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ON A CLASS OF VERTEX OPERATOR ALGEBRAS HAVING A FAITHFUL S_{n+1}-ACTION

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Abstract. By using the lattice VOA $V_{\sqrt{2}A_n}$, we construct a class of vertex operator algebras $\{M^{(n)} | \, n = 2, 3, 4, \dots\}$ as coset subalgebras. We show that the VOA $M = M^{(n)}$ is generated by its weight 2 subspace and the symmetric group S_{n+1}, which is isomorphic to the Weyl group $W(A_n)$ of the root system of type A_n, acts faithfully on M. Moreover, some irreducible modules of M are constructed using the coset construction.

1. Introduction

Let A_n be a rank n root lattice of type A. It was shown in Dong et al. [4] that the Virasoro element ω of the lattice vertex operator algebra (VOA) $V_{\sqrt{2}A_n}$ can be decomposed into a sum of $n+1$ mutually orthogonal conformal vectors ω^i, $1 \le i \le n+1$ and the central charge c_i of the conformal vector ω^i is given by

$$c_i = 1 - 6/(i+2)(i+3) \quad \text{for } 1 \le i \le n \quad \text{and} \quad c_{n+1} = 2n/(n+3).$$

In other words, the lattice vertex operator algebra $V_{\sqrt{2}A_n}$ contains a subalgebra $T = T_n$ which is isomorphic to the tensor product of $n+1$ simple Virasoro VOAs $\otimes_{i=1}^{n+1} L(c_i, 0)$. Moreover, $V_{\sqrt{2}A_n}$ is a direct sum of irreducible T-submodules.

Note that $c_i = 1 - 6/(i+2)(i+3)$ for $1 \le i \le n$ are members of the unitary series and c_{n+1} is the central charge of the parafermion algebra. In fact, it was shown in [20] (see also [1]) that the conformal vector ω^{n+1} actually corresponds to a coset subalgebra isomorphic to the parafermion algebra $W_{n+1}(2n/(n+3))$ inside $V_{\sqrt{2}A_n}$. In addition, the complete decomposition of $V_{\sqrt{2}A_n}$ as a direct sum of irreducible modules of

$$\mathcal{W} = L(c_1, 0) \otimes L(c_2, 0) \otimes \cdots \otimes L(c_n, 0) \otimes W_{n+1}(2n/(n+3))$$

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is obtained.

For small n dividing 24, namely, $n = 1, 2, 3, 4$, there are evidences to show that the parafermion algebra $W_{n+1}(2n/(n+3))$ is actually contained in the Moonshine vertex operator algebra $V^\natural$ and the $\mathbb{Z}_{n+1}$-symmetry of $W_{n+1}(2n/(n+3))$ will induce an automorphism of order $n+1$ on $V^\natural$, which should correspond to the $2A, 3A, 4A$ and $5A$ elements of the Monster [13, 15, 16, 21, 24]. On the other hand, by using pure group theory, Glauberman and Norton [9] observed that there are some interesting relations between the centralizers of the $2A, 3A, 4A$ and $5A$ elements of the Monster simple group with the Weyl group of the type A_1, A_2, A_3 and A_4, respectively.

In this article, we shall study the commutant (or coset) subalgebra

$$M^{(n)} = \left\{ v \in V_{\sqrt{2}A_n} \,|\, u_k v = 0 \text{ for all } k \geq 0 \text{ and } u \in W_{n+1}\left(\frac{2n}{n+3}\right)\right\}$$

of $W_{n+1}(2n/(n+3))$ in $V_{\sqrt{2}A_n}$. As our main result, we shall show that the VOA $M = M^{(n)}$ is generated by its weight 2 subspace and the Weyl group $W(A_n)$ ($\cong S_{n+1}$) acts faithfully on M. Moreover, some irreducible modules of M will be constructed using the coset construction.

We shall note that for any n dividing 24, the tensor product VOA $M^{\otimes 24/n}$ can be embedded into the orbifold VOA V_Λ^+, where V_Λ^+ is the fixed point subspace of the Leech lattice VOA V_Λ associated with the automorphism θ induced by the isometry $\alpha \mapsto -\alpha$ for $\alpha \in \Lambda$ (cf. [4, 13]). Hence $M^{\otimes 24/n}$ is also contained in the famous Moonshine VOA $V^\natural$. With respect to a suitable embedding, we believe that the S_{n+1}-action on M can actually be lifted to some automorphism subgroup of $V^\natural$, which is in fact the main motivation for the present work.

2. Conformal Vectors in the Lattice VOA $V_{\sqrt{2}A_n}$

In this section, we review the construction of certain conformal vectors in $V_{\sqrt{2}A_n}$ from [4]. First we shall consider a chain of root systems

$$\Phi = \Phi_n \supset \Phi_{n-1} \supset \cdots \supset \Phi_1$$

such that Φ_i is a root system of type A_i. Let Φ_i^+ be a set of all positive roots in Φ_i and let $\Phi_i^- = -\Phi_i^+$ be the set of all negative roots in Φ_i. Then we have

$$\Phi_i = \Phi_i^+ \cup \Phi_i^- = \Phi_i^+ \cup (-\Phi_i^+).$$

For any $i = 1, 2, \ldots, n$, define

$$s^i = \frac{1}{2(i+3)} \sum_{\alpha \in \Phi_i^+} \left(\alpha(-1)^2 \cdot 1 - 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})\right)$$

and

$$\omega = \frac{1}{2(n+1)} \sum_{\alpha \in \Phi_n^+} \alpha(-1)^2 \cdot 1.$$

It was shown by Dong et al. [4] that ω is the Virasoro element of $V_{\sqrt{2}A_n}$ and the elements

$$\omega^1 = s^1, \qquad \omega^i = s^i - s^{i-1},\ 2 \le i \le n, \qquad \omega^{n+1} = \omega - s^n \tag{2.1}$$

are mutually orthogonal conformal vectors in $V_{\sqrt{2}A_n}$. Moreover, the central charges $c(\omega^i)$ of ω^i are given by

$$c(\omega^i) = 1 - \frac{6}{(i+2)(i+3)} \qquad \text{for } 1 \le i \le n,$$

and

$$c(\omega^{n+1}) = \frac{2n}{n+3}.$$

Note that $c_i = c(\omega^i)$, $1 \le i \le n$, are members of the unitary series and c_{n+1} is the central charge of the parafermion algebra. In fact, it was shown in [20] that $V_{\sqrt{2}A_n}$ actually contains a subalgebra isomorphic to the parafermion algebra $W_{n+1}(2n/(n+3))$. Moreover, we have the following decomposition.

Theorem 2.1. ([20]). *As a module of* $L(c_1,0)\otimes\cdots\otimes L(c_n,0)\otimes W_{n+1}(2n/(n+3))$,

$$V_{\sqrt{2}A_n} \cong \bigoplus_{\substack{0\le k_j\le j+1,\\ j=0,\dots,n\\ k_j\equiv 0 \bmod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n, h^n_{k_{n-1}+1,k_n+1}) \otimes W_{n+1}(0,k_n), \tag{2.2}$$

where $W_{n+1}(0,k)$ *are irreducible* $W_{n+1}(2n/(n+3))$*-submodules* (*see Section 3.2 for the definition*) *and*

$$h^m_{r,s} = \frac{[r\,(m+3) - s\,(m+2)]^2 - 1}{4\,(m+2)\,(m+3)}$$

for any $1 \le r \le m+1, 1 \le s \le m+2$.

In this article, we are interested in the commutant subalgebra of the parafermion algebra $W_{n+1}(2n/(n+3))$ in $V_{\sqrt{2}A_n}$, that is the commutant subalgebra

$$\begin{aligned} M &= \left\{ v \in V_{\sqrt{2}A_n} \mid u_k v = 0 \text{ for all } k \ge 0 \text{ and } u \in W_{n+1}\left(\frac{2n}{n+3}\right) \right\} \\ &\cong \bigoplus_{\substack{0\le k_j\le j+1,\\ j=0,\dots,n-1\\ k_j\equiv 0 \bmod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n, h^n_{k_{n-1}+1,1}). \end{aligned}$$

Remark 2.2. Note that the Weyl group $W(A_n)$ of the root system A_n induces a natural action on the lattice VOA $V_{\sqrt{2}A_n}$. By our construction (cf. [20]), the parafermion algebra $W_{n+1}(2n/(n+3))$ is actually fixed under the action of $W(A_n)(\cong S_{n+1})$ and the commutant algebra M is $W(A_n)$-invariant.

3. Construction of Irreducible Modules for M

In this section, we shall construct some irreducible modules for M using the lattice VOA $V_{\sqrt{2}A_n}$. First we shall recall certain arguments used in Lam and Yamada [20].

3.1. GKO construction of unitary Virasoro VOA

We shall first review the famous GKO construction for unitary Virasoro vertex operator algebras. We shall also study a certain decomposition of the lattice vertex operator algebra $V_{{A_1}^{n+1}}$ and its relation with the lattice VOA $V_{\sqrt{2}A_n}$.

Let $\mathfrak{g}$ be the Lie algebra $sl_2(\mathbb{C})$ with generators e, f, α such that $[e, f] = \alpha$, $[\alpha, e] = 2e$, $[\alpha, f] = -2f$ and $\tilde{\mathfrak{g}} = sl_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ the corresponding affine Lie algebra of type $A_1^{(1)}$. We shall denote $\hat{\mathfrak{g}} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = sl_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$. For any $\hat{\mathfrak{g}}$-module M, $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$, we denote the action of $x \otimes t^m$ on M by $x(m)$ and identify $\mathfrak{g} \otimes t^0$ with $\mathfrak{g}$. Let $\Lambda_0 = d$ and $\Lambda_1 = d + \frac{1}{2}\alpha$ be the fundamental weights for $\tilde{\mathfrak{g}}$. Then the dominant integral weights of $\tilde{\mathfrak{g}}$ for which d vanishes are given by

$$P_+ = \left\{ (m-j)\Lambda_0 + j\Lambda_1 = md + \frac{1}{2}j\alpha \,\middle|\, m \in \mathbb{Z}^+, j \in \mathbb{Z}^+ \cup \{0\}, j \le m \right\}.$$

Let $\mathcal{L}(m, j) = \mathcal{L}((m-j)\Lambda_0 + j\Lambda_1)$ be the irreducible highest weight module of $\tilde{\mathfrak{g}}$ of weight $(m-j)\Lambda_0 + j\Lambda_1 \in P_+$. By the Sugawara construction, $\mathcal{L}(m, j)$ has a natural Virasoro action given by the operators

$$\begin{aligned} L_k^{\mathfrak{g},m} =& \frac{1}{4(m+2)} \sum_{j\in\mathbb{Z}} : \alpha(-j)\alpha(k+j) : \\ &+ \frac{1}{2(m+2)} \sum_{j\in\mathbb{Z}} \big(: e(-j)f(k+j) : + : f(-j)e(k+j) : \big) \end{aligned}$$

with central charge $3m/(m+2)$, where : : denotes the normal ordered product.

Let $\mathcal{L}(\Lambda)$ and $\mathcal{L}(\Lambda')$ be two integrable highest weight representations of $\tilde{\mathfrak{g}}$ with level 1 and m respectively. Then $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ acts on the tensor product $\mathcal{L}(\Lambda) \otimes \mathcal{L}(\Lambda')$ by

$$(x(m) \oplus y(n))(v \otimes w) = (x(m)v) \otimes w + v \otimes (y(n)w),$$

for any $x(n), y(m) \in \hat{\mathfrak{g}}$ and $v \otimes w \in \mathcal{L}(\Lambda)\otimes\mathcal{L}(\Lambda')$. Now let $L_k^{\mathfrak{p}} = L_k^{\mathfrak{g},1}\otimes 1 + 1\otimes L_k^{\mathfrak{g},m}$ be an operator on $\mathcal{L}(\Lambda)\otimes\mathcal{L}(\Lambda')$. Then $L_k^{\mathfrak{p}}$, $k \in \mathbb{Z}$, form a representation of the Virasoro algebra with central charge $1 + 3m/(m+2)$ on $\mathcal{L}(\Lambda)\otimes\mathcal{L}(\Lambda')$. On the other hand, $\hat{\mathfrak{g}}$ acts on $\mathcal{L}(\Lambda)\otimes\mathcal{L}(\Lambda')$ by the diagonal action

$$x(n)(v\otimes w) = (x(n)v)\otimes w + v\otimes(x(n)w),$$

for any $x(n) \in \hat{\mathfrak{g}}$ and $v\otimes w \in \mathcal{L}(\Lambda)\otimes\mathcal{L}(\Lambda')$. This gives a level $m+1$ representation of $\hat{\mathfrak{g}}$ and the Sugawara operators $L_k^{\mathfrak{g},m+1}$ form the Virasoro algebra on $\mathcal{L}(\Lambda)\otimes\mathcal{L}(\Lambda')$ with central charge $3(m+1)/(m+3)$. Let $L_k = L_k^{\mathfrak{p}} - L_k^{\mathfrak{g},m+1}$. It is well known (cf. [8, 11]) that $L_k, k \in \mathbb{Z}$, commute with the diagonal Virasoro operators $L_n^{\mathfrak{g},m+1}$ for all $n \in \mathbb{Z}$ and they give rise to a representation the Virasoro algebra $Vir = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}\,c$ on $\mathcal{L}(\Lambda)\otimes\mathcal{L}(\Lambda')$ with central charge $c_m = 1 - 6/(m+2)(m+3)$. Moreover, $\mathcal{L}(\Lambda)\otimes\mathcal{L}(\Lambda')$ is completely reducible as a module of $Vir \oplus \hat{\mathfrak{g}}$.

By using the theory of character, the explicit decomposition of $\mathcal{L}(\Lambda)\otimes\mathcal{L}(\Lambda')$ as a $Vir\oplus\hat{\mathfrak{g}}$-module is known [8, 11, 25]. It is given by

$$\begin{aligned}\mathcal{L}(m,n)\otimes\mathcal{L}(1,\epsilon) &= \bigoplus_{\substack{0\le s\le n\\ s\equiv n+\epsilon \mod 2}} L(c_m, h^m_{n+1,s+1})\otimes\mathcal{L}(m+1,s)\\ &\quad \oplus \bigoplus_{\substack{n+1\le s\le m+1\\ s\equiv n+\epsilon \mod 2}} L(c_m, h^m_{m-n+1,m+2-s})\otimes\mathcal{L}(m+1,s)\\ &= \bigoplus_{\substack{0\le s\le m+1\\ s\equiv n+\epsilon \mod 2}} L(c_m, h^m_{n+1,s+1})\otimes\mathcal{L}(m+1,s),\end{aligned} \tag{3.1}$$

for any $\epsilon = 0, 1$, and $0 \le n \le m$.

Let ${A_1}^{n+1} = \mathbb{Z}\alpha^0 \oplus \mathbb{Z}\alpha^1 \oplus \cdots \oplus \mathbb{Z}\alpha^n$ be the orthogonal sum of $n+1$ copies of A_1 and $V_{{A_1}^{n+1}}$ the lattice vertex operator algebra associated with the lattice ${A_1}^{n+1}$. Then we have

$$V_{{A_1}^{n+1}} \cong V_{A_1}\otimes\cdots\otimes V_{A_1} \cong \mathcal{L}(1,0)\otimes\cdots\otimes\mathcal{L}(1,0)$$

as a vertex operator algebra and

$$V_{\gamma_a + {A_1}^{n+1}} \cong \mathcal{L}(1,a_0)\otimes\cdots\otimes\mathcal{L}(1,a_n)$$

as a module of $\mathcal{L}(1,0)\otimes\cdots\otimes\mathcal{L}(1,0)$, where $a = (a_0, a_1, \ldots, a_n) \in \{0,1\}^{n+1}$ and $\gamma_a = \frac{1}{2}\sum_{i=0}^n a_i\alpha^i$.

For each $0 \le j \le n+1$, let $H^j = \alpha^0(-1)1 + \cdots + \alpha^j(-1)1$, $E^j = e^{\alpha^0} + \cdots + e^{\alpha^j}$, and $F^j = e^{-\alpha^0} + \cdots + e^{-\alpha^j}$. Then $\mathrm{span}_{\mathbb{C}}\{H^j, E^j, F^j\}$ forms a simple Lie algebra $sl_2(\mathbb{C})$ inside the weight one space of $V_{{A_1}^{m+1}}$ under the 0-th product, i.e., $[x, y] = x_0 y$ for $x, y \in (V_{{A_1}^{m+1}})_1$. Moreover, $\{H^j, E^j, F^j\}$ generates a simple VOA $\mathcal{L}(j+1, 0)$ of level $j+1$ and the Virasoro elements of $\mathcal{L}(j+1, 0)$ is given by

$$\begin{aligned}\Omega^j &= \frac{1}{2(j+3)}\left(\frac{1}{2}H^j_{-1}H^j + E^j_{-1}F^j + F^j_{-1}E^j\right)\\ &= \frac{1}{2(j+3)}\left\{\frac{3}{2}\sum_{p=0}^{j}\alpha^p(-1)^2 1 + \frac{1}{2}\sum_{\substack{0\le p,\, q\le j\\ p\ne q}}\alpha^p(-1)\alpha^q(-1)1\right.\\ &\quad \left. +2\sum_{\substack{0\le p,\, q\le j\\ p\ne q}} e^{\alpha^p-\alpha^q}\right\}\end{aligned}$$

and the central charges of Ω^j is $3(j+1)/(j+3)$ [2, 6]. On the other hand, the Virasoro element of the lattice subVOA $V_{\mathbb{Z}\alpha^j}$ ($\cong V_{A_1}$) is given by $\frac{1}{4}\alpha^j(-1)^2 1$. By using the GKO construction, $\tilde{w}^j = \frac{1}{4}\alpha^j(-1)^2\cdot 1 + \Omega^{j-1} - \Omega^j$ generates a Virasoro subVOA $L(c_j, 0)$ with central charge $c_j = 1 - 6/(j+2)(j+3)$. Thus by induction, we have the following theorem.

Lemma 3.1. [cf. [11, 18, 25]] *The lattice VOA $V_{{A_1}^{n+1}}$ contains a subVOA isomorphic to $U = L(c_1, 0)\otimes L(c_2, 0)\otimes\cdots\otimes L(c_n, 0)\otimes\mathcal{L}(n+1, 0)$. Moreover,*

$$V_{\gamma_a + {A_1}^{n+1}} \cong \bigoplus_{\substack{0\le k_j\le j+1,\\ j=0,\dots,n\\ k_j\equiv b_j \bmod 2}} L(c_1, h^1_{k_0+1,k_1+1})\otimes\cdots\otimes L(c_n, h^n_{k_{n-1}+1,k_n+1})\otimes\mathcal{L}(n+1, k_n)$$

as a U-module for any $a = (a_0, a_1, \dots, a_n)\in\{0,1\}^{n+1}$, where $b_j = \sum_{i=0}^{j} a_i$.

3.2. A construction of parafermion algebras and their modules

Now let us recall a construction of parafermion algebras from [2]. We shall then use this construction to obtain decompositions for irreducible $V_{\sqrt{2}A_n}$-modules with respect to the subalgebra

$$\mathcal{W} = L(c_1, 0)\otimes L(c_2, 0)\otimes\cdots\otimes L(c_n, 0)\otimes W_{n+1}(2n/(n+3)).$$

Recall that $H^n = \alpha^0(-1)1 + \cdots + \alpha^n(-1)1$, $E^n = e^{\alpha^0} + \cdots + e^{\alpha^n}$, and $F^n = e^{-\alpha^0} + \cdots + e^{-\alpha^n}$ generate a subVOA isomorphic to a level $n+1$ representation

$\mathcal{L}(n+1,0)$ (cf. [2]). Let $\gamma = \alpha^0 + \cdots + \alpha^n$. Then $\gamma(-1)1 = H^n$ and it is easy to check that

$$e^{\gamma} = \frac{1}{(n+1)!}(E^n_{-1})^n E^n.$$

Thus $\mathcal{L}(n+1,0)$ contains a subalgebra isomorphic to the lattice VOA $V_{\mathbb{Z}\gamma}$.

Let $W_{n+1} = \{v \in \mathcal{L}(n+1,0) |\ u_n v = 0 \text{ for all } u \in V_{\mathbb{Z}\gamma} \text{ and } n \geq 0\}$ be the commutant subalgebra of $V_{\mathbb{Z}\gamma}$ in $\mathcal{L}(n+1,0)$. Then, for any $1 \leq j \leq n+1$, $\mathcal{L}(n+1,j)$ is a $V_{\mathbb{Z}\gamma} \otimes W_{n+1}$-module.

Now let

$$\mathcal{L}(n+1,j) = \bigoplus_{s=0}^{2n+1} V_{\mathbb{Z}\gamma + \frac{s}{2(n+1)}\gamma} \otimes W_{n+1}(j,s)$$

be the decomposition of $\mathcal{L}(n+1,j)$ as $V_{\mathbb{Z}\gamma} \otimes W_{n+1}$-modules. It is shown in [2] that

$$W_{n+1}(j,s) = 0 \qquad \text{if } j+s \equiv 1 \mod 2$$

and

$$\mathcal{L}(n+1,j) = \begin{cases} \bigoplus_{s=0}^{n} V_{\mathbb{Z}\gamma + \frac{s}{n+1}\gamma} \otimes W_{n+1}(j,2s) & \text{if } j \text{ is even,} \\ \bigoplus_{s=0}^{n} V_{\mathbb{Z}\gamma + \frac{2s+1}{2(n+1)}\gamma} \otimes W_{n+1}(j,2s+1) & \text{if } j \text{ is odd.} \end{cases} \tag{3.2}$$

Proposition 3.2. [cf. Dong-Lepowsky [2]] *All $W_{n+1}(j,s)$, $0 \leq j \leq n+1$, $0 \leq s \leq 2n+1$, $j \equiv s \mod 2$, are irreducible W_{n+1}-modules.*

Let $N = \mathrm{span}_{\mathbb{Z}}\{-\alpha^0 + \alpha^1, -\alpha^1 + \alpha^2, \ldots, -\alpha^{n-1} + \alpha^n\} \subset {A_1}^{n+1}$, $\gamma = \alpha^0 + \cdots + \alpha^n$ and $\eta = \frac{1}{n+1}(-\alpha^0 - \cdots - \alpha^{n-1} + n\alpha^n)$. Then N is isomorphic to $\sqrt{2}A_n$ and the dual lattice of N is

$$\begin{aligned} N^* &= \{x \in \mathbb{Q} \otimes_{\mathbb{Z}} N |\ \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in N\} \\ &\cong \frac{1}{\sqrt{2}}(A_n^*). \end{aligned}$$

Note that $\langle N, \gamma\rangle = 0$, $|N^*/N| = 2^n \cdot (n+1)$ and $\eta + N$ is a generator of the quotient group $2N^*/N$. In addition, we have the following lemma.

Lemma 3.3. *Let $a = (a_0, \ldots, a_n) \in \{0,1\}^{n+1}$ be a binary word. We shall denote*

$$\gamma_a = \frac{1}{2}\sum_{i=0}^{n} a_i \alpha^i \quad \textit{and} \quad \beta_a = \frac{1}{2}\sum_{i=0}^{n} a_i(\alpha^i - \alpha^n).$$

Then we have

$$\gamma_a + {A_1}^{n+1} = \begin{cases} \bigcup_{s=0}^{n} \left\{ (\beta_a + s\eta + N) + \left(\frac{s}{n+1}\gamma + \mathbb{Z}\gamma \right) \right\} & \text{if } |a| \text{ is even,} \\ \bigcup_{s=0}^{n} \left\{ \left(\beta_a + \frac{2s+1}{2}\eta + N \right) + \left(\frac{2s+1}{2(n+1)}\gamma + \mathbb{Z}\gamma \right) \right\} & \text{if } |a| \text{ is odd,} \end{cases}$$

where $|a| = \sum_{i=0}^{n} a_i$ *is the weight of the binary word* a.

Proof. First we shall show that

$$\mathcal{A} = \bigcup_{s=0}^{n} \left\{ (s\eta + N) + \left(\frac{s}{n+1}\gamma + \mathbb{Z}\gamma \right) \right\} = {A_1}^{n+1}.$$

Clearly, $\mathcal{A}$ is closed under addition and it forms a sublattice of ${A_1}^{n+1}$. Note also that

$$\eta^s = \frac{1}{n+1} \left(-s \sum_{i=0}^{n-s} \alpha^i + (n+1-s) \sum_{i=n+1-s}^{n} \alpha^i \right) \in s\eta + N$$

and

$$\eta^s + \frac{s}{n+1}\gamma = \sum_{i=n+1-s}^{n} \alpha^i.$$

Hence, $\mathcal{A}$ contains all α^i for $i = 0, \ldots, n$ and thus $\mathcal{A} = {A_1}^{n+1}$.

Now let $a = (a_0, \ldots, a_n) \in \{0,1\}^{n+1}$. Then

$$\gamma_a = \frac{1}{2} \sum_{i=0}^{n} a_i \alpha^i = \frac{1}{2} \sum_{i=0}^{n} a_i (\alpha^i - \alpha^n) + \frac{|a|}{2}\alpha^n = \beta_a + \frac{|a|}{2}\alpha^n.$$

If $|a|$ is even, then $\frac{|a|}{2}\alpha^n$ is in ${A_1}^{n+1}$ and thus we have

$$\gamma_a + {A_1}^{n+1} = \bigcup_{s=0}^{n} \left\{ (\beta_a + s\eta + N) + \left(\frac{s}{n+1}\gamma + \mathbb{Z}\gamma \right) \right\}.$$

If $|a|$ is odd, then $\gamma_a + {A_1}^{n+1} = (\beta_a + \frac{\alpha^n}{2}) + {A_1}^{n+1}$. On the other hand,

$$\frac{\alpha^n}{2} = \frac{1}{2}\eta + \frac{1}{2(n+1)}\gamma,$$

and thus

$$\gamma_a + {A_1}^{n+1} = \bigcup_{s=0}^{n} \left\{ \left(\beta_a + \frac{2s+1}{2}\eta + N\right) + \left(\frac{2s+1}{2(n+1)}\gamma + \mathbb{Z}\gamma\right) \right\}$$

when $|a|$ is odd. ■

As a corollary of Lemma 3.3, we have the following proposition.

Proposition 3.4. *Let* $\delta = (\delta_0, \delta_1, \dots, \delta_{n-1}) \in \mathbb{Z}_2^n$ *and denote*

$$\beta_\delta = \frac{1}{2}\sum_{i=0}^{n-1} \delta_i(\alpha^i - \alpha^n).$$

Then, for any $s = 0, \dots, n$, *we have the following decompositions:*

$$V_{\beta_\delta + s\eta + N} \cong \bigoplus_{\substack{0\le k_j \le j+1,\\ j=0,\dots,n\\ k_j \equiv b_j \bmod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n, h^n_{k_{n-1}+1,k_n+1}) \otimes W_{n+1}(k_n, 2s), \tag{3.3}$$

where $b_j = \sum_{i=0}^{j} \delta_j$ *for* $j = 0, 1, , \dots, n-1$ *and*

$$b_n = \begin{cases} |\delta| & \text{if } |\delta| \text{ is even}, \\ |\delta| + 1 & \text{if } |\delta| \text{ is odd}. \end{cases}$$

and

$$V_{\beta_\delta + \frac{2s+1}{2}\eta + N} \cong \bigoplus_{\substack{0\le k_j \le j+1,\\ j=0,\dots,n\\ k_j \equiv d_j \bmod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n, h^n_{k_{n-1}+1,k_n+1}) \otimes W_{n+1}(k_n, 2s+1), \tag{3.4}$$

where $d_j = b_j = \sum_{i=0}^{j} \delta_j$ *for* $j = 0, 1, , \dots, n-1$ *and*

$$d_n = \begin{cases} |\delta| + 1 & \text{if } |\delta| \text{ is even}, \\ |\delta| & \text{if } |\delta| \text{ is odd}. \end{cases}$$

Proof. For $\delta = (\delta_0, \dots, \delta_{n-1}) \in \mathbb{Z}_2^n$, denote

$$\tilde{\delta} = \begin{cases} (\delta_0, \dots, \delta_{n-1}, 0) & \text{if } |\delta| \text{ is even}, \\ (\delta_0, \dots, \delta_{n-1}, 1) & \text{if } |\delta| \text{ is odd}. \end{cases}$$

Then $\tilde{\delta}$ is always even and $\hat{\delta} = \tilde{\delta} + (0, \dots, 0, 1)$ is always odd. Thus, by Lemma 3.3, we have

$$\gamma_{\tilde{\delta}} + {A_1}^{n+1} = \bigcup_{s=0}^{n} \left\{ \left(\beta_{\tilde{\delta}} + s\eta + N\right) + \left(\frac{s}{n+1}\gamma + \mathbb{Z}\gamma\right) \right\}$$

and

$$\gamma_{\hat{\delta}} + {A_1}^{n+1} = \bigcup_{s=0}^{n} \left\{ \left(\beta_{\hat{\delta}} + \frac{2s+1}{2}\eta + N\right) + \left(\frac{2s+1}{2(n+1)}\gamma + \mathbb{Z}\gamma\right) \right\}$$

Note that $\beta_\delta = \beta_{\tilde{\delta}} = \beta_{\hat{\delta}}$ and we have

$$V_{\gamma_{\tilde{\delta}} + {A_1}^{n+1}} = \bigoplus_{s=0}^{n} \left(V_{\beta_\delta + s\eta + N} \otimes V_{\frac{s}{n+1}\gamma + \mathbb{Z}\gamma} \right)$$

and

$$V_{\gamma_{\hat{\delta}} + {A_1}^{n+1}} = \bigoplus_{s=0}^{n} \left(V_{\beta_\delta + \frac{2s+1}{2}\eta + N} \otimes V_{\frac{2s+1}{2(n+1)}\gamma + \mathbb{Z}\gamma} \right).$$

Now by Lemma 3.1 and (3.2), we immediately have the desired results. ∎

Let $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{Z}_2^{n+1}$. Then

$$\beta_{\mathbf{1}} = \frac{1}{2}\sum_{i=0}^{n-1}(\alpha^i - \alpha^n) = -\frac{n+1}{2}\eta$$

and we have

$$\beta_{\mathbf{1}+a} + N = \beta_a + \beta_{\mathbf{1}} + N = \beta_a - \frac{n+1}{2}\eta + N,$$

for any $a = (a_0, \dots, a_n) \in \mathbb{Z}_2^{n+1}$. Hence we have

$$\gamma_{a+\mathbf{1}} + {A_1}^{n+1} = \begin{cases} \bigcup_{s=0}^{n} \left\{ \left(\beta_a + \frac{2s-n-1}{2}\eta + N\right) + \left(\frac{s}{n+1}\gamma + \mathbb{Z}\gamma\right) \right\} & \text{if } |a+\mathbf{1}| \text{ is even,} \\ \bigcup_{s=0}^{n} \left\{ \left(\beta_a + \frac{2s-n}{2}\eta + N\right) + \left(\frac{2s+1}{2(n+1)}\gamma + \mathbb{Z}\gamma\right) \right\} & \text{if } |a+\mathbf{1}| \text{ is odd.} \end{cases} \tag{3.5}$$

Proposition 3.5. *Let* $0 \le j \le n+1$ *and* $0 \le s \le 2n+1$. *Then we have*

$$W_{n+1}(j, s) \cong W_{n+1}(n+1-j, s')$$

as a W_{n+1}-module, where $s' \equiv s+n+1 \mod 2(n+1)$.

Proof. For $0 \le j \le n+1$, define $a=(a_0,\ldots,a_n) \in \mathbb{Z}_2^{n+1}$ by

$$a_i = \begin{cases} 1 & \text{if } i<j, \\ 0 & \text{otherwise.} \end{cases}$$

Then by Lemma 3.3 and (3.5), we have

$$V_{\beta_a+\frac{s}{2}\eta+N} \cong \bigoplus_{\substack{0\le k_\ell \le \ell+1,\\ \ell=0,\ldots,n\\ k_\ell \equiv b_\ell \bmod 2}} L(c_1,h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n,h^n_{k_{n-1}+1,k_n+1}) \otimes W_{n+1}(k_n,s),$$

and

$$V_{\beta_a+\frac{s'-n-1}{2}\eta+N} \cong \bigoplus_{\substack{0\le k'_\ell \le \ell+1,\\ \ell=0,\ldots,n\\ k'_\ell \equiv b'_\ell \bmod 2}} L(c_1,h^1_{k'_0+1,k'_1+1}) \otimes \cdots \otimes L(c_n,h^n_{k'_{n-1}+1,k'_n+1}) \otimes W_{n+1}(k'_n,s'),$$

for any $0 \le s,s' \le 2n+1$, where $b_\ell = \sum_{i=0}^{\ell} a_i$ for $\ell = 0,1,,\ldots,n$ and $b'_\ell = \ell+1-b_\ell$.

Now suppose $s = s'-n-1 \mod 2(n+1)$. Then we have

$$\begin{aligned} V_{\beta_a+\frac{s}{2}\eta+N} &\cong \bigoplus_{\substack{0\le k_\ell \le \ell+1,\\ \ell=0,\ldots,n\\ k_\ell \equiv b_\ell \bmod 2}} L(c_1,h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n,h^n_{k_{n-1}+1,k_n+1}) \otimes W_{n+1}(k_n,s), \\ &\cong \bigoplus_{\substack{0\le k'_\ell \le \ell+1,\\ \ell=0,\ldots,n\\ k'_\ell \equiv b'_\ell \bmod 2}} L(c_1,h^1_{k'_0+1,k'_1+1}) \otimes \cdots \otimes L(c_n,h^n_{k'_{n-1}+1,k'_n+1}) \otimes W_{n+1}(k'_n,s'). \end{aligned}$$

Note that $h^m_{r,s} = h^m_{m+2-r,m+3-s}$ for any m,r and s and we have

$$h^\ell_{k'_{\ell-1}+1,k'_\ell+1} = h^\ell_{(\ell-k'_{\ell-1})+1,(\ell+1-k'_\ell)+1}.$$

Recall that

$$k'_\ell \equiv b'_\ell = \ell+1-b_\ell \mod 2.$$

Hence, we have

$$\ell + 1 - k'_\ell \equiv (\ell+1) - (\ell+1) + b_\ell \equiv b_\ell \mod 2$$

and

$$\begin{aligned} & V_{\beta_a + \frac{s}{2}\eta + N} \\ \cong & \bigoplus_{\substack{0\le k_\ell \le \ell+1,\\ \ell=0,\dots,n\\ k_\ell \equiv b_\ell \bmod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n, h^n_{k_{n-1}+1,k_n+1}) \otimes W_{n+1}(k_n, s), \\ \cong & \bigoplus_{\substack{0\le k_\ell \le \ell+1,\\ \ell=0,\dots,n\\ k_\ell \equiv b_\ell \bmod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n, h^n_{k_{n-1}+1,k_n+1}) \otimes W_{n+1}(n+1-k_n, s'). \end{aligned}$$

Therefore, $W_{n+1}(j,s) \cong W_{n+1}(n+1-j, s')$ as desired. ■

Next we shall construct some irreducible modules for the coset algebra

$$M = M^{(n)} = \left\{ v \in V_{\sqrt{2}A_n} \,|\, u_n v = 0 \text{ for all } n \ge 0 \text{ and } u \in W_{n+1}\left(\frac{2n}{n+3}\right) \right\}.$$

Note that M is also contained in the lattice VOA $V_{{A_1}^{n+1}}$ and we have

$$\begin{aligned} M &\cong \left\{ v \in V_{{A_1}^{n+1}} \,|\, \Omega^{n+1}_1 v = 0 \right\} \\ &\cong \bigoplus_{\substack{0\le k_j \le j+1,\\ j=0,\dots,n-1\\ k_j \equiv 0 \bmod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n, h^n_{k_{n-1}+1,1}), \end{aligned}$$

where Ω^{n+1} is the Virasoro element of the VOA $\mathcal{L}(n+1, 0)$.

Definition 3.6. For any $\delta = (\delta_0, \dots, \delta_{n-1}) \in \mathbb{Z}_2^n$ and $0 \le k \le n+1$, denote

$$\delta' = \begin{cases} (\delta_0, \dots, \delta_{n-1}, 0) & \text{if } |\delta| \equiv k \mod 2, \\ (\delta_0, \dots, \delta_{n-1}, 1) & \text{if } |\delta| \equiv k+1 \mod 2. \end{cases}$$

We define

$$M^\delta(k) = \left\{ u \in V_{\gamma_{\delta'} + {A_1}^{n+1}} \;\middle|\; \begin{array}{c} (\Omega^{n+1})_i u = 0 \text{ for all } i \ge 2, (E^n)_0 u = 0 \\ \text{and } (\Omega^{n+1})_1 u = \frac{k(k+2)}{4(n+3)} u \end{array} \right\}.$$

In other words, $M^{\delta}(k)$ corresponds to the multiplicity of $\mathcal{L}(n+1,k)$ in $V_{\gamma_{\delta'}+A_1{}^{n+1}}$ and hence we have

$$M^{\delta}(k) \cong \bigoplus_{\substack{0\le k_j\le j+1,\\ j=0,\dots,n-1\\ k_j\equiv b_j \bmod 2}} L(c_1,h^1_{k_0+1,k_1+1})\otimes\cdots\otimes L(c_n,h^n_{k_{n-1}+1,k+1}),$$

where $b_j=\sum_{i=0}^{j}\delta_j,\ j=0,\dots,n-1$.

By using the similar argument as Proposition 3.5, we also have the following theorem.

Theorem 3.7. *Let* $\mathbf{1}=(1,\dots,1)\in\mathbb{Z}_2^n$. *For any* $\delta=(\delta_0,\dots,\delta_{n-1})\in\mathbb{Z}_2^n$ *and* $0\le k\le n+1$, *we have* $M^{\delta}(k)\cong M^{\delta+\mathbf{1}}(n+1-k)$.

Proof. By using Lemma 3.1, (3.2) and Lemma 3.3, it is clear that

$$V_{\beta_\delta+\frac{s}{2}\eta+N}\cong\bigoplus_{\substack{0\le k\le n+1\\ k\equiv s \bmod 2}} M^{\delta}(k)\otimes W_{n+1}(k,s),$$

for any $0\le s\le 2n+1$, where $\beta_\delta=\frac{1}{2}\sum_{i=0}^{n-1}\delta_i(\alpha_i-\alpha_n)$. On the other hand,

$$\begin{aligned} V_{\beta_{\delta+\mathbf{1}}+\frac{s}{2}\eta+N} &= V_{\beta_\delta+\frac{s-n-1}{2}\eta+N}\\ &\cong \bigoplus_{\substack{0\le k\le n+1\\ k\equiv s'' \bmod 2}} M^{\delta}(k)\otimes W_{n+1}(k,s'')\end{aligned}$$

where $0\le s''\le 2n+1$ and $s''\equiv s-n-1 \mod 2(n+1)$. Thus

$$\begin{aligned} V_{\beta_{\delta+\mathbf{1}}+\frac{s}{2}\eta+N} &\cong \bigoplus_{\substack{0\le k'\le n+1\\ k'\equiv s \bmod 2}} M^{\delta+\mathbf{1}}(k')\otimes W_{n+1}(k',s),\\ &\cong \bigoplus_{\substack{0\le k\le n+1\\ k\equiv s'' \bmod 2}} M^{\delta}(k)\otimes W_{n+1}(k,s'').\end{aligned}$$

Since $W_{n+1}(k',s)\cong W_{n+1}(n+1-k',s'')$, we have $M^{\delta+\mathbf{1}}(k')\cong M^{\delta}(k)$ if $k=n+1-k'$ and thus $M^{\delta}(k)\cong M^{\delta+\mathbf{1}}(n+1-k)$ as desired. ■

3.3. Inequivalence of Irreducible modules

In this section, we shall show that $M^{\delta}(k)$ and $M^{\sigma}(\ell)$ are inequivalent except for the cases:
(1) $\delta=\sigma$ and $k=\ell$ and **(2)** $\delta=\sigma+\mathbf{1}$ and $k=n+1-\ell$.

First we shall recall that for any $\delta \in \mathbb{Z}_2^n$ and $1 \le k \le n+1$,

$$M^\delta(k) = \left\{ u \in V_{\gamma_{\delta'}+A_1{}^{n+1}} \;\middle|\; \begin{array}{c} (\Omega^{n+1})_i u = 0 \text{ for all } i \ge 2, (E^n)_0 u = 0 \\ \text{and } (\Omega^{n+1})_1 u = \frac{k(k+2)}{4(n+3)} u \end{array} \right\},$$

where δ' is defined by

$$\delta' = \begin{cases} (\delta_0, \ldots, \delta_{n-1}, 0) & \text{if } |\delta| \equiv k \mod 2, \\ (\delta_0, \ldots, \delta_{n-1}, 1) & \text{if } |\delta| \equiv k+1 \mod 2, \end{cases}$$

and $\gamma_a = \frac{1}{2}\sum_{i=0}^n a_i \alpha^i$ for any $a = (a_0, \ldots, a_n) \in \mathbb{Z}_2^{n+1}$.

Lemma 3.8. *For any $\delta = (\delta_0, \ldots, \delta_{n-1}) \in \mathbb{Z}_2^n$ and $0 \le k \le n+1$, we have*

$$M^\delta(k) = \bigoplus_{\substack{0 \le k' \le n \\ k' \equiv b \bmod 2}} M^{\bar{\delta}}(k') \otimes L(c_n, h^n_{k'+1,k+1}), \tag{3.2}$$

where $b = \sum_{i=0}^{n-1} \delta_i$ and $\bar{\delta} = (\delta_0, \ldots, \delta_{n-2})$.

Proof. First we shall note that

$$V_{\gamma_{\delta'}+A_1{}^{n+1}} \cong V_{\gamma_\delta+A_1{}^n} \otimes V_{\frac{1}{2}\delta'_n\alpha^n+A_1}.$$

By the definition of $M^\delta(k)$, we also have

$$V_{\gamma_\delta+A_1{}^n} \cong \bigoplus_{\substack{0 \le k' \le n \\ k' \equiv b \mod 2}} M^{\bar{\delta}}(k') \otimes \mathcal{L}(n, k').$$

Moreover, we have

$$\mathcal{L}(n, k') \otimes \mathcal{L}(1, \delta') \cong \bigoplus_{\substack{0 \le s \le n+1 \\ s \equiv k'+\delta' \mod 2}} L(c_n, h^n_{k'+1,s+1}) \otimes \mathcal{L}(n+1, s).$$

Hence,

$$\begin{aligned} &V_{\gamma_{\delta'}+A_1{}^{n+1}} \\ &\cong \bigoplus_{\substack{0 \le s \le n+1 \\ s \equiv k'+\delta' \mod 2}} \left(\bigoplus_{\substack{0 \le k' \le n \\ k' \equiv b \mod 2}} M^{\bar{\delta}}(k') \otimes L(c_n, h^n_{k'+1,s+1}) \right) \otimes \mathcal{L}(n+1, s) \end{aligned}$$

and we have

$$M^{\delta}(k) = \bigoplus_{\substack{0\le k'\le n\\ k'\cong b \bmod 2}} M^{\bar{\delta}}(k') \otimes L(c_n, h^n_{k'+1,k+1}),$$

as required. ∎

Theorem 3.9. *Let $\delta, \sigma \in \mathbb{Z}_2^n$ and $0 \le k, \ell \le n+1$. Suppose that $M^{\delta}(k) \cong M^{\sigma}(\ell)$. Then we have either (1) $k = \ell$ and $\delta = \sigma$ or (2) $k = n+1-\ell$ and $\delta = \sigma + \mathbf{1}$.*

Proof. We shall prove the theorem by induction on n. For $n = 1$, $M^{(1)} \cong L(1/2, 0)$. The theorem clearly holds. The case for $n = 2$ has also been proved in [19].

Now let $n > 2$ and denote $b = \sum_{i=0}^{n-1} \delta_i$ and $c = \sum_{i=0}^{n-1} \sigma_i$. Since $M^{\delta}(k) \cong M^{\sigma}(\ell)$, by the previous lemma, for any $0 \le k' \le n$ with $k' \equiv b \mod 2$, there is $0 \le \ell' \le n$ with $\ell' \equiv c \mod 2$ such that

$$M^{\bar{\delta}}(k') \cong M^{\bar{\sigma}}(\ell') \quad \text{and} \quad h^n_{k'+1,k+1} = h^n_{\ell'+1,\ell+1}.$$

Since $n \ge 3$, there is k' such that $k' \ne n-k'$. For such a k', we have either (1) $\bar{\delta} = \bar{\sigma}$ and $\ell' = k' \ne n-k'$ or (2) $\bar{\delta} = \bar{\sigma} + \mathbf{1}$ and $\ell = n-k' \ne k'$ by the induction hypothesis.

Case 1. $\bar{\delta} = \bar{\sigma}$ and $\ell' = k' \ne n-k'$.
In this case, $b \equiv k' = \ell' \equiv c \mod 2$ and thus $\delta = \sigma$. Moreover,$h^n_{k'+1,k+1} = h^n_{\ell'+1,\ell+1}$ and $k' = \ell'$ implies $k = \ell$.

Case 2. $\bar{\delta} = \bar{\sigma} + \mathbf{1}$ and $\ell' = n-k' \ne k'$.
In this case, we have $h^n_{k'+1,k+1} = h^n_{n-k'+1,\ell+1}$ and thus $\ell = n+1-k$. Moreover, $k' = n-\ell \equiv n+c \mod 2$. Thus,

$$b = \sum_{i=0}^{n-1} \delta_i \equiv n + \sum_{i=0}^{n-1} \sigma_i \mod 2$$

and we have $\delta_{n-1} \equiv \sigma_{n-1} + 1 \mod 2$ and $\delta = \sigma + \mathbf{1}$. Note that $\sum_{i=0}^{n-2} \delta_i \equiv \sum_{i=0}^{n-2} \sigma_i + n - 1 \mod 2$ as $\bar{\delta} = \bar{\sigma} + \mathbf{1}$. ∎

We believe that $M^{\delta}(k)$'s are all the irreducible modules for M and end this section with the following conjecture.

Conjecture 3.10. When n is an even integer,

$$\{M^{\delta}(2k) \mid \delta \in \mathbb{Z}_2^n, 0 \le 2k \le n+1\}$$

is a complete set of all inequivalent irreducible modules for M. On the other hand, if n is odd, then

$$\{M^{\delta}(k) \mid 0 \leq k \leq n+1, \delta \in \mathbb{Z}_2^n \text{ with } |\delta| \equiv k \bmod 2\}$$

is a complete set of all inequivalent irreducible modules for M.

4. The Symmetric Group S_{n+1} and Automorphisms of M

In this section, we shall discuss the automorphisms of M. We shall show that the Weyl group $W(A_n)(\cong S_{n+1})$ acts faithfully on M and the VOA M is generated by its weight 2 subspace.

4.1. The action of $W(A_n)$ on M

Let ${A_1}^{n+1} = \mathbb{Z}\alpha^0 \oplus \mathbb{Z}\alpha^1 \oplus \cdots \oplus \mathbb{Z}\alpha^n$ be the orthogonal sum of $n+1$ copies of A_1. Denote

$$N = \mathrm{span}_{\mathbb{Z}}\{-\alpha^0 + \alpha^1, -\alpha^1 + \alpha^2, \ldots, -\alpha^{n-1} + \alpha^n\}$$

and

$$\Phi = \left\{ \left. \frac{\pm(\alpha^i - \alpha^j)}{\sqrt{2}} \right| \ 0 \leq i < j \leq n \right\}.$$

Then N is isomorphic to the lattice $\sqrt{2}A_n$ and Φ is a root system of type A_n.

Let S_{n+1} be the symmetry group on the set $\{\alpha^0, \alpha^1, \ldots, \alpha^n\}$. Then S_{n+1} acts naturally on Φ and N. Actually, S_{n+1} is exactly the Weyl group of Φ and $S_{n+1} \cong W(\Phi) = W(A_n)$. Note that the action of S_{n+1} on N also induces an action on the lattice VOA V_N by defining

$$\begin{aligned}&\sigma(\beta_1(-i_1)\beta_2(-i_2)\cdots\beta_k(-i_k) \otimes e^{\beta})\\ =&(\sigma\beta_1)(-i_1)(\sigma\beta_2)(-i_2)\cdots(\sigma\beta_k)(-i_k) \otimes e^{\sigma\beta}\end{aligned}$$

for any $\sigma \in S_{n+1}$ and $\beta_1(-i_1)\beta_2(-i_2)\cdots\beta_k(-i_k) \otimes e^{\beta} \in V_N$.

Lemma 4.1. *For any $\sigma \in S_{n+1}$ and $u \in M$, we have $\sigma u \in M$. Hence M is S_{n+1}-invariant and S_{n+1} acts on M.*

Proof. Recall that

$$\begin{aligned} M &= \left\{ v \in V_{\sqrt{2}A_n} \left| \, u_k v = 0 \text{ for all } k \geq 0 \text{ and } u \in W_{n+1}\left(\frac{2n}{n+3}\right) \right. \right\} \\ &= \left\{ v \in V_{\sqrt{2}A_n} \,|\, \omega_1^{n+1} u = 0 \right\}, \end{aligned}$$

where

$$\begin{aligned}\omega^{n+1} =&\omega - \frac{1}{2(n+3)} \sum_{\alpha\in\Phi^+} \left(\alpha(-1)^2\cdot 1 - 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})\right)\\ =&\frac{1}{n+3}\left(2\omega + \sum_{\alpha\in\Phi^+}(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})\right).\end{aligned}$$

Note that ω^{n+1} is fixed by S_{n+1} and thus for any $\sigma \in S_{n+1}$ and $u \in M$, we have

$$\omega_1^{n+1}(\sigma u) = (\sigma\omega^{n+1})_1(\sigma u) = \sigma(\omega_1^{n+1}u) = 0.$$

Hence, $\sigma u \in M$. ∎

Next we shall consider certain conformal vectors of central charge $1/2$ in M.

Lemma 4.2. *For any $\alpha \in \Phi$, define*

$$\omega(\alpha) = \frac{1}{8}\alpha(-1)^2\cdot 1 - \frac{1}{4}(e^{\sqrt{2}\alpha} + e^{\sqrt{2}\alpha}).$$

Then $\omega(\alpha)$ is a conformal vector of central charge $1/2$ in M.

Proof. Since $\langle\sqrt{2}\alpha, \sqrt{2}\alpha\rangle = 4$, it is well known (cf. [5, 23]) that

$$\omega(\alpha) = \frac{1}{8}\alpha(-1)^2\cdot 1 - \frac{1}{4}(e^{\sqrt{2}\alpha} + e^{\sqrt{2}\alpha})$$

is a conformal vector of central charge $1/2$. In addition,

$$\begin{aligned}&\omega_3^{n+1}\omega(\alpha) = \langle\omega^{n+1}, \omega(\alpha)\rangle\\ =&\frac{1}{4(n+3)}\langle 2\omega + \sum_{\beta\in\Phi^+}(e^{\sqrt{2}\beta} + e^{-\sqrt{2}\beta}), \frac{1}{2}\alpha(-1)^2\cdot 1 - (e^{\sqrt{2}\alpha} + e^{\sqrt{2}\alpha})\rangle\\ =&\frac{1}{4(n+3)}(\frac{1}{2}\langle\alpha,\alpha\rangle^2 - 2) = 0.\end{aligned}$$

Hence ω^{n+1} and $\omega(\alpha)$ are mutually orthogonal. Thus $\omega_1^{n+1}\omega(\alpha) = 0$ and $\omega(\alpha) \in M$. ∎

Proposition 4.3. *For $n \geq 2$, the action of S_{n+1} on M is faithful and hence* $\mathrm{Aut}\, M$ *contains a subgroup isomorphic to S_{n+1}.*

Proof. By the previous lemma, the set $\{\omega(\alpha) | \, \alpha \in \Phi^+\}$ is contained in M. Moreover, it is clear that $\sigma(\omega(\alpha)) = \omega(\sigma\alpha)$ for any $\alpha \in \Phi^+$ and $\sigma \in S_{n+1}$. Note that $\omega(\alpha) = \omega(-\alpha)$ and we shall identify $\sigma\alpha$ with $-\sigma\alpha$ if $\sigma\alpha \in \Phi^-$. Since S_{n+1} acts faithfully on Φ, using the above identification, the action of S_{n+1} on Φ^+ is still faithful for $n \geq 2$. Hence the action of S_{n+1} on M is also faithful. ∎

Next we shall show that M is generated by $\{\omega(\alpha) | \alpha \in \Phi^+\}$.

Lemma 4.4. *For any* $n \geq 1$, $\dim M_2 = n(n+1)/2$.

Proof. First we shall recall that

$$M = M^{(n)} \cong \bigoplus_{\substack{0 \leq k_j \leq j+1, \\ j=0,\ldots,n-1 \\ k_j \equiv 0 \bmod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_n, h^n_{k_{n-1}+1,1}).$$

Note that

$$h^m_{r,s} = \frac{[r\,(m+3) - s\,(m+2)]^2 - 1}{4\,(m+2)\,(m+3)}$$

and thus we have

$$h^m_{2k+1,1} = \frac{k(k(m+3)+1)}{m+2} = k^2 + \frac{k(k+1)}{m+2}$$

and

$$h^m_{2k+1,3} = (k-1)^2 + \frac{k(k+1)}{m+2} - \frac{2}{m+3}.$$

First, we shall show that

$$h^1_{2k_0+1,2k_1+1} + \cdots + h^n_{2k_{n-1}+1,1} \gneqq 2$$

if there exists any $k_i > 1$.

Suppose $k_i > 1$ for some $1 \leq i \leq n-1$. Let ℓ be the largest integer such that $k = k_\ell > 1$ and let $j > \ell$ be the smallest integer such that $k_j = 0$. Then $k_i = 1$ for all $\ell < i < j$. In this case,

$$\begin{aligned}
&h^{\ell+1}_{2k_\ell+1,2k_{\ell+1}+1} + \cdots + h^j_{2k_{j-1}+1,2k_j+1} \\
&= h^{\ell+1}_{2k+1,3} + \cdots + h^j_{3,1} \\
&=((k-1)^2 + \frac{k(k+1)}{\ell+3} - \frac{2}{\ell+4}) + (\frac{2}{\ell+2+2} - \frac{2}{\ell+2+3}) + \cdots + (1 + \frac{2}{j+3}) \\
&=(k-1)^2 + \frac{k(k+1)}{\ell+3} + 1 \gneqq 2
\end{aligned}$$

and hence $h^1_{2k_0+1,2k_1+1} + \cdots + h^n_{2k_{n-1}+1,1} \gneqq 2$

Similarly, if there exists $0 \le i < j \le n-1$ such that $k_{i-1} = 0, k_i = \cdots = k_{j-1} = 1,$ and $k_j = 0$, then

$$\begin{aligned}&h^i_{1,3} + h^{i+1}_{3,3} + \cdots h^{j-1}_{3,3} + h^j_{3,1}\\ =&(1 - \frac{2}{i+3}) + (\frac{2}{i+1+2} - \frac{2}{i+1+3}) + \cdots\\ &+ (\frac{2}{j-1+2} - \frac{2}{j-1+3}) + (1 + \frac{2}{j+2}) = 2\end{aligned}$$

Therefore,

$$h^1_{2k_0+1,2k_1+1} + \cdots + h^n_{2k_{n-1}+1,1} = 2$$

if and only if there exists $0 \le i < j \le n-1$ such that

$$k_0 = \cdots = k_{i-1} = 0,\ k_i = \cdots = k_{j-1} = 1, \quad \text{and} \quad k_j = \cdots = k_{n-1} = 0.$$

Hence, there are exactly $n(n-1)/2$ highest weight vectors of weight 2 in M and we have

$$\dim M_2 = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

as desired. ■

Proposition 4.5. *The Griess algebra M_2 is spanned by $\{\omega(\alpha) \mid \alpha \in \Phi^+\}$.*

Proof. By definition, it is clear that $\{\omega(\alpha) \mid \alpha \in \Phi^+\}$ is linearly independent over $\mathbb{C}$. Note that $|\Phi^+| = (n+1)n/2 = \dim M_2$ and hence we have $M_2 = \mathrm{span}_{\mathbb{C}}\{\omega(\alpha) \mid \alpha \in \Phi^+\}$. ■

Proposition 4.6. *The VOA M is generated by its weight 2 subspace M_2 and hence the VOA M is generated by $\{\omega(\alpha) \mid \alpha \in \Phi^+\}$.*

We shall divide the proof into several steps. First we shall review the notion of Neveu-Schwarz vertex operator superalgebras (SVOAs).

Let $\mathbf{NS} = Vir \oplus (\oplus_{m\in\frac{1}{2}+\mathbb{Z}}\mathbb{C}G_m)$ be the Neveu-Schwarz $N=1$ conformal algebra which has commutation relations:

$$\begin{aligned}[G_m, L_n] &= \left(m - \frac{n}{2}\right) G_{m+n},\\ [G_m, G_{m'}]_+ &= 2L_{m+m'} + \frac{1}{3}\left(m + \frac{1}{2}\right)\left(m - \frac{1}{2}\right)\delta_{m+m',0}c,\\ [c, \mathbf{NS}] &= 0,\end{aligned}$$

for $n \in \mathbb{Z}$ and $m, m' \in \frac{1}{2} + \mathbb{Z}$. For complex numbers c and h, let $N(c,h)$ be the irreducible highest weight $\mathbf{NS}$-module with the central charge c and the highest weight h. Then, $N(c, 0)$ has a SVOA structure and is generated by the Virasoro element and $G_{-3/2}\mathbf{1} \in N(c, 0)_{3/2}$ (cf. [22]).

We consider the tensor product of $\mathcal{L}(m, k)$ and $\mathcal{L}(2, 0) \oplus \mathcal{L}(2, 2)$. It is known [12] that $\mathcal{L}(m, k) \otimes (\mathcal{L}(2, 0) \oplus \mathcal{L}(2, 2))$ is a **NS**-module with the central charge

$$c'_m = \frac{3}{2}\left(1 - \frac{8}{(m+2)(m+4)}\right)$$

such that the action of **NS** commutes with the diagonal action of $\hat{sl}_2$. The decomposition of $\mathcal{L}(m, k) \otimes (\mathcal{L}(2, 0) \oplus \mathcal{L}(2, 2))$ as a $\hat{sl}_2 \oplus \mathbf{NS}$-module is determined in [12]. It is given by

$$\text{(4.1)} \quad \mathcal{L}(m, k) \otimes (\mathcal{L}(2, 0) \oplus \mathcal{L}(2, 2)) \cong \bigoplus_{\substack{0 \le k' \le m+2 \\ k' \equiv k \bmod 2}} \mathcal{L}(m+2, k') \otimes N(c'_m, h'^m_{k+1, k'+1}),$$

where

$$h'^m_{r,s} = \frac{\{r(m+4) - s(m+2)\}^2 - 4}{8(m+2)(m+4)}.$$

The SVOA $N(c'_m, 0)$ is the commutant subalgebra of $\mathcal{L}(m+2, 0)$ in the SVOA $\mathcal{L}(m, 0) \otimes (\mathcal{L}(2, 0) \oplus \mathcal{L}(2, 2))$. We shall denote the even (resp. odd) part of $N(c'_m, 0)$ by $N^0_{c'_m}$ (resp. $N^1_{c'_m}$). Note that

$$N^i_{c'_m} = N(c'_m, 0) \cap \Big(\mathcal{L}(m, 0) \otimes \mathcal{L}(2, 2i)\Big)$$

for $i = 0, 1$.

Now, let

$$X = \left\{ u \in M \,\middle|\, w_k u = 0 \text{ for all } w \in M^{(0^{n-2})}(0), k \ge 0 \right\}$$

be the commutant subalgebra of $M^{(0^{n-2})}(0)$ in $M = M^{(0^n)}(0)$, where (0^m) denotes the codeword $(0, \ldots, 0) \in \mathbb{Z}_2^m$. By the definition of M and $V_{A_1^{n+1}} = V_{A_1^{n-1}} \otimes V_{A_1} \otimes V_{A_1}$, X is also the commutant subalgebra of $\mathcal{L}(n+1, 0)$ in $\mathcal{L}(n-1, 0) \otimes \mathcal{L}(1, 0) \otimes \mathcal{L}(1, 0)$. By using the GKO construction, we have

$$\begin{aligned}
&\mathcal{L}(n-1, 0) \otimes \mathcal{L}(1, 0) \otimes \mathcal{L}(1, 0) \\
&= \bigoplus_{\substack{0 \le k \le n \\ k \equiv 0 \bmod 2}} L(c_{n-1}, h^{n-1}_{1,k+1}) \otimes \mathcal{L}(n, k) \otimes \mathcal{L}(1, 0) \\
&= \bigoplus_{\substack{0 \le k' \le n+1 \\ k \equiv 0 \bmod 2}} \left(\bigoplus_{\substack{0 \le k \le n \\ k \equiv 0 \bmod 2}} L(c_{n-1}, h^{n-1}_{1,k+1}) \otimes L(c_n, h^n_{k+1,k'+1}) \right) \otimes \mathcal{L}(n+1, k')
\end{aligned}$$

and hence

$$X = \bigoplus_{\substack{0 \le k \le n \\ k \equiv 0 \bmod 2}} L(c_{n-1}, h^{n-1}_{1,k+1}) \otimes L(c_n, h^n_{k+1,1}).$$

Note that $h^{n-1}_{1,k+1} + h^n_{k+1,1} = k^2/2$ and so $\dim X_2 = 3$.

Lemma 4.7.

(1) *The VOA X contains a subalgebra isomorphic to the tensor product $N^{\,0}_{c'_{n-1}} \otimes L(1/2, 0)$ and*

(4.2) $$X = N^0_{c'_{n-1}} \otimes L(1/2, 0) \oplus N^1_{c'_{n-1}} \otimes L(1/2, 1/2).$$

(2) *X is generated by the weight 2 subspace X_2.*

Proof. (1) By using the GKO construction,

$$\mathcal{L}(1,0) \otimes \mathcal{L}(1,0) = \mathcal{L}(2,0) \otimes L(1/2,0) \oplus \mathcal{L}(2,2) \otimes L(1/2,1/2)$$

and so

$$\begin{aligned} &\mathcal{L}(n-1,0) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0) \\ &= \mathcal{L}(n-1,0) \otimes \mathcal{L}(2,0) \otimes L(1/2,0) \oplus \mathcal{L}(n-1,0) \otimes \mathcal{L}(2,2) \otimes L(1/2,1/2) \end{aligned}$$

For $i = 0, 1$, by (4.1), $\mathcal{L}(n-1,0) \otimes \mathcal{L}(2,2i)$ is a direct sum of $\mathcal{L}(n+1,0) \otimes N^0_{c'_{n-1}}$-modules:

$$\mathcal{L}(n-1,0) \otimes \mathcal{L}(2,2i) = \bigoplus_{\substack{0 \le k \le n+1 \\ k \equiv 0 \bmod 2}} \mathcal{L}(n+1,k) \otimes N^i_{c'_{n-1}}(k)$$

where $N^0_{c'_{n-1}}(k) \oplus N^1_{c'_{n-1}}(k) = N(c'_{n-1}, h'^{n-1}_{1,k+1})$ and $N^i_{c'_{n-1}}(k)$ is an $N^0_{c'_{n-1}}$-module. Then,

$$\begin{aligned} &\mathcal{L}(n-1,0) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0) \\ &= \bigoplus_{\substack{0 \le k \le n+1 \\ k \equiv 0 \bmod 2}} \mathcal{L}(n+1,k) \otimes \left(N^0_{c'_{n-1}}(k) \otimes L(1/2,0) \oplus N^1_{c'_{n-1}}(k) \otimes L(1/2,1/2) \right). \end{aligned}$$

Hence,

$$X = N^0_{c'_{n-1}} \otimes L(1/2,0) \oplus N^1_{c'_{n-1}} \otimes L(1/2,1/2).$$

(2) First, we shall note that $N(c'_{n-1}, 0) = N^0_{c'_{n-1}} \oplus N^1_{c'_{n-1}}$ is generated by its Virasoro element and the element $G_{-3/2}\mathbf{1}$ as a SVOA. By (1), we have

$$X = N^0_{c'_{n-1}} \otimes L(1/2,0) \oplus N^1_{c'_{n-1}} \otimes L(1/2,1/2).$$

and hence X is generated by the Virasoro element of $N^0_{c'_{n-1}}$, $q \otimes G_{-3/2}\mathbf{1}$ and the Virasoro of $L(1/2, 0)$, where q is a highest weight vector of weight $1/2$ in $L(\frac{1}{2}, \frac{1}{2})$. As they are all of weight 2, X is generated by X_2.

Proof of Proposition 4.6. Finally, we shall show that $M = M^{(0^n)}$ is generated by the weight 2 subspace M_2 by induction on n.

Since

$$M^{(0)}(0) = L(\frac{1}{2}, 0),$$
$$M^{(0,0)}(0) = L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}),$$

M is generated by M_2 for $n = 1, 2$. Assume that $n \geq 3$, by (3.2), we have

$$M^{(0^n)}(0) = \bigoplus_{\substack{0 \leq k \leq n \\ k \equiv 0 \bmod 2}} M^{(0^{n-1})}(k) \otimes L(c_n, h^n_{k+1,1}).$$

Since $M^{(0^{n-1})}(k)$ contains $M^{(0^{n-2})}(0) \otimes L(c_{n-1}, h^{n-1}_{1,k+1})$ for each k, we have $M^{(0^{n-1})}(k) \otimes L(c_n, h^n_{k+1,1})$ is generated by $L(c_{n-1}, h^{n-1}_{1,k+1}) \otimes L(c_n, h^n_{k+1,1}) \subset X$ as an $M^{(0^{n-1})}(0) \otimes L(c_n, 0)$-module. Hence, $M^{(0^n)}(0)$ is generated by $M^{(0^{n-1})}(0)$ and X.

Now, by induction on n, we know that $M^{(0^{n-1})}(0)$ is generated by its weight 2 subspace $[M^{(0^{n-1})}(0)]_2$. On the other hand, X is generated by X_2 by Lemma 4.7. Therefore, $M = M^{(0^n)}(0)$ is generated by the weight 2 subspace M_2. ■

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