}$. By compactness of $A_i$'s we conclude that $A_i^*$'s are compact. We show that $A_i^* \neq \varnothing$ for every $i$. To see this, let $A_j^* = \varnothing$ for one $j \in \mathbb{N}$. Then $r < \varphi(b)$ for every $b \in A_j$ and so $r \le \varphi(b)$ for every $b \in \overline{co}A_j$. But $a \in \bigcap_{i=1}^{\infty}(\overline{co}A_i)$, hence $a \in \overline{co}(A_j)$ and we have $r \le \varphi(a)$; but this is a contradiction to (1). Therefore

$A_1^* \supseteq A_2^* \supseteq \cdots \supseteq A_i^* \supseteq \cdots$ is a decreasing sequence of nonempty compact subsets of $E$. Hence, $(\bigcap_i A_i) \bigcap H_r = \bigcap_i (A_i \bigcap H_r) = \bigcap_i A_i^* \neq \emptyset$. But, for $x \in (\bigcap_i A_i) \bigcap H_r$ we have $\varphi(x) \leq r$. This contradicts (1), and hence the assertion follows. ■

It should be noted that in general the convex hull of a compact set is not even closed (see [1, p. 173]. In a normed vector space, it is possible to apply Lemma 3.1 with the norm and weak topologies.

The following definition is well known:

**Definition 3.2.** Let $(M, \rho)$ be a complete metric space and $\Omega$ denotes the family of all nonempty, bounded closed subsets of $M$. For $X, Y \in \Omega$, set $d(X, Y) = \sup\{dist(y, X) : y \in Y\}$, $d(Y, X) = \sup\{dist(x, Y) : x \in X\}$ and let $D(X, Y) = \max\{d(X, Y), d(Y, X)\}$. Then $D$ provides a metric for $\Omega$ called the *Hausdorff metric*.

Let $C$ be a nonempty closed convex subset of a Banach space $E$ and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. It is easy to verify that $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$ as $n \to \infty$ iff $dist(x_n, F(T)) \to 0$ as $n \to \infty$, for all sequences $\{x_n\}$ with $x_n \in \overline{co}F_{\frac{1}{n}}(T), \forall n$. So, if $C$ is compact, it is easy to see that $D(\overline{co}F_{\frac{1}{n}}(T), \bigcap_i \overline{co}F_{\frac{1}{i}}(T)) \to 0$ as $n \to \infty$; and applying Lemma 3.1, we have $\bigcap_i \overline{co}F_{\frac{1}{i}}(T) = \overline{co}F(T)$. Therefore, we have shown

$$D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$$

as $n \to \infty$, in case that $F(T)$ is convex.

In this stage, we give some examples satisfying the convergence property above, however $C$ is not compact.

**Example 3.3.**

$(i)$ Let $C = \prod_{i \in \mathbb{N}} [0, 1] \subset l^\infty$. Then $C$ is not compact!. Now, let $T : C \to C$ be a nonexpansive mapping defined by $T(x_1, x_2, x_3, \ldots) = (f(x_1), 0, 0, \ldots)$, where $f : [0, 1] \to [0, 1]$ is an arbitrary nonexpansive mapping. Since $\mathbb{R}$ is strictly convex and $[0, 1]$ is compact, $f$ is of type $(\gamma)$ and $D(\overline{co}F_{\frac{1}{n}}(f), F(f)) \to 0$. On the other hand, $F_{\frac{1}{n}}(T) = F_{\frac{1}{n}}(f) \times (\prod_{i \in \mathbb{N}-\{1\}} [0, \frac{1}{n}])$. So $\overline{co}F_{\frac{1}{n}}(T) = \overline{co}F_{\frac{1}{n}}(f) \times (\prod_{i \in \mathbb{N}-\{1\}} [0, \frac{1}{n}])$ and $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, since $F(T) = \{(x_1, 0, 0, \ldots) : x_1 \in F(f)\}$. Also it is easy to see that $F(T)$ is compact and $T$ is of type $(\gamma)$.

$(ii)$ Let $C = \prod_{i \in \mathbb{N}} [0, \frac{1}{2}] \subset l^\infty$ and $T(x_1, x_2, \ldots) = (\frac{{x_1}^2}{2}, \frac{{x_2}^2}{2}, \ldots)$. One notes that both $C$ and $T(C)$ are not compact!. Obviously $T$ is a nonexpansive mapping on $C$ and $F_{\frac{1}{n}}(T) = \prod_{i\in\mathbb{N}}[0, 1 - \sqrt{1-\frac{2}{n}}\ ]$, for $n \geq 2$. Therefore, $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, where $F(T) = \{0\}$. The mapping $T$ is of type $(\gamma)$, where $\gamma$ is the identity mapping: Let $x, y \in C$ and $0 \leq \lambda \leq 1$. Then $\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\| = \sup_i(\frac{1}{2}|\lambda x_i^2 + (1-\lambda)y_i^2 - (\lambda x_i + (1-\lambda) y_i)^2|) = \frac{\lambda(1-\lambda)}{2}\sup_i(x_i - y_i)^2$. So, $\|Tx - Ty\| + \|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\| = \frac{1}{2}\sup_i|x_i^2 - y_i^2| + \frac{\lambda(1-\lambda)}{2}\sup_i(x_i - y_i)^2 \leq \frac{1}{2}\sup_i|x_i - y_i| + \frac{1}{2}\sup_i|x_i - y_i| = \|x - y\|$, since $0 \leq x_i, y_i \leq \frac{1}{2}$ for each $i \in \mathbb{N}$. Therefore $T$ is of type $(\gamma)$, where $\gamma$ is the identity mapping. By an elementary computation we can show $T^n$ is also of type $(\gamma)$, for which $\gamma$ is the identity mapping.

$(iii)$ Let $f : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ be an nonexpansive mapping of type $(\gamma)$, where $\gamma$ is the identity mapping, and $C$ be as in (ii). Define $T : C \to C$ by $T(x_1, x_2, \dots) = (f(x_1), \frac{{x_2}^2}{2}, \frac{{x_3}^2}{2}, \dots)$. As in (ii), it is easy to show that for each $n$, $T^n$ is a nonexpansive mapping of type $(\gamma)$ such that $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, where $F(T) = \{(x_1, 0, 0, \dots) : x_1 \in F(f)\}$.

$(iv)$ Let $C$ be as in (i) and $T : C \to C$ be a nonexpansive mapping defined by $T(x_1, x_2, x_3, \dots) = (x_1, \frac{x_2^2}{2}, \frac{x_3^2}{2}, \dots)$. Then we have $F(T) = \{(x_1, 0, 0, \dots) : x_1 \in [0, 1]\}$ is a compact convex set. Also, we have $F_{\frac{1}{n}}(T) = [0, 1] \times (\prod_{i\in\mathbb{N}-\{1\}}[0, 1 - \sqrt{1-\frac{2}{n}}\ ])$, for $n \geq 2$. Hence, it is easy to note $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, as $n \to \infty$.

In the above examples $\overline{co}F_{\frac{1}{n}}(T)$'s are not compact; however, $F(T)$'s are compact and we have $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, as $n \to \infty$. We can apply some results of this paper to examples like above.

## 4. Cluster Point of Means

The following lemmas are essential to our purpose.

**Lemma 4.1.** *Let $C$ be a nonempty closed, convex subset of a Banach space $E$ and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \varnothing$ and $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, as $n \to \infty$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{co}F_\delta(T) \subset F_\varepsilon(T)$.*

*Proof.* Let $\varepsilon > 0$. Since $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$ there exists $\delta > 0$ such that $\overline{co}F_\delta(T) \subset F(T) + B_{\frac{\varepsilon}{2}}$. On the other hand, we have $F(T) + B_{\frac{\varepsilon}{2}} \subset F_\varepsilon(T)$. Hence the assertion follows. ∎

**Lemma 4.2.** *Let $C, E$ and $T$ be as in Lemma 4.1. If $C$ is bounded and $T$ is of type $(\gamma)$, then for each $\eta > 0$ there exists $\delta > 0$ and $N > 0$, such that for every sequence $\{x_n\}$ in $C$ satisfying $\|x_{n+1} - Tx_n\| \le \delta$ for all $n$,*

$$\frac{1}{n}\sum_{i=1}^{n} x_i \in F_\eta(T)$$

*for all $n \ge N$.*

*Proof.* The proof is essentially the same as Theorem 1.3 of [8]. First, choose $\varepsilon > 0$ using Lemma 4.1, so that $\overline{co}F_\varepsilon(T) \subset F_{\frac{\eta}{3}}(T)$ and $\varepsilon d < \frac{\eta}{6}$ where $d = \text{diam}C$ . We choose a natural number $p$ such that $d < p\frac{\varepsilon^2}{2}$. Next, put $q(t) = \gamma^{-1}(2t) + t$ and $q_n(t) = \gamma^{-1}(\frac{d}{n} + 2t) + t$ and choose $0 < \delta < \frac{\eta}{3}$ so small that $q^{p-1}(\delta) < \frac{\varepsilon^2}{2}$. Finally, choose $N$ so large that $\frac{p}{N} < \varepsilon$ and $q_n^{p-1}(\delta) < \frac{\varepsilon^2}{2}$ for all $n \ge N$. Put $w_i = \frac{1}{p}\sum_{j=0}^{p-1} x_{j+i}$. Paralleling the proof of Lemma 1.5 of [7], we find $\frac{1}{n}\sum_{j=0}^{n-1}\|w_{j+1} - Tw_j\| \le q_n^{p-1}(\delta)$ provided $\|x_{i+1} - Tx_i\| \le \delta$ for all $i$. Obviously $\|w_{i+1} - w_i\| \le \frac{d}{p}$ for all $i$. So by using the triangle inequality we have,

$$\frac{1}{n}\sum_{i=0}^{n-1}\|w_i - Tw_i\| \le \varepsilon^2 \tag{2}$$

for every $n \ge N$. Put $A(n) = \{i \in \mathbb{Z} : 0 \le i \le n-1\ ,\ \|w_i - Tw_i\| \ge \varepsilon\}$ and $B(n) = \{0, 1, \ldots, n-1\} - A(n)$. Then $\frac{|A(n)|}{n} \le \varepsilon$ by $(2)$. Also we have,

$$\frac{1}{n}\sum_{i=0}^{n-1} x_i = \frac{1}{n}\sum_{i=0}^{n-1} w_i + \frac{1}{np}\sum_{i=1}^{p-1}(p-i)[x_{i-1} - x_{n+i-1}] \tag{3}$$

and $p\frac{d}{n} \le p\frac{d}{N} < d\varepsilon$ for every $n \ge N$. Therefore,

$$\|\frac{1}{np}\sum_{i=1}^{p-1}(p-i)[x_i - x_{n+i-1}]\| \le \frac{1}{np}p^2 d < d\varepsilon < \frac{\eta}{6}$$

and so, $\frac{1}{n}\sum\limits_{i=0}^{n-1} x_i \in [\frac{1}{n}\sum\limits_{i=0}^{n-1} w_i] + B_{\frac{\eta}{6}}$. Fix $f \in F_\varepsilon(T)$. Then,

$$\frac{1}{n}\sum_{i=0}^{n-1} w_i = [\frac{1}{n}|A(n)|f + \frac{1}{n}\sum_{i\in B(n)} w_i] + [\frac{1}{n}\sum_{i\in A(n)}(w_i - f)]$$

and $\|\frac{1}{n}\sum_{i\in A(n)}(w_i - f)\| \le \frac{|A(n)|}{n}d < \varepsilon d < \frac{\eta}{6}$. So,

$$\frac{1}{n}\sum_{i=0}^{n-1} x_i \in coF_\varepsilon(T) + B_{\frac{\eta}{6}} + B_{\frac{\eta}{6}} \subset F_{\frac{\eta}{3}}(T) + B_{\frac{\eta}{3}} \subset F_\eta(T)$$

for every $n \geq N$. This completes the proof. ■

**Lemma 4.3.** *In Lemma 4.2 put* $S_n = \frac{1}{n}(I + T + \cdots + T^{n-1})$. *Then* $\lim_n \|S_n(y) - TS_n(y)\| = 0$ *uniformly in* $y \in C$. *Moreover, if* $F(T)$ *is compact (weakly compact), then for every sequence* $\{y_n\}_{n\geq 1}$ *in* $C$, $\{S_n(y_n)\}_{n\geq 1}$ *has a cluster point (weak cluster point) in* $F(T)$.

*Proof.* Set $x_n = T^n y$ for each $n$ and $y \in C$, and apply Lemma 4.2 to conclude the first assertion. For the second assertion, let $F(T)$ be compact (weakly compact) and $\{y_n\}$ be an arbitrary sequence in $C$. We note $dist(S_n(y_n), F(T)) \to 0$, as $n \to \infty$ by using the first part of this lemma. Then for each $k \geq 1$ there exist $n_k > k$ and $f_k \in F(T)$ with $\|S_{n_k}(y_{n_k}) - f_k\| \leq \frac{1}{k}$. Since $F(T)$ is compact (weakly compact), without lose of generality we can assume that $f_k \to f$ ($f_k \rightharpoonup f$) for some $f$, as $k \to \infty$. It is enough to conclude the result. ■

## 5. Ergodic Theorems

By studying the proofs of Lemmas 2.2, 2.3 and 3.1 in [2], we obtain the following lemma:

**Lemma 5.1.** *Let* $C$ *be a nonempty bounded closed convex subset of a Banach space* $E$, $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \neq \varnothing$ *and* $T^n$ *is of type* $(\gamma)$ *for all* $n$. *Let* $x \in C$. *Then, there exists a sequence* $\{i_n\}$ *in* $\mathbb{N}$ *such that for each* $z \in F(T)$,

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x - z\| \text{ exists.}$$

*Moreover if* $\{i'_n\}$ *is a sequence in* $\mathbb{N}$ *such that* $i'_n \geq i_n$ *for each* $n \geq N$, *then for every* $z \in F(T)$,

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x - z\| = \lim_{n\to\infty} \|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+i'_n}x - z\|.$$

Recall that $E$ is said to satisfy Opial's condition, if for each sequence $\{x_n\}$ in $E$, the condition that the sequence $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

**Theorem 5.2.** *Let $C$ be a nonempty closed convex subset of a Banach space $E$ and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \varnothing$ and $T^n$ is of type $(\gamma)$ for all $n$ and $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, as $n \to \infty$. Let $x \in C$. Then,*

- $(i)$ *If $F(T)$ is compact, then $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$ converges strongly to a fixed point of $T$ uniformly in $h \geq 0$.*
- $(ii)$ *If $F(T)$ is weakly compact and $E$ satisfies Opial's condition, then $\frac{1}{n}\sum_{k=0}^{n} T^{k+i_n}x$ converges weakly to some $y \in F(T)$, for a sequence $\{i_n\}$ like the sequence in Lemma 5.1.*

*Proof.* Let $z$ be an arbitrary element of $F(T)$. Set $D = \{y \in C : \|y - z\| \leq \|x-z\|\}$. We note that $x \in D$, $T(D) \subset D$ and $D$ is a bounded closed convex subset of $C$. So we can assume that $C$ is bounded. By Lemma 5.1, there exists a sequence $\{i_n\}$ in $\mathbb{N}$ such that for each $f \in F(T)$, $\lim_{n\to\infty} \|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x - f\|$ exists. Now put $\{\Phi_n\} = \{\frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x\}$. We first prove (i). If $F(T)$ is compact, then $\{\Phi_n\}$ has a cluster point $y_0$ in $F(T)$ by Lemma 4.3.

Consequently, we have $\Phi_n \to y_0$; and from the last part of Lemma 5.1, $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+h+i_n}x$ converges strongly to $y_0$ uniformly in $h \geq 0$. Let $\varepsilon > 0$. Then, there exists $m \in \mathbb{N}$ such that $\|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+h+i_n}x - y_0\| < \varepsilon$ for every $n \geq m$ and $h \in \mathbb{N} \cup \{0\}$. Then, it follows from the equality (3) that $\|\frac{1}{n}\sum\limits_{i=0}^{n-1} T^{i+h}x - y_0\|$

$$= \|[\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{m}\sum_{j=0}^{m-1} T^{i+j+h}x - y_0] + [\frac{1}{nm}\sum_{i=1}^{m-1}(m-i)(T^{i+h-1}x - T^{i+h+n-1}x)]\|$$

$$\leq \frac{1}{n}\sum_{i=0}^{n-1}\|\frac{1}{m}\sum_{j=0}^{m-1} T^{j+h+i}x - y_0\| + \frac{1}{nm}\sum_{i=1}^{m-1}(m-i)\|T^{i+h-1}x - T^{i+h+n-1}x\|$$

$$= \frac{1}{n}\sum_{i=0}^{i_m-1}\|\frac{1}{m}\sum_{j=0}^{m-1} T^{j+h+i}x - y_0\| + \frac{1}{n}\sum_{i=0}^{n-i_m-1}\|\frac{1}{m}\sum_{j=0}^{m-1} T^{j+h+i+i_m}x - y_0\|$$

$$+ \frac{1}{nm}\sum_{i=1}^{m-1}(m-i)\|T^{i+h-1}x - T^{i+h+n-1}x\| \leq \frac{i_m M}{n} + \frac{(n-i_m)\varepsilon}{n} + \frac{mM}{n},$$

for every $n > i_m$ and $h \in \mathbb{N} \cup \{0\}$, where

$$M = \sup\{\|T^i x - y_0\| : j \in \mathbb{N} \cup \{0\}\}.$$

Since $\varepsilon > 0$ is arbitrary, $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$ converges strongly to $y_0$ uniformly in $h \in \mathbb{N} \cup \{0\}$, and so the proof of (i) is completed. To prove (ii) we assume $F(T)$ is weakly compact and $E$ satisfies Opial's condition. Then $\{\Phi_n\}$ has a weak cluster point $f$ in $F(T)$, by Lemma 4.3. We show $\Phi_n \rightharpoonup f$ as $n \to \infty$. Let $\Phi_{n_k} \rightharpoonup f_1$, $\Phi_{m_k} \rightharpoonup f_2$ and $f_1 \neq f_2$. Since $f_1, f_2 \in F(T)$, we put $r_1 := \lim_{n\to\infty} \|\Phi_n - f_1\|$

and $r_2 := \lim_{n\to\infty} \|\Phi_n - f_2\|$. By Opial's condition, we conclude

$$\begin{aligned} r_1 = \lim_{k\to\infty} \|\Phi_{n_k} - f_1\| &< \lim_{k\to\infty} \|\Phi_{n_k} - f_2\| = r_2 \\ = \lim_{k\to\infty} \|\Phi_{m_k} - f_2\| &< \lim_{k\to\infty} \|\Phi_{m_k} - f_1\| = r_1, \end{aligned}$$

which is a contradiction. It means that $f_1 = f_2$. This leads to the desired conclusion. ■

The following example shows that the condition $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$ in Theorem 5.2 can not be omitted.

**Example 5.3.** Let $C$ and $E$ be as in Example 3.3 (i). Define $T : C \to C$ by $T(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$, where $0 \le \lambda_i < 1$ for each $i \in \mathbb{N}$, and $\lim_{i\to\infty} \lambda_i = 1$. Then $T$ is a nonexpansive mapping such that $T^n$ is of type $(\gamma)$ for all $n$ and $F(T) = \{0\}$ which is compact. Also, $F_{\frac{1}{n}}(T) = \prod_{i=1}^{\infty}([0, \frac{1}{n(1-\lambda_i)}] \cap [0,1])$. So $\overline{co}F_{\frac{1}{n}}(T) \overset{d}{\nrightarrow} F(T)$. Now, by considering $x = (1, 1, \dots)$ in $C$ we have $\|\frac{1}{n}\sum_{i=1}^{n} T^i x\| = \sup_k(\frac{1}{n}\sum_{i=1}^{n}\lambda_k^i) = 1$, since $\lim_{k\to\infty}\lambda_k = 1$. So $\frac{1}{n}\sum_{i=1}^{n} T^i x$ does not converge to a member of $F(T)$. ■

## 6. A Strong Convergence Theorem of Mann's Type

In this section, using the iterative method of Mann's Type [11], we study how to find a fixed point of a nonexpansive mapping as in Theorem 5.2. Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $T$ be a nonexpansive mapping on $C$ with $F(T) \neq \varnothing$. Consider the following iteration scheme:

$$x_1 = x \in C \text{ and } x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n(x_n) \tag{4}$$

for every $n \in \mathbb{N}$, where $S_n = \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1})$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. For any $\omega \in F(T)$ we can prove

$$\|x_{n+1} - \omega\| \le \|x_n - \omega\| \tag{5}$$

for every $n \in \mathbb{N}$ and hence $\lim_{n\to\infty} \|x_n - \omega\|$ exists (see [3]).
The following lemma is essential.

**Lemma 6.1.** *Let $C$ be a nonempty closed convex subset of a Banach space $E$ and $T : C \to C$ be a nonexpansive mapping of type $(\gamma)$ such that $F(T) \neq \varnothing$ and $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, as $n \to \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$. Suppose that $x_1 = x \in C$ and let $\{x_n\}$ be as in (4). Then*

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

*Proof.* As in Theorem 5.2 we can assume that $C$ is bounded. Fix $\varepsilon > 0$ and set $M_0 = \sup\{\|z\| : z \in C\}$. Then, by Lemma 4.1, there exists $\delta > 0$ such that $\overline{co}F_\delta(T) \subset F_\varepsilon(T)$. From Lemma 4.3, there exists $M \in \mathbb{N}$ such that $\|S_n(y) - TS_n(y)\| < \delta$ for every $n \geq M$ and $y \in C$. Thus

$$S_n(x_n) \in F_\delta(T) \tag{6}$$

for every $n \geq M$. We have for each $k \in \mathbb{N}$,

$$x_{M+k} = (\prod_{i=M}^{M+k-1} \alpha_i)x_M + (1 - \prod_{i=M}^{M+k-1} \alpha_i)y_k \tag{7}$$

where

$$y_k = \frac{1}{1 - \prod_{i=M}^{M+k-1} \alpha_i}(\sum_{j=M}^{M+k-2} ((\prod_{i=j+1}^{M+k-1} \alpha_i)(1-\alpha_j)S_j(x_j)) \\ +(1-\alpha_{M+k-1})S_{M+k-1}(x_{M+k-1}))$$

(see [3, 4]). Now, from

$$\sum_{j=M}^{M+k-2} ((\prod_{i=j+1}^{M+k-1} \alpha_i)(1-\alpha_j)) + (1-\alpha_{M+k-1}) = 1 - \prod_{i=M}^{M+k-1} \alpha_i,$$

it follows that $y_k \in co\{S_n(x_n) : n \geq M\}$ and hence $y_k \in coF_\delta(T) \subset F_\varepsilon(T)$ for each $k \in \mathbb{N}$, by (6). From the Abel-Dini theorem and $\sum_{i=M}^{\infty}(1-\alpha_i) = \infty$, there exists $p \in \mathbb{N}$ such that $\prod_{i=M}^{M+k-1} \alpha_i < \frac{\varepsilon}{2M_0}$ for all $k \geq p$. From (7) we obtain

$$\|x_{M+k} - y_k\| = \prod_{i=M}^{M+k-1} \alpha_i \|x_M - y_k\| < \frac{\varepsilon}{2M_0} 2M_0 = \varepsilon$$

for each $k \geq p$. Hence $\|Tx_{M+k} - x_{M+k}\| \leq \|Tx_{M+k} - Ty_k\| + \|Ty_k - y_k\| + \|y_k - x_{M+k}\| \leq 2\|x_{M+k} - y_k\| + \|Ty_k - y_k\| \leq 2\varepsilon + \varepsilon = 3\varepsilon$
for every $k \geq p$. So $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. ■

**Theorem 6.2.** *Let $C$ be a nonempty closed convex subset of a Banach space $E$ and $T : C \to C$ be a nonexpansive mapping of type $(\gamma)$ such that $F(T) \neq \varnothing$ and $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, as $n \to \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Suppose that $x_1 = x \in C$ and $x_{n+1} = \alpha_n x_n + (1-\alpha_n)S_n(x_n)$ for every $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* By Lemma 6.1, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. From the assumption $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, we have $\lim d(x_n, F(T)) \to 0$ as $n \to \infty$. Hereafter, we will prove that $\{x_n\}$ is a Cauchy sequence. For all $\epsilon > 0$, there exists a natural number $N$ such that when $n \geq N$ $d(x_n, F(T)) < \frac{\epsilon}{4}$. Specifically, $d(x_N, F(T)) < \frac{\epsilon}{4}$. Thus there exists a point $y_0$ in $F(T)$ such that $\|x_n - y_0\| \leq \|x_N - y_0\| < \frac{\epsilon}{2}$ for each $n \geq N$, using (5) and the definition of $d(x_N, F(T))$. It follows that for each $n \geq N$ and $m$ in $\mathbb{N}$, $\|x_n - x_{n+m}\| \leq \|x_n - y_0\| + \|x_{n+m} - y_0\| < \epsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. Because the space is complete, the sequence $\{x_n\}$ is convergent to a point that is a fixed point of $T$. ■

**Remark 6.3.** It is not assumed in Theorem 6.2 that $C$ be bounded nor $F(T)$ be compact.

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