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# PERTURBATION OF EIGENVALUES FOR PERIODIC MATRIX PAIRS VIA JOINT SPECTRUM

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**Abstract.** We shall link the well-established concept of joint spectrum with the interesting and practical periodic eigenvalue problems (PEVPs). A Bauer-Fike perturbation theory, incorporating a Clifford algebra technique, for joint spectrum is applied to PEVPs, producing new perturbation results.

## 1. INTRODUCTION

The Bauer-Fike technique has been applied to the ordinary eigenvalue problem (OEP) [6]

$$A\mathbf{x} = \lambda\mathbf{x}$$

Let $\lambda(\cdot)$ denote the spectrum and $\|\cdot\|$ be any Hölder norm. For OEPs, we have the perturbation result that for any perturbed eigenvalue $\tilde{\lambda} \in \lambda(\tilde{A})$ (with `\tilde` $(\tilde{\cdot})$ indicating perturbation from now on),

$$\min_j |\tilde{\lambda} - \lambda_j| \leq \max\{\theta, \theta^{1/q}\} \tag{1}$$

where

$$\theta = C\|X\| \cdot \|Y\| \cdot \|\delta A\| \ , \quad \delta A \equiv \tilde{A} - A$$

with $C$ being some constant, $X$ and $Y = X^{-H}$ denoting respectively matrices containing right- and left-eigenvectors in their columns, and $q$ being the size of the largest Jordan block for $A$. The result applies to perturbations of any size and eigenvalues of any multiplicity and structure. The quantity $\kappa_X \equiv \|X\| \cdot \|Y\|$ can then be considered to be a condition number for the OEP. With small (asymptotic)

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perturbations, the result still holds for individual or clusters of eigenvalues, where $X$ and $Y$ are respectively replaced by individual or group of corresponding eigenvectors, and $q$ is the size of the largest Jordan block for the cluster. These results are consistent with many well-known results (see, e.g., [28, 31]).

For the generalized eigenvalue problem (GEP) [7]

$$A\mathbf{x} = \lambda B\mathbf{x}$$

and the matrix polynomial eigenvalue problem (MPEP) [10]

$$\sum_{k=0}^{m} \lambda^k A_k \mathbf{x} = \mathbf{0}$$

the possibility of infinite eigenvalues requires eigenvalues to be represented by ordered pairs $(\alpha_i, \beta_i)$ (assume w.l.o.g. that $\alpha_i^2 + \beta_i^2 = 1$), with the traditional eigenvalues represented by the quotients $\alpha_i/\beta_i$. The MPEP generalizes to

$$\sum_{k=0}^{m} \alpha^k \beta^{m-k} A_k \mathbf{x} = \mathbf{0} \tag{2}$$

If we use the more convenient scaling convention

$$(\alpha_i, \beta_i) = (\sin\phi_i, \cos\phi_i)\ , \quad \phi_i \in \left[0, \frac{\pi}{2}\right]$$

and measure distances by the chordal metric $\rho$ [26, 27], the resulting perturbation results will be simpler. Assume that both the original and perturbed matrix pencils are regular (to exclude the possibility of continuous spectra), we have, for any perturbed eigenvalue $(\tilde{\alpha}, \tilde{\beta}) = (\sin\tilde{\phi}, \cos\tilde{\phi})$,

$$\begin{aligned} &\min_j \rho\{(\sin\tilde{\phi}, \cos\tilde{\phi}); (\sin\phi_j, \cos\phi_j)\} \\ &= \|\tilde{\alpha}\beta_k - \tilde{\beta}\alpha_k| = \sin|\tilde{\phi} - \phi_k| \le \max\{\theta, \theta^{1/q}\} \end{aligned} \tag{3}$$

where

$$\theta = C\|X\| \cdot \|Y\| \cdot \|(\delta A_0, \cdots, \delta A_m)\|\ , \quad \delta A_i \equiv \tilde{A}_i - A_i$$

with $C$, $X$ and $Y$ defined as in (1), and $q$ equals the size of the largest Jordan block. (Obviously, the GEP is a special case of the MPEP, with $A_0 = -B$ and $A_1 = A$ in (2); see other details, like the exact form of $C$, $X$ and $Y$, in [7, 10].) Again, the result applies to a general spectrum for perturbations of any size, and to individual or clusters of eigenvalues for asymptotic perturbations. The product of the norms of the whole/part of the eigenvector matrices $X$ and $Y$ can again be considered a condition number for individual/clusters of eigenvalues (see also [26, 27, 29, 30] for related results).

In [12], the traditional Bauer-Fike technique was applied to periodic matrix pairs [4, 20, 21, 25]. The result looks essentially the same as that in (1) or (3), with $\theta$ on the right-hand-side (RHS) somewhat modified (see §2 for more details).

The perturbation results have been applied to inverse eigenvalue or pole assignment problems in [9, 8, 11]. It is hoped that the result in this paper can be applied to similar robust pole assignment problems for periodic control systems [4, 14, 16, 25].

In this paper, we shall generalized the concept of joint spectrum to regular PEVPs. The perturbation results by Freedman [15] can then be applied to produce new perturbation results for PEVPs. We have to overcome two main obstacles. Firstly, the concept of joint spectrum has to be generalized to cope with infinite eigenvalues in PEVPs. This is attempted in §4.2. Secondly, the traditional concept of joint spectrum requires common eigenvectors, which are absent in PEVPs. This issue is addressed in §4.1.

The plan of the paper is as follows. We shall describe PEVPs in §2, including some perturbation results from [12, 20]. The basic results of joint spectrum and its perturbation are included in §3. The necessary generalizations of the concept of joint spectrum and the application to PEVPs are contained in §4. The paper is concluded in §5.

Some words of warning before proceeding. Comparison of perturbation results is a risky business. Typically, error bounds and condition numbers are simplified upper bounds of more complicated quantities and a better (worse) upper bound does not always imply a smaller (larger) error for the eigenvalue in question. Furthermore, optimization of such upper bounds are often possible but seldom performed because of costs and convenience, making such comparison of perturbation results even more perilous. Consequently, we do not claim to have found the "best" perturbation result. Quite often, perturbation results are applied qualitatively, indicating when things go wrong. Nevertheless, our new perturbation results via joint spectrum provide a new tool of investigating periodic eigenvalues.

## 2. Periodic Matrix Pairs

We shall first introduce the basics of the periodic eigenvalue problemin this Section. More details can be found in [12, 20, 21].

Let $E_j,\ A_j \in \mathcal{C}^{n\times n}$ ($j = 1, \cdots, p$), where $E_{j+p} = E_j$ and $A_{j+p} = A_j$ for all $j$. We shall denote the periodic matrix pairs of periodicity $p$ by $\{(A_j,\ E_j)\}_{j=1}^{p}$. In this paper, the indices $j$ for all periodic coefficient matrices are chosen in $\{1, \cdots, p\}$ modulo $p$. The equation

$$\beta_j A_j \mathbf{x}_{j-1} = \alpha_j E_j \mathbf{x}_j \tag{4}$$

defines the nonzero right-eigenvectors $\mathbf{x}_j$ for complex variables $(\alpha_j,\ \beta_j)$. Similarly, the equation

$$\beta_{j-1}\mathbf{y}_j^H A_j = \alpha_j \mathbf{y}_{j-1}^H E_{j-1} \tag{5}$$

defines the nonzero left-eigenvectors $\mathbf{y}_j$. The ordered pairs of products $(\pi_\alpha,\ \pi_\beta) = (\prod_{j=1}^p \alpha_j,\ \prod_{j=1}^p \beta_j)$ then constitute the spectrum, with the traditional eigenvalues being the quotients $\pi_\alpha/\pi_\beta$. Because of the possibility of infinite eigenvalues, we shall deal with spectra in their ordered pair representation, with equality interpreted in the sense of the corresponding equivalence relationship for quotients.

Using the notation $\mathrm{col}[\mathbf{x}_j]_{j=1}^p \equiv \left[\mathbf{x}_1^T, \cdots, \mathbf{x}_p^T\right]^T$, the eigenvalue equations (4) and (5) can also be written as

$$C\begin{pmatrix} \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_p \end{pmatrix} \mathrm{col}[\mathbf{x}_j]_{j=1}^p =$$

$$\begin{bmatrix} \alpha_1 E_1 & & & & -\beta_1 A_1 \\ -\beta_2 A_2 & \alpha_2 E_2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\beta_p A_p & \alpha_p E_p \end{bmatrix} \mathrm{col}[\mathbf{x}_j]_{j=1}^p = \mathbf{0}$$

and

$$\left\{\mathrm{col}[\mathbf{y}_j]_{j=1}^p\right\}^H \tilde{C}\begin{pmatrix} \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_p \end{pmatrix} =$$

$$\left\{\mathrm{col}[\mathbf{y}_j]_{j=1}^p\right\}^H \begin{bmatrix} \alpha_2 E_1 & & & & -\beta_p A_1 \\ -\beta_1 A_2 & \alpha_3 E_2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\beta_{p-1} A_p & \alpha_1 E_p \end{bmatrix} = \mathbf{0}^T$$

In this paper, we shall avoid continuous spectrum by considering only regular matrix pairs, for which

$$\det C\begin{pmatrix} \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_p \end{pmatrix} = \sum_{k=0}^{n} c_k \pi_\alpha^k \pi_\beta^{n-k} \not\equiv 0 \tag{6}$$

and consequently $(\pi_\alpha,\ \pi_\beta) \neq (0, 0)$. For regular matrix pairs, at least one of the coefficients $c_k \neq 0$ and there are exactly $n$ eigenvalues for $\{(A_j,\ E_j)\}_{j=1}^p$, counting multiplicity.

The eigenvalue problem (4) reflects the behaviour of the linear discrete-time periodic system [4, 14, 16, 25]

$$E_j \mathbf{z}_{j+1} = A_j \mathbf{z}_j \tag{7}$$

in terms of its solvability and stability.

There has been much recent interest in periodic systems. A large variety of processes can be modelled through periodic systems, including multirate sampled-data systems, chemical processes, periodic time-varying filters and networks, and seasonal phenomena. Applications include the helicopter ground resonance damping problem and the satellite attitude control problem. Please refer to [4, 14] and the references therein for further information.

The solvability [25] of (7) is equivalent to the regularity of the pencil

$$\alpha\mathcal{E} - \beta\mathcal{A} \equiv \begin{bmatrix} \alpha E_1 & & & & -\beta A_1 \\ -\beta A_2 & \alpha E_2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\beta A_p & \alpha E_p \end{bmatrix}$$

From the characteristic polynomial in (6), it is easy to check that

$$\lambda\left(\{(A_j,\ E_j)\}_{j=1}^{p}\right) = \{(\alpha^p,\ \beta^p) |\det(\alpha\mathcal{E} - \beta\mathcal{A}) = 0\} \tag{8}$$

However, the periodic eigenvectors $\mathbf{x}_j$ and $\mathbf{y}_j$ cannot be solved via the generalized eigenvalue problem of the pencil $\alpha\mathcal{E} - \beta\mathcal{A}$. For an example [21], consider the following $n = 1,\ p = 3$ case:

$$(\alpha\mathcal{E} - \beta\mathcal{A})\,\mathrm{col}[x_i]_{i=1}^3 = \begin{bmatrix} \alpha & 0 & 0 \\ -\beta & \alpha & 0 \\ 0 & -\beta & \alpha \end{bmatrix} \mathrm{col}[x_i]_{i=1}^3 = 0 \tag{9}$$

The characteristic polynomial $\det(\alpha\mathcal{E} - \beta\mathcal{A}) = \alpha^3$, indicating a regular system. The system in (9) is satisfied by the unique eigenvalue $(\alpha,\ \beta) = (0,\ 1)$ and the corresponding eigenvector (for the generalized pencil) $[x_1, x_2, x_3] = [\gamma, 0, 0]$ for some nonzero constant $\gamma$. Thus it will be impossible to find nonzero $x_2$ and $x_3$ so that (9) holds. However, $x_j = 1\ (j = 1, 2, 3)$ defines an eigenvector sequence for $\{(A_j,\ E_j)\}_{j=1}^3$ corresponding to the zero eigenvalue $(\alpha,\ \beta) = (0,\ 1)$, with $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 1$.

The situation is summarized by the following theorem from [21]:

**Theorem 2.1.** *Let $\alpha\mathcal{E} - \beta\mathcal{A}$ be a regular pencil. If $(\alpha,\ \beta) \in \lambda(\alpha\mathcal{E} - \beta\mathcal{A})$ then there exist complex numbers $\alpha_j,\ \beta_j\ (j = 1, \cdots, p)$ and nonzero vectors $\{\mathbf{x}_j\}_{j=1}^p$ for which (4) holds, and $(\pi_\alpha,\ \pi_\beta) = (\alpha^p,\ \beta^p)$.*

We also have a periodic Schur decomposition: [5, 21]

**Theorem 2.2.** (Periodic Schur Theorem). *Let* $\{(A_j,\ E_j)\}_{j=1}^p$ *be regular matrix pairs.*
*There exist unitary matrices* $Q_j,\ Z_j\ (j=1,\cdots,p)$ *such that*

$$Q_j^H A_j Z_{j-1} = \hat{A}_j\ ,\quad Q_j^H E_j Z_j = \hat{E}_j \quad (j=1,\cdots,p)$$

*are all upper triangular, with* $Z_0 = Z_p$. *Moreover, the diagonal parts*

$$\{[diag(\alpha_{j1},\cdots,\alpha_{jn}),\ diag(\beta_{j1},\cdots,\beta_{jn})]\}_{j=1}^p$$

*of* $\{(\hat{A}_j, \hat{E}_j)\}_{j=1}^p$ *determine all eigenvalues* $\left\{\left(\Pi_{j=1}^p \alpha_{jk},\ \Pi_{j=1}^p \beta_{jk}\right)\right\}_{j=1}^p$ *of* $\{(A_j, E_j)\}_{j=1}^p$, *and the eigenvalues can be arranged to appear in any order.*

We can also generalized the concept of eigenspaces as follows:

**Definition.** Let $\mathcal{X}_j$, $\mathcal{Y}_j$ $(j=1,\cdots,p)$ be subspaces in $\mathcal{C}^n$ of equal dimension. The pairs $\{(\mathcal{X}_j,\ \mathcal{Y}_j)\}_{j=1}^p$ are called periodic deflating subspaces of $\{(A_j,\ E_j)\}_{j=1}^p$ if

$$A_j\mathcal{X}_j \subset \mathcal{Y}_j\ ,\quad E_j\mathcal{X}_{j-1} \subset \mathcal{Y}_j \quad (j=1,\cdots,p)$$

Furthermore, the subspaces $\{\mathcal{X}_j\}_{j=1}^p$ are called periodic invariant subspaces of $\{(A_j,\ E_j)\}_{j=1}^p$.

We list some more further results as follows:

(i) Theorem 2.1 implies that $\lambda\left(\{(A_j,\ E_j)\}_{j=1}^p\right) = \lambda\left(\{(A_j^T,\ E_j^T)\}_{j=1}^p\right)$

(ii) An eigenvalue is said to be simple if it is in a linear factor in the characteristic polynomial.

(iii) Let $Z_1^{(j)},\ Q_1^{(j)} \in \mathcal{C}^{n\times r}$ be unitary and span, respectively, $\mathcal{X}_j$ and $\mathcal{Y}_j$. It can be verified [21] that $\{(\mathcal{X}_j,\ \mathcal{Y}_j)\}_{j=1}^p$ are periodic deflating subspaces of the regular matrix pairs $\{(A_j,\ E_j)\}_{j=1}^p$ if and only if the unitary matrices $Z_j = \left[Z_1^{(j)},\ Z_2^{(j)}\right]$, $Q_j = \left[Q_1^{(j)},\ Q_2^{(j)}\right] \in \mathcal{C}^{n\times n}$ satisfy

$$Q_j^H A_j Z_{j-1} = \begin{bmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ 0 & A_{22}^{(j)} \end{bmatrix}\ ,\ Q_j^H E_j Z_j = \begin{bmatrix} E_{11}^{(j)} & E_{12}^{(j)} \\ 0 & E_{22}^{(j)} \end{bmatrix}\ (j=1,\cdots,p)$$

where $A_{11}^{(j)},\ E_{11}^{(j)} \in \mathcal{C}^{r\times r}$, and both $\{(A_{11}^{(j)},\ E_{11}^{(j)})\}_{j=1}^p$ and $\{(A_{22}^{(j)},\ E_{22}^{(j)})\}_{j=1}^p$ are regular. Furthermore, if the intersection of the spectra of the two sub-matrix pairs is empty, the periodic deflation subspaces $\{(\mathcal{X}_j,\ \mathcal{Y}_j)\}_{j=1}^p$ are called simple periodic deflating subspaces, and $\{\mathcal{X}_j\}_{j=1}^p$ the simple periodic invariant subspaces.

### 2.1. Periodic Kronecker Canonical Form

From the periodic Schur decomposition in Theorem 2.2, we obtain a periodic Kronecker canonical form [19]:

**Theorem 2.3.** (Periodic Kronecker Canonical Form). *Suppose that the periodic matrix pairs* $\{(A_j,\ E_j)\}_{j=1}^{p}$ *are regular. Then there exist nonsingular matrices* $X_j$ *and* $Y_j$ $(j = 1, \cdots, p)$ *such that*

$$Y_j^H E_j X_j = \begin{bmatrix} I & 0 \\ 0 & E_j^0 \end{bmatrix}, \quad Y_j^H A_j X_{j-1} = \begin{bmatrix} A_j^f & 0 \\ 0 & I \end{bmatrix} \tag{10}$$

*where* $X_0 = X_p$*; and for* $j = 1, \cdots, p$*,*

$$J^{(j)} \equiv A_{j+p-1}^f A_{j+p-2}^f \cdots A_j^f$$

*is a Jordan matrix corresponding to the finite eigenvalues of* $\{(A_j,\ E_j)\}_{j=1}^{p}$*, and*

$$N^{(j)} \equiv E_j^0 E_{j+1}^0 \cdots E_{j+p-1}^0$$

*is a nilpotent Jordan matrix corresponding to the infinite eigenvalues.*

**Remark** It is clear from the proof of Theorem 2.3 in [19] that $A_j^f$ and $E_j^0$ are upper triangular. In addition, from Section 2.2 in [20], these matrices can be further reduced to be block-upper triangular. Each individual diagonal block in $A_j^f$ or $E_j^0$ relates to the corresponding Jordan block corresponding to a multiple eigenvalue in $\{(A_j,\ E_j)\}_{j=1}^{p}$.

Note also that, for different values of $j$, the Jordan matrices $J^{(j)}$ and $N^{(j)}$ in Theorem 2.3 may have different structures. Thus an eigenvalue, of a certain algebraic multiplicity, may have different geometric multiplicities dependent on $j$.

In many applications such as optimal control [14] and pole assignment [16], a sequence of periodic stable invariance subspaces for the positive semidefinite solution set of a periodic discrete-time Riccati equation [5] is needed. Here stability means the spectrum staying within the unit circle. Thus, the importance in studying the sensitivities of deflating subspaces and eigenvalues is self-evident.

Next we quote the Bauer-Fike perturbation result [12, Theorem 3.1]. We first design a symmetric set of notation. The periodic Kronecker canonical form in (10) now reads

$$Y_j^H E_j X_j = \Lambda_{\beta,j} = \begin{bmatrix} \Lambda_{\beta,j}^{(1)} & 0 \\ 0 & \Lambda_{\beta,j}^{(2)} \end{bmatrix}, \; Y_j^H A_j X_{j-1} = \Lambda_{\alpha,j} = \begin{bmatrix} \Lambda_{\alpha,j}^{(1)} & 0 \\ 0 & \Lambda_{\alpha,j}^{(2)} \end{bmatrix} \tag{11}$$

with identity matrices $\Lambda^{(1)}_{\beta,j}$ (associated by finite eigenvalues) and $\Lambda^{(2)}_{\alpha,j}$ (associated with infinite eigenvalues). The roles of $E_j$ and $A_j$, or finite and infinite eigenvalues, are then symmetric in the canonical form in (11). The final Kronecker canonical form of $\{(A_j,\ E_j)\}_{j=1}^{p}$ then involves the Jordan matrices, for $j = 1, \cdots, p$,

$$J^{(j)} \equiv \Lambda^{(1)}_{\alpha,j+p-1}\Lambda^{(1)}_{\alpha,j+p-2}\cdots\Lambda^{(1)}_{\alpha,j}\ , \quad N^{(j)} \equiv \Lambda^{(2)}_{\beta,j}\Lambda^{(2)}_{\beta,j+1}\cdots\Lambda^{(2)}_{\beta,j+p-1}$$

We do not need to distinguish between finite and infinite pairs $(\alpha_j,\ \beta_j)$ in the development that follows. The symmetric notation will be more convenient for analyzing clustering in Section 4 later.

From (11), we have

$$Y_j = [Y_{j1},\ Y_{j2}]\ , \quad X_j = [X_{j1},\ X_{j2}]$$

with

$$Y_{jk}^H E_j X_{jk} = \Lambda^{(k)}_{\beta,j}\ , \quad Y_{jk}^H A_j X_{j-1,k} = \Lambda^{(k)}_{\alpha,j} \tag{12}$$

for $j = 1, \cdots, p$ and $k = 1, 2$.

Define the perturbations

$$\begin{aligned} &\delta\Lambda^{(j)}_{\alpha} \\ &\equiv \tilde{\Lambda}^{(j)}_{\alpha} - \Lambda^{(j)}_{\alpha} = \tilde{\Lambda}_{\alpha,j+p-1}\tilde{\Lambda}_{\alpha,j+p-2}\cdots\tilde{\Lambda}_{\alpha,j} - \Lambda_{\alpha,j+p-1}\Lambda_{\alpha,j+p-2}\cdots\Lambda_{\alpha,j} \\ &= Y_{j+p-1}^H(A_{j+p-1} + \delta A_{j+p-1})X_{j+p-2}\cdots Y_j^H(A_j + \delta A_j)X_{j-1} \\ &\quad - Y_{j+p-1}^H A_{j+p-1}X_{j+p-2}\cdots Y_j^H A_j X_{j-1} \end{aligned} \tag{13}$$

and

$$\begin{aligned} \delta\Lambda^{(j)}_{\beta} &\equiv \tilde{\Lambda}^{(j)}_{\beta} - \Lambda^{(j)}_{\beta} = \tilde{\Lambda}_{\beta,j}\tilde{\Lambda}_{\beta,j+1}\cdots\tilde{\Lambda}_{\beta,j+p-1} - \Lambda_{\beta,j}\Lambda_{\beta,j+1}\cdots\Lambda_{\beta,j+p-1} \\ &= Y_j^H(E_j + \delta E_j)X_j\cdots Y_{j+p-1}^H(E_{j+p-1} + \delta E_{j+p-1})X_{j+p-1} \\ &\quad - Y_j^H E_j X_j\cdots Y_{j+p-1}^H E_{j+p-1}X_{j+p-1} \end{aligned} \tag{14}$$

Also, denote

$$\Delta_j \equiv \left\| \tilde{\pi}_\beta \delta\Lambda^{(j)}_{\alpha} - \tilde{\pi}_\alpha \delta\Lambda^{(j)}_{\beta} \right\| \tag{15}$$

We have the following Bauer-Fike Theorem [20, Theorem 2.3, Corollary 2.4]:

**Theorem 2.4.** (Bauer-Fike Theorem). *With the assumptions in this Section, we have the following three cases:*

**Case I.** (Large Perturbation). When $\min_{(\pi_\alpha,\ \pi_\beta)} |\tilde{\pi}_\alpha \pi_\beta - \pi_\alpha \tilde{\pi}_\beta| \geq 1$, we have

$$\min_{(\pi_\alpha,\ \pi_\beta)} \rho\{(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta),\ (\pi_\alpha,\ \pi_\beta)\} \leq \frac{\theta_1}{\|(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta)\|_2 \cdot \|(\pi_\alpha,\ \pi_\beta)\|_2} \tag{16}$$

where

$$\theta_1 \equiv \max_j \{c_1 \Delta_j\}\ , \quad c_1 \equiv \min\left\{\frac{2q_j+1}{q_j+1}, q_j\right\}$$

and $q_j$ is the size of the largest Jordan block in $J^{(j)}$ or $N^{(j)}$.

**Case II.** (Small Perturbation). When $\Delta_j$ are sufficiently small, let $\min_{(\pi_\alpha,\pi_\beta)}$ $|\tilde{\pi}_\alpha \pi_\beta - \pi_\alpha \tilde{\pi}_\beta| \leq 1$ with the minimum occurring at a Jordan block $B_{jk} \equiv \tilde{\pi}_\beta J_{\alpha k}^{(j)} - \tilde{\pi}_\alpha J_{\beta k}^{(j)}$ of size $\hat{q}_j$. Denote by $P_j$ the columns of the identity matrix $I_n$ which pick out the Jordan block

$$B_{jk} = P_j^T \left[\tilde{\pi}_\beta \Lambda_\alpha^{(j)} - \tilde{\pi}_\alpha \Lambda_\beta^{(j)}\right] P_j$$

(ignoring the dependence of $\hat{q}_j$ and $P_j$ on $k$ to simplify the notation.) We then have

$$\min_{(\pi_\alpha,\ \pi_\beta)} \rho\{(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta),\ (\pi_\alpha,\ \pi_\beta)\} \leq \frac{\theta_2}{\|(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta)\|_2 \cdot \|(\pi_\alpha,\ \pi_\beta)\|_2} + O(\hat{\Delta}_j^2) \tag{17}$$

with

$$\theta_2 \equiv \max_j \left\{\left(c_1 \hat{\Delta}_j\right)^{1/\hat{q}_j}\right\}\ , \quad \hat{\Delta}_j \equiv \left\|P_j^T \left[\tilde{\pi}_\beta \delta\Lambda_\alpha^{(j)} - \tilde{\pi}_\alpha \delta\Lambda_\beta^{(j)}\right] P_j\right\| \tag{18}$$

and

$$c_1 \equiv \min\left\{\frac{2\hat{q}_j+1}{\hat{q}_j+1}, \hat{q}_j\right\}$$

**Case III.** (Intermediate Perturbation). For any other perturbation, we have

$$\min_{(\pi_\alpha,\ \pi_\beta)} \rho\{(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta),\ (\pi_\alpha,\ \pi_\beta)\} \leq \frac{\theta_3}{\|(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta)\|_2 \cdot \|(\pi_\alpha,\ \pi_\beta)\|_2} \tag{19}$$

where

$$\theta_3 = \max_j \{\phi_j,\ \phi_j^{1/q_j}\}\ , \quad \phi_j \equiv c_1 \Delta_j\ , \quad c_1 \equiv \min\left\{\frac{2q_j+1}{q_j+1}, q_j\right\}\ ,$$

and $q_j$ is the size of the largest Jordan block in $J^{(j)}$ or $N^{(j)}$.

**Remarks.**

(1) The coefficient $c_1 = 1$ when $q_j, \hat{q}_j = 1$ and $c_1 < 2$ otherwise.

(2) We can expand the expressions in $\Delta_j$ and $\hat{\Delta}_j$, showing the results in Theorem 2.4 in terms of the $X_j$, $Y_j$, $\delta A_j$ and $\delta E_j$, using the definitions in (13) and (14). The expressions will be tedious. However, it is clear that condition numbers, in terms of products of norms of whole or part of $X_j$ and $Y_j$, can be obtained. We shall attempt this exercise for the case when a simple eigenvalues is perturbed with a small perturbation in §2.2.

(3) Case III is analogous to [28, Theorem 1.12, p. 174].

(4) Notice that the chordal metric $\rho$ in (16), (17) and (19) are independent of scaling of the eigenvalues $(\pi_\alpha, \pi_\beta)$ and $(\tilde{\pi}_\alpha, \tilde{\pi}_\beta)$. The corresponding error bounds can be made independent of $(\pi_\alpha, \pi_\beta)$ and $(\tilde{\pi}_\alpha, \tilde{\pi}_\beta)$ by the scaling $\|(\pi_\alpha, \pi_\beta)\| = \|(\tilde{\pi}_\alpha, \tilde{\pi}_\beta)\| = 1$, together with

$$\|\Delta_j\| \leq \|(\tilde{\pi}_\alpha, \tilde{\pi}_\beta)\| \left\| \begin{bmatrix} \delta\Lambda_\alpha^{(j)} \\ \delta\Lambda_\beta^{(j)} \end{bmatrix} \right\| , \quad \|\hat{\Delta}_j\| \leq \|(\tilde{\pi}_\alpha, \tilde{\pi}_\beta)\| \left\| \begin{bmatrix} P_j^T \delta\Lambda_\alpha^{(j)} P_j \\ P_j^T \delta\Lambda_\beta^{(j)} P_j \end{bmatrix} \right\|$$

When $q_j, \hat{q}_j = 1$ in (17) or (19), we require only $\|(\pi_\alpha, \pi_\beta)\| = 1$ because of cancellation (see, e.g., (25) and (26)). Similarly, when a simple eigenvalue is perturbed asymptotically in Section 4, the error bounds is dependent only on $(\pi_\alpha, \pi_\beta)$.

(5) The chordal metric $\rho\{(\tilde{\pi}_\alpha, \tilde{\pi}_\beta); (\pi_\alpha, \pi_\beta)\}$ is unchanged when the original and the perturbed eigenvalues, or the eigenvalues and their reciprocals, are swapped. Consequently, we need to consider only those eigenvalues on or within the unit circle. For eigenvalues outside the unit circle, we can consider the reciprocals of the eigenvalues, i.e., interchanging $\pi_\alpha$ and $\tilde{\pi}_\alpha$ with $\pi_\beta$ and $\tilde{\pi}_\beta$, respectively. Combine with the observation in the last remark, we need only to consider finite eigenvalues with $\tilde{\pi}_\beta = 1$ in proving Theorem 2.4.

(6) In Lin and Sun [20, Theorem 2.3, Corollary 2.4], the implicit function theorem has been applied to obtain perturbation expansions for simple eigenvalues, their associated simple periodic eigenvectors and simple periodic deflating subspaces (see also [2]) for related results). The perturbation result for simple eigenvalues under asymptotic perturbations is similar to those in §2.2 below.

(7) The pessimistic Case III always holds and we have the condition for which Case I applies. Case II corresponds to perturbations which are (asymptotically) small, as in [20]. Detailed conditions for this case can be written down but are seldom checked. In general, perturbation results indicate pitfalls. Error estimation usually requires too much information or computing resources. When perturbations are too large, it is usually too much to ask for perturbation

results. Bauer-Fike results are valid for perturbations of *any* size but the trade off is the lack of results for deflating subspaces (they will be jumbled up). Note that other approaches usually requires perturbations to be asymptotically small.

### 2.2. Simple Eigenvalues

For a simple eigenvalue $(\pi_\alpha,\ \pi_\beta) = (\prod_j \alpha_{j1},\ \prod_j \beta_{j1})$ perturbed asymptotically to $(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta)$, as in Case II in Theorem 2.4. Assume for convenience and without loss of generality that $(\pi_\alpha,\ \pi_\beta)$ appears at the (1,1) position. The definitions in (13), (14) and (18) imply

$$\begin{aligned}\hat{\Delta}_j &= \left|\mathbf{e}_1^T \left[\tilde{\pi}_\beta \delta\Lambda_\alpha^{(j)} - \tilde{\pi}_\alpha \delta\Lambda_\beta^{(j)}\right] \mathbf{e}_1\right| \\ &= \left|\mathbf{e}_1^T \left[\tilde{\pi}_\beta \left(\prod_{k=j+p-1}^{j} \tilde{\Lambda}_{\alpha,k} - \prod_{k=j+p-1}^{j} \Lambda_{\alpha,k}\right)\right.\right. \\ &\qquad \left.\left. -\tilde{\pi}_\alpha \left(\prod_{k=j}^{j+p-1} \tilde{\Lambda}_{\beta,k} - \prod_{k=j}^{j+p-1} \Lambda_{\beta,k}\right)\right] \mathbf{e}_1\right|\end{aligned}$$

Note that the terms linear in the perturbations matrices $\delta A_k$ (or $\delta E_k$) are products of block-diagonal matrices $\Lambda_{\alpha,k}$ (or $\Lambda_{\beta,k}$) with one single $Y_k^H \delta A_k X_{k-1}$ (or $Y_k^H \delta E_k X_k$). Denote

$$\delta\alpha_k \equiv \mathbf{y}_k^H \delta A_k \mathbf{x}_{k-1}\ , \quad \delta\beta_k \equiv \mathbf{y}_k^H \delta E_k \mathbf{x}_k \tag{20}$$

and ignore higher order terms, we arrive at

$$\begin{aligned}\hat{\Delta}_j &\preceq \left|\tilde{\pi}_\beta \sum_{l=j}^{j+p-1} \left(\prod_{k\neq l} \alpha_k\right) \delta\alpha_l - \tilde{\pi}_\alpha \sum_{l=j}^{j+p-1} \left(\prod_{k\neq l} \beta_k\right) \delta\beta_l\right| \\ &\le |\tilde{\pi}_\beta \pi_\alpha| \sum_{l=j}^{j+p-1} \left|\frac{\delta\alpha_l}{\alpha_l}\right| + |\tilde{\pi}_\alpha \pi_\beta| \sum_{l=j}^{j+p-1} \left|\frac{\delta\beta_l}{\beta_l}\right| \qquad (21) \\ &\le \|(\tilde{\pi}_\beta \pi_\alpha,\ \tilde{\pi}_\alpha \pi_\beta)\|_2 \cdot \sum_{l=j}^{j+p-1} \left\|\begin{bmatrix} \delta\alpha_l/\alpha_l \\ \delta\beta_l/\beta_l \end{bmatrix}\right\|_2\end{aligned}$$

Now let $\delta \equiv \max\{\|\delta A_j\|_2,\ \|\delta E_j\|_2\}$ and $|\alpha|_j,\ |\beta_j| \neq 0$. We obtain, using the definitions in (20) and the properties of norms,

$$\hat{\Delta}_j \preceq \|(\pi_\alpha,\ \pi_\beta)\|_2 \cdot \|(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta)\|_2 \cdot \sum_{j=1}^{p} \left[\|\mathbf{y}_j\|_2 \left(\frac{\|\mathbf{x}_{j-1}\|_2^2}{|\alpha_j|^2} + \frac{\|\mathbf{x}_j\|_2^2}{|\beta_j|^2}\right)^{1/2}\right] \cdot \delta$$

Substitute into (17) and with $c_1 = q_j = 1$, we obtain the perturbation result

$$\min_{(\pi_\alpha,\pi_\beta)} \rho\{(\tilde{\pi}_\alpha, \tilde{\pi}_\beta), (\pi_\alpha, \pi_\beta)\} \preceq \sum_{j=1}^{p} \left[ \|\mathbf{y}_j\|_2 \left( \frac{\|\mathbf{x}_{j-1}\|_2^2}{|\alpha_j|^2} + \frac{\|\mathbf{x}_j\|_2^2}{|\beta_j|^2} \right)^{1/2} \right] \cdot \delta \tag{22}$$

Alternatively, we can replace the two $\|\cdot\|_2$ in (21) with, respectively, $\|\cdot\|_\infty$ and $\|\cdot\|_1$. The result (22) now has the form

$$\begin{aligned} &\min_{(\pi_\alpha,\pi_\beta)} \rho\{(\tilde{\pi}_\alpha, \tilde{\pi}_\beta), (\pi_\alpha,\ \pi_\beta)\} \\ &\preceq \frac{|\pi_\alpha \pi_\beta|}{|\pi_\alpha|^2 + |\pi_\beta|^2} \cdot \sum_{j=1}^{p} \left[ \|\mathbf{y}_j\|_2 \left( \frac{\|\mathbf{x}_{j-1}\|_2}{|\alpha_j|} + \frac{\|\mathbf{x}_j\|_2}{|\beta_j|} \right) \right] \cdot \delta \end{aligned} \tag{23}$$

(We have replaced the perturbed eigenvalue with the original one on the right-hand-side of the final result.) The result in (23) is identical to that in [20, Theorem 3.2], when $\tau_j, \sigma_j = 1$.

When $\alpha_{l_0} = 0$, $\beta_j \neq 0$ for all $j$ (because of regularity) and (21) degenerates to

$$\begin{aligned} \hat{\Delta}_j &\preceq \left| \tilde{\pi}_\beta \left( \prod_{k \neq l_0} \alpha_k \right) \delta\alpha_{l_0} - \tilde{\pi}_\alpha \sum_{l=j}^{j+p-1} \left( \prod_{k \neq l} \beta_k \right) \delta\beta_{l_0} \right| \\ &\leq \left| \left( \tilde{\pi}_\beta \prod_{k \neq l_0} \alpha_k,\ -\tilde{\pi}_\alpha \pi_\beta \right) \begin{bmatrix} \delta\alpha_{l_0} \\ \sum_{l=j}^{j+p-1} \delta\beta_l/\beta_l \end{bmatrix} \right| \end{aligned}$$

Similar to (23), we obtain

$$\begin{aligned} &\min_{(\pi_\alpha,\ \pi_\beta)} \rho\{(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta),\ (\pi_\alpha,\ \pi_\beta) \\ &\preceq \left( \frac{\prod_{k \neq l_0} |\alpha_k|}{|\pi_\beta|} \cdot \|\mathbf{x}_{l_0-1}\|_2 \|\mathbf{y}_{l_0}\|_2 + \frac{|\pi_\alpha|}{|\pi_\beta|} \sum_{j=1}^{p} \frac{\|\mathbf{x}_j\|_2 \|\mathbf{y}_j\|_2}{|\beta_j|} \right) \cdot \delta \end{aligned} \tag{24}$$

There is also the possibility of other $\alpha_k = 0$, eliminating the first term in (24). The result for $\pi_\beta = 0$ is similar, interchanging the $\alpha$s and $\beta$s in (24).

From the above discussion, the asymptotic error bounds for individual pair $(\alpha_i,\ \beta_i)$ $(j = 1, \cdots, p)$ can be shown, in a similar fashion, to be

$$\min_{(\alpha_j,\ \beta_j)} \rho\{(\tilde{\alpha}_j,\ \tilde{\beta}_j),\ (\alpha_j,\ \beta_j)\} \preceq \frac{\|\mathbf{y}_j\|_2 \left( |\beta_j| \|\mathbf{x}_{j-1}\|_2 + |\alpha_j| \|\mathbf{x}_j\|_2 \right)}{|\alpha_j|^2 + |\beta_j|^2} \cdot \delta$$

(which holds for all values of $\alpha_j$ and $\beta_j$).

When $p = 1$, the above results reduce to the case for GEPs, with (22) now reads

$$\min_{(\pi_\alpha,\ \pi_\beta)} \rho\{(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta),\ (\pi_\alpha,\ \pi_\beta)\} \preceq \frac{\sqrt{|\pi_\alpha|^2+|\pi_\beta|^2}}{|\pi_\alpha \pi_\beta|} \cdot \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2 \cdot \delta$$

When $(\pi_\alpha, \pi_\beta) = (\sin\phi, \cos\phi)$, the RHS becomes $(2|\csc 2\phi|\ \|\mathbf{x}\|_2\ \|\mathbf{y}\|_2\ \delta)$, which may be large. A better result comes from (23), which becomes

(25) $$\min_{(\pi_\alpha,\ \pi_\beta)} \rho\{(\tilde{\pi}_\alpha,\ \tilde{\pi}_\beta),\ (\pi_\alpha,\ \pi_\beta)\} \preceq \frac{|\pi_\alpha|+|\pi_\beta|}{|\pi_\alpha|^2+|\pi_\beta|^2} \cdot \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2 \cdot \delta$$

(This result coincides with that of [30, Chapter 4, §4.2-2] with $\gamma_A = \gamma_B = 1$ and $p = \infty$.) With $(\pi_\alpha, \pi_\beta) = (\sin\phi, \cos\phi)$, the RHS equals

(26) $$|\sin\phi + \cos\phi|\ \|\mathbf{x}\|_2\ \|\mathbf{y}\|_2\ \delta \leq \sqrt{2}\ \|\mathbf{x}\|_2\ \|\mathbf{y}\|_2\ \delta$$

### 2.3. Clusters of Eigenvalues

We do not need to distinguish between finite and infinite pairs $(\alpha_j,\ \beta_j)$ in the development in Section 3. Indeed, we may have $(\Lambda_{\alpha 1}^{(j)},\ \Lambda_{\beta 1}^{(j)})$ and $(\Lambda_{\alpha 2}^{(j)},\ \Lambda_{\beta 2}^{(j)})$ representing different clusters of eigenvalues, so long as the intersection of the subspectra is empty and the diagonal assumption for $\Lambda_{\beta 1}^{(j)}$ and $\Lambda_{\alpha 1}^{(j)}$ is dropped. Although the results in Case II of Theorem 2.4 considers a multiple eigenvalue, it is straight forward to generalize the result to a cluster of eigenvalues. In (17), $P_j$ is then selected to extracting the appropriate cluster and $\hat{q}_j$ is the size of the largest Jordan block associated with the cluster. For details, see [12].

## 3. Joint Spectrum

We then introduce the concept of a joint spectrum for commuting matrices (see [3, 13, 22, 24] and the references therein for details). Let $A = (A_1, \cdots, A_m) \in (\mathcal{C}^{n\times n})^m$ be a $m$-tuple of $n \times n$ matrices with complex entries. A joint eigenvalue of $A$ is a vector $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_m) \in \mathcal{C}^m$ such that $A_j\mathbf{x} = \lambda_j\mathbf{x}$ for $j = 1, \cdots, m$ for some $\mathbf{x} \in \mathcal{C}^n \backslash \{\mathbf{0}\}$. Such an $\mathbf{x}$ is called a joint eigenvector of $A$. If $A_i$ are commuting there exists at least one joint eigenvalue. The joint spectrum $\lambda(A)$ of $A$ is the set of joint eigenvalues of $A$.

Clifford algebra [3, 22, 24] was used as a tool to study joint spectra and we summarize the technique here. Let $\Re_{(m)}$ denote the Clifford algebra generated by $\mathbf{e}_1, \cdots, \mathbf{e}_m$ with the relations

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad (i \neq j)\ , \quad \mathbf{e}_i^2 = -1 \quad (\forall i)$$

Then $\Re_{(m)}$ is an associate algebra over $\Re$ of dimension $2^m$. The elements $\mathbf{e}_S$, where $S$ runs over all subsets of $\{1, \cdots, m\}$ form a basis of $\Re_{(m)}$ if we define $\mathbf{e}_\emptyset = \mathbf{e}_0 = 1$ and $\mathbf{e}_S = \mathbf{e}_{s_1} \cdots \mathbf{e}_{s_k}$ when $S = \{s_1, \cdots, s_k\}$ with $1 \leq s_1 < s_2 < \cdots < s_k \leq m$.

Let $L(X)$ be the space of bounded linear operators on a vector space $X$. The Clifford operator $\text{Cliff}(A) \in \mathcal{C}^{n\times n} \otimes \Re_{(m)}$ of an $m$-tuple $A = (A_1, \cdots, A_m) \in (\mathcal{C}^{n\times n})^m$ is defined as

$$\text{Cliff}(A) \equiv i \sum_{j=1}^{m} A_j \otimes \mathbf{e}_j$$

Each $T = \sum_S T_S \otimes \mathbf{e}_S \in \mathcal{C}^{n\times n} \otimes \Re_{(m)}$ acts on $\mathbf{x} \in \sum_S \mathbf{x}_S \otimes \mathbf{e}_S \in \mathcal{C}^n \otimes \Re_{(m)}$ by

$$T(\mathbf{x}) = \sum_{S,S'} T_S(\mathbf{x}_{S'}) \otimes \mathbf{e}_S \mathbf{e}_{S'}$$

with $\text{Cliff}(A) \in \mathcal{C}^{n\times n} \otimes \Re_{(m)} \subseteq L(\mathcal{C}^n \otimes \Re_{(m)})$. We embed $\mathcal{C}^{n\times n}$ into $\mathcal{C}^n \otimes \Re_{(m)}$ via the map $A \to A \otimes \mathbf{e}_0$ and let $\|\text{Cliff}(A)\|$ denote the operator norm of $\text{Cliff}(A)$ as an element of $L(\mathcal{C}^n \otimes \Re_{(m)})$.

For any matrix $B$, the Schur decomposition [17] $U^H BU = D + N$ where $U$ is unitary, $D$ is diagonal and $N$ is strict upper triangular. Henrici [18] defined the measure of non-normality as $\Delta(B) = \inf \|N\|$, where the infimum is taken over all choices of unitary $U$. For $r, \Delta > 0$ and natural number $n$ let $g = g_n(\Delta/r)$ be the unique positive solution of $g + \cdots + g^n = \Delta/r$ and put $S_n(\Delta, r) = \Delta/g_n(\Delta/r)$.

We described $A$ as being *simultaneously Schur blockable* (by $X$) when $X^{-1}AX = \text{diag}(A^1, \cdots, A^r)$ where $A^i = \boldsymbol{\lambda}^i I + N^i = (\lambda_1^i I + N_1^i, \cdots, \lambda_m^i I + N_m^i)$ for nilpotent $N_j^i$ of identical dimensions for all $j$. We shall use the abbreviation SSB for such a property from now on.

We shall quote the main result (with proof) in [15, Theorem 6.1] for joint spectrum:

**Theorem 3.1.** *Let $A = (A_1, \cdots, A_m)$ and $\tilde{A} \equiv A + \delta A$, with $\delta A = (\delta A_1, \cdots, \delta A_m)$, be commuting $m$-tuples of $n \times n$ matrices with real spectra. Suppose the $A_j$ are simultaneously Schur blocked by $X$ and put $A = A^1 \oplus \cdots \oplus A^r$ where $A^j = \boldsymbol{\lambda}^j I + N^j$ as above. Then for all $\tilde{\boldsymbol{\lambda}} \in \lambda(\tilde{A})$ we have*

$$\min_{\boldsymbol{\lambda} \in \lambda(A)} \left|\tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}\right| \leq S_p\left(\left\|\text{Cliff}(N^k)\right\|,\ \kappa(X)\left\|\text{Cliff}(\delta A)\right\|\right)$$

*where* $\max_j \left\|\text{Cliff}(A^j - \tilde{\boldsymbol{\lambda}} I)\right\|$ *occurs at* $j = k$, $p = \dim(N^k)$ *and* $\kappa(X) \equiv \|X^{-1}\|\,\|X\|$.

*Proof.* Let $\tilde{\boldsymbol{\lambda}} \in \lambda(\tilde{A})$. If $\tilde{\boldsymbol{\lambda}} \in \lambda(A)$ the inequality is trivial. Otherwse

$\tilde{\boldsymbol{\lambda}} \notin \lambda(A) = \gamma(A)$ where

$$\gamma(A) \equiv \left\{ \boldsymbol{\lambda} \in \Re^m : \mathbf{0} \in \lambda\left( \sum_{j=1}^{m} (A_j - \lambda_j I)^2 \right) \right\}$$

Let $\hat{A} = (X^{-1}A_1X, \cdots, X^{-1}A_mX)$ be in Schur block form. As the $A_j$ commute therefore by [24, Theorem 3.1] $0 \notin \lambda\left(X\text{Cliff}(\hat{A} - \tilde{\boldsymbol{\lambda}}I)X^{-1}\right) = \lambda\left(\text{Cliff}(\hat{A} - \tilde{\boldsymbol{\lambda}}I)\right)$, thus $\text{Cliff}(\hat{A} - \tilde{\boldsymbol{\lambda}}I)$ is invertible. Now

$X^{-1}\text{Cliff}(\tilde{A} - \tilde{\boldsymbol{\lambda}}I)X = \text{Cliff}(\hat{A} - \tilde{\boldsymbol{\lambda}}I) + X^{-1}\text{Cliff}(\tilde{A} - A)X = \text{Cliff}(\hat{A} - \tilde{\boldsymbol{\lambda}}I)(I \otimes \mathbf{e}_0 + M)$

where $M \equiv [\text{Cliff}(\hat{A} - \tilde{\boldsymbol{\lambda}}I)]^{-1}X^{-1}\text{Cliff}(\delta A)X$ with $\delta A \equiv \tilde{A} - A$. Since $\tilde{\boldsymbol{\lambda}} \in \lambda(\tilde{A}) = \gamma(\tilde{A})$ therefore $\text{Cliff}(A - \tilde{\boldsymbol{\lambda}}I)$ is not invertible, nor is $I \otimes \mathbf{e}_0 + M$. As $I \otimes \mathbf{e}_0$ is the unit in $\mathcal{C}^{n\times n} \otimes \Re_{(m)}$ it follows that $\|M\| \geq 1$, so

$$\begin{aligned} 1 &\leq \|[\text{Cliff}(\hat{A} - \tilde{\boldsymbol{\lambda}}I)]^{-1}\|\ \kappa(X)\ \|\text{Cliff}(\delta A)\| \\ &= \|[\text{Cliff}(\hat{A^k} - \tilde{\boldsymbol{\lambda}}I)]^{-1}\|\ \kappa(X)\ \|\text{Cliff}(\delta A)\| \qquad (27) \\ &= \|[\text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I + N^k)]^{-1}\|\ \kappa(X)\ \|\text{Cliff}(\delta A)\| \quad A^k = \boldsymbol{\lambda}^k I + N^k \end{aligned}$$

Now

$\text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I + N^k) = \text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I) + \text{Cliff}(N^k) = \text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I)(I \otimes \mathbf{e}_0 - Z)$

where $Z = -[\text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I)]^{-1}\text{Cliff}(N^k)$. Thus

$$\begin{aligned} \left[\text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I + N^k)\right]^{-1} &= (I \otimes \mathbf{e}_0 - Z)^{-1}\left[\text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I)\right]^{-1} \\ &= (I \otimes \mathbf{e}_0 + Z + \cdots + Z^{p-1})\left[\text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I)\right]^{-1} \end{aligned}$$

since $Z^p = 0$ with $p = \dim(N^k)$ [3, page 10]. Therefore

$$\begin{aligned} &\left\|\text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I + N^k)\right\| \\ \leq\ & \left\|\left[\text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I)\right]^{-1}\right\|\ (1 + \|Z\| + \cdots + \|Z\|^{p-1}) \\ \leq\ & \eta^{-1}\left[1 + \eta^{-1}\|\text{Cliff}(N^k)\| + \cdots + (\eta^{-1}\|\text{Cliff}(N^k)\|)^{p-1}\right] \end{aligned}$$

where $\eta \equiv \left\|\left[\text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I)\right]^{-1}\right\|$. So from (27)

$$\begin{aligned} &\eta^{-1}\left[1 + \eta^{-1}\|\text{Cliff}(N^k)\| + \cdots + (\eta^{-1}\|\text{Cliff}(N^k)\|)^{p-1}\right] \\ &\geq [\kappa(X)\ \|\text{Cliff}(\delta A)\|]^{-1}\ \eta^{-1}\|\text{Cliff}(N^k)\| + \cdots + (\eta^{-1}\|\text{Cliff}(N^k)\|)^p \\ &\geq \frac{\|\text{Cliff}(N^k)\|}{\kappa(X)\ \|\text{Cliff}(\delta A)\|} = g + \cdots + g^p \end{aligned}$$

where $g = g_p \left( \|\text{Cliff}(N^k)\| / \kappa(X) \right) \|\text{Cliff}(\delta A)\|)$. Since $g > 0$ and $\|\text{Cliff}(N^k)\|/\eta > 0$ we have $g \leq \|\text{Cliff}(N^k)\|/\eta$ and $\eta \leq \|\text{Cliff}(N^k)\|/g$. It remains to show that $\eta \geq \min_{\boldsymbol{\lambda} \in \lambda(A)} \left| \tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda} \right|$. By [3, Proposition 2.2]

$$\left[ \text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I) \right]^{-1} = \left[ \sum_{j=1}^{m} \left( (\lambda_j^k - \tilde{\lambda}_j)I \right)^2 \right]^{-1} \text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I)$$

$$= \left[ |\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}}|^2 I \right]^{-1} \text{Cliff}((\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I) = \text{Cliff} \left( |\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}}|^{-2} \, (\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I \right)$$

So with $r(\cdot)$ denoting the spectral radius and by [24, Proposition 3.2]

$$\begin{aligned} \eta &= \left\| \text{Cliff} \left( |\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}}|^{-2} \, (\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I \right) \right\|^{-1} = \left[ r \left( |\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}}|^{-2} \, (\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}})I \right) \right]^{-1} \\ &= |\boldsymbol{\lambda}^k - \tilde{\boldsymbol{\lambda}}| \geq \min_{\boldsymbol{\lambda} \in \lambda(A)} \left| \tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda} \right| \end{aligned}$$

■

For a complex joint spectrum, let $A_j = A_{1j} + iA_{2j}$ and consider the partition

$$\pi(A) \equiv (A_{11}, \cdots, A_{1m}; A_{21}, \cdots, A_{2m})$$

We have the result [15, Corollary 6.2] for complex joint spectrum:

**Corollary 3.1.** *Let $A$, $\tilde{A}$, $k$ and $p$ be as in Theorem 3.1 except allow the $A_i$ and $\tilde{A}_i$ to have complex spectra. Let $\pi(A) = (A_{11}, \cdots, A_{1m}; A_{21}, \cdots, A_{2m})$ be a partition of $A$ and suppose the $A_{jk}$ are simultaneously Schur blocked by $X$. Then for all $\tilde{\boldsymbol{\lambda}} \in \lambda(\tilde{A})$ we have*

$$\min_{\boldsymbol{\lambda} \in \lambda(A)} \left| \tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda} \right| \leq S_p \left( \left\| \text{Cliff}(N^k) \right\|, \; \kappa(X) \, \|\text{Cliff}(\delta A)\| \right)$$

*with $\delta A \equiv \pi(\tilde{A}) - \pi(A)$.*

## 4. Generalizing Joint Spectrum

### 4.1. Joint Spectrum without Common Eigenvectors

For an $m$-tupe $A = (A_1, \cdots, A_m)$ *without* common eigenvectors, assume that the corresponding Jordan (or Schur) form $J_j$ share the same Jordan structures. Then Theorem 3.1 and Corollary 3.1 can be applied to $J = (J_1, \cdots, J_m)$ which is SSB. Similar to the perturbation bound in Theorem 3.1, the corresponding result will then be

$$\min_{\boldsymbol{\lambda} \in \lambda(A)} \left| \tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda} \right| \leq S_p \left( \left\| \text{Cliff}(N^k) \right\|, \; \|\text{Cliff}(X^{-1} \delta A X)\| \right)$$

Finally, the assumption that the $m$-tuple $A = (A_1, \cdots, A_m)$ possesses identical Jordan structures is not necessary, because we only require existence of a joint spectrum. One possibility is that $A_i$ shares the same eigenvalues and the eigenstructures, so that the associated Jordan blocks can be grouped in some way to form $N_j^i$ in Theorem 3.1 (similar to the construction for periodic eigenvalue problems in Section 5). The worst case will be to group all the blocks for a particular eigenvalue together, thus creating larger $N_j^i$ and worse error bounds.

### 4.2. Generalized Joint Spectrum

We shall generalize further the concept of a joint spectrum, allowing infinite eigenvalues. Let $E = (E_1, \cdots, E_m),\ A = (A_1, \cdots, A_m) \in (\mathcal{C}^{n\times n})^m$ be $m$-tuples of $n \times n$ matrices with complex entries. A generalized joint eigenvalue of $(E, A)$ is a vector $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha_1, \cdots, \alpha_m;\ \beta_1, \cdots, \beta_m) \in \mathcal{C}^{2m}$ such that $\beta_j A_j \mathbf{x} = \alpha_j E_j \mathbf{x}$ and $\beta_j \mathbf{y}^H A_j = \alpha_j \mathbf{y}^H E_j$ for $j = 1, \cdots, m$ for some $\mathbf{x} \in \mathcal{C}^n \backslash \{\mathbf{0}\}$. As in (2), the traditional eigenvalue $\lambda_j = \alpha_j/\beta_j$ can be infinite. We shall assume that the matrix pencils $\beta A_j - \alpha E_j$ are regular (i.e., $\det(\beta A_j - \alpha E_j)$ are not identically zero) to avoid continuous spectra. Such an $\mathbf{x}$ is called a joint eigenvector of $(E, A)$. The joint spectrum $\lambda(E, A)$ is the set of joint eigenvalues of $(E, A)$.

If $\{(\alpha E_i + \beta A_i)^{-1}(\gamma E_i + \delta A_i)\}$ (for some $\{\alpha, \beta, \gamma, \delta\}$) are commuting there exists at least one generalized joint eigenvalue. In the periodic eigenvalue problem in §2, we have another example of a generalized joint spectrum independent of commutativity (as we shall be working with canonical forms).

We first define $(E, A)$ as simultaneously Schur blockable (SSB) by $(X, Y)$ when $(E, B) = (E^1 \oplus \cdots \oplus E^r, A^1 \oplus \cdots \oplus A^r)$ where $(E^j, A^j) = (\boldsymbol{\beta}^j I + N_1^j, \boldsymbol{\alpha}^j I + N_2^j)$, with $N_k^i$ sharing the same dimensions for all $i$ and $k$. We shall share the same abbreviation SBB for both joint spectra (as defined before Theorem 3.1) and generalized joint spectra, without any possibility of confusion.

**Theorem 4.1.** *Let $(E, A) = (E_1, \cdots, E_m; A_1, \cdots, A_m)$, $\tilde{E} \equiv E + \delta E$ and $\tilde{A} \equiv A + \delta A$, with $\delta E = (\delta E_1, \cdots, \delta E_m)$ and $\delta A = (\delta A_1, \cdots, \delta A_m)$, be generalized $m$-tuples of $n \times n$ matrices with real spectra. Suppose the $(E_j, A_j)$ is SSB by $(X, Y)$. Then for all $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \in \lambda(\tilde{E}, \tilde{A})$ we have*

$$\min_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\lambda(E,A)} \left|\tilde{\boldsymbol{\alpha}}\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\boldsymbol{\alpha}\right| \le S_p\left(\max_i \left\|\mathrm{Cliff}(N_i^k)\right\|,\ \kappa(X)\left\|\mathrm{Cliff}(\tilde{\boldsymbol{\alpha}}\ \delta E - \tilde{\boldsymbol{\beta}}\ \delta A)\right\|\right)$$

*where $\tilde{\boldsymbol{\alpha}}\boldsymbol{\beta}–\tilde{\boldsymbol{\beta}}\boldsymbol{\alpha} \equiv (\tilde{\alpha}_1\beta_1 - \tilde{\beta}_1\alpha_1, \cdots, \tilde{\alpha}_m\beta_m - \tilde{\beta}_m\alpha_m)$, $\max_j \left\|\mathrm{Cliff}(\tilde{\boldsymbol{\alpha}}E^j - \tilde{\boldsymbol{\beta}}A^j)\right\|$ occurs at $j = k$, $p = \dim(N_i^k)$ and $\kappa(X) \equiv \|Y\|\ \|X\|$.*

*Proof.* The proof is exactly the same as that for Theorem 3.1, except for some minor adaptation. The inverse $X^{-1}$ has to be replaced by the left eigenvector matrix

$Y$, which simultaneously (Schur-)block diagonalize $(E, A)$ such that

$$\hat{E} \equiv Y^H E X = \text{daig}\{\hat{E}^1, \cdots, \hat{E}^r\}\ , \quad \hat{A} \equiv Y^H A X = \text{daig}\{\hat{A}^1, \cdots, \hat{A}^r\}$$

with the components of $\alpha \hat{E}^i - \beta \hat{A}^i$ $(i = 1, \cdots, r)$ being parts of the generalized Schur/Kronecker canonical form of $\alpha E_j - \beta A_j$ $(j = 1, \cdots, m)$. Also, the $m$-tuple $A = (A_1, \cdots, A_m)$ has to be replaced by the $2m$-tuple $(E, A) = (E_1, \cdots, E_m; A_1, \cdots, A_m)$.

Finally, it is easy to see that the generalized spectral set and the generalized spectrum are equal, as

$$\begin{aligned} &\gamma(E, A) \\ \equiv &\left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Re^m \times \Re^m : \mathbf{0} \in \lambda \left( \sum_{j=1}^m (\beta_j A_j - \alpha_j E_j)^2 \right) \right\} \\ = &\left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Re^m \times \Re^m : \mathbf{0} \in \lambda \left( \sum_{j=1}^m \left[ Y^{-H} (\beta_j \hat{A}_j - \alpha_j \hat{E}_j) X^{-1} \right]^2 \right) \right\} \\ = &\left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Re^m \times \Re^m : \mathbf{0} \in \lambda \left( Y^{-H} \left( \sum_{j=1}^m Z^j X^{-1} Y^{-H} Z^j \right) X^{-1} \right) \right\} \end{aligned}$$

with $Z^j \equiv (\beta_j \hat{A}_j - \alpha_j \hat{E}_j)$. Thus, we have $\gamma(E, A) = \{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Re^m \times \Re^m : \mathbf{0} \in \lambda\,(Z)\}$, where

$$Z \equiv \left[ Z^1, \cdots, Z^m \right]\ \text{diag}\{X^{-1} Y^{-H}, \cdots, X^{-1} Y^{-H}\} \begin{bmatrix} Z^1 \\ \vdots \\ Z^m \end{bmatrix}$$

As the parameters $(\alpha_j, \beta_j)$ for which $Z$ is singular is the same as those for

$$\tilde{Z} \equiv \left[ Z^1, \cdots, Z^m \right] \begin{bmatrix} Z^1 \\ \vdots \\ Z^m \end{bmatrix} = \sum_{j=1}^m (Z^j)^2 = \sum_{j=1}^m \left[ (\beta_j \hat{A}_j - \alpha_j \hat{E}_j) \right]^2$$

with $(\hat{E}, \hat{A})$ in Schur/Kronecker canonical form, we deduced that $\gamma(E, A) = \lambda(\hat{E}, \hat{A}) = \lambda(E, A)$. ■

For complex joint spectrum, let $E_j = E_{1j} + iE_{2j}$ and $A_j = A_{1j} + iA_{2j}$, and consider a similar partition $\pi(\cdot)$ as in Corollary 2.1. We have the result for complex generalized joint spectrum:

**Corollary 4.1.** *Let* $(E, A)$, $(\tilde{E}, \tilde{A})$, $k$ *and* $p$ *be as in Theorem 4.1 except allow the* $(E_i, A_i)$ *and* $(\tilde{E}_i\tilde{A}_i)$ *to have complex spectra. Let* $\pi(\cdot)$ *be a partition of* $(E, A)$ *and suppose the* $(E_{jk}, A_{jk})$ *are simultaneously Schur blocked by* $(X, Y)$*. Then for all* $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \in \lambda(\tilde{E}, \tilde{A})$ *we have*

$$\min_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\lambda(E,A)} \left|\tilde{\boldsymbol{\alpha}}\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\boldsymbol{\alpha}\right| \leq S_p\left(\max_i \left\|\text{Cliff}(N_i^k)\right\|,\ \kappa(X)\left\|\text{Cliff}(\tilde{\boldsymbol{\alpha}}\delta E - \tilde{\boldsymbol{\beta}}\delta A)\right\|\right)$$

*with* $\delta E \equiv \pi(\tilde{E}) - \pi(E)$ *and* $\delta A \equiv \pi(\tilde{A}) - \pi(A)$.

*Proof.* Similar to the proof of Corollary 2.1, with the real and complex parts of $(E, A)$ forming a $4m$-tuple, in place of the $2m$-tuple in Corollary 2.1. ■

The perturbation of generalized joint spectra with common generalized Schur /Kronecker structure (without common eigenvectors) can then be considered. Similar to the bound in Theorem 4.1, the result will then have the form:

$$\begin{aligned} &\min_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\lambda(E,A)} \left|\tilde{\boldsymbol{\alpha}}\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\boldsymbol{\alpha}\right| \\ &\leq S_p\left(\max_i \left\|\text{Cliff}(N_i^k)\right\|,\ \left\|\text{Cliff}\left(Y^H(\tilde{\boldsymbol{\alpha}}\delta E - \tilde{\boldsymbol{\beta}}\delta A)X\right)\right\|\right) \end{aligned} \tag{28}$$

### 4.3. Perturbation to Periodic Eigenvalues

Consider a periodic eigenvalue problem of periodicity $m$. Let $(\hat{E}_j, \hat{A}_j) \equiv (Y_j^H E_j X_j, Y_j^H A_j X_{j-1})$ be the periodic Schur/Kronecker forms (Theorem 2.3) of $(E_j, A_j)$, and denote $E^{(j)} \equiv \hat{E}_j\hat{E}_{j+1}\cdots\hat{E}_{j+m-1}$, $A^{(j)} \equiv \hat{A}_{j+m-1}\hat{A}_{j+m-2}\cdots\hat{A}_j$. Consider the $m$-tuples $(E, A)$ where $E = \left(E^{(1)}, \cdots, E^{(m)}\right)$, $A = \left(A^{(1)}, \cdots, A^{(m)}\right)$, assuming that the Schur/Jordan blocks associated with a certain eigenvalue $(\pi_\alpha, \pi_\beta)$ appear at the same position in $(E^{(j)}, A^{(j)})$. As $(E^{(j)}, A^{(j)})$ (for all $j$) share the same eigenvalues and eigenvectors (all columns of the identity matrix), we can consider the joint spectrum of $(E, A)$. Thus we can apply our results on generalized joint spectrum to $(E, A)$, which is obviously simultaneously Schur blockable (see §1). Note that there may be more than one way of grouping the eigenvalues together and writing down $N_i^k$ in Theorem 4.1. The worst case will be to group all the blocks associated with a given eigenvalue together, making the dimensions of the corresponding $N_i^k$ and the error bounds large.

For any perturbed eigenvalue $(\tilde{\pi}_\alpha, \tilde{\pi}_\beta)$, (28) now reads

$$\min_{(\pi_\alpha,\pi_\beta)} |\tilde{\pi}_\alpha\pi_\beta - \tilde{\pi}_\beta\pi_\alpha| \leq S_p\left(\max_i \left\|\text{Cliff}(N_i^k)\right\|, \left\|\text{Cliff}(\tilde{\pi}_\alpha\delta\hat{E} - \tilde{\pi}_\beta\delta\hat{A})\right\|\right)$$

where $\delta\hat{E} \equiv (\delta\hat{E}^{(1)}, \cdots, \delta\hat{E}^{(m)})$ and $\delta\hat{A} \equiv (\delta\hat{A}^{(1)}, \cdots, \delta\hat{A}^{(m)})$ are respectively the changes in $\hat{E}$ and $\hat{A}$ after perturbing $(E_j, A_j)$ to $(E_j + \delta E_j, A_j + \delta A_j)$.

When the Schur forms degenerate to Jordan forms, we have $\left\|\text{Cliff}(N_i^k)\right\| \leq m$ and $\mathbf{\Delta} \equiv (\Delta_1, \cdots, \Delta_m) = \tilde{\pi}_\alpha \delta\hat{E} - \tilde{\pi}_\beta \delta\hat{A}$ with $\Delta_j$ as defined in (15) in §2. The perturbation result then degenerates to

$$\min_{(\pi_\alpha, \pi_\beta)} |\tilde{\pi}_\alpha \pi_\beta - \tilde{\pi}_\beta \pi_\alpha| \leq S_p\left(m, \|\text{Cliff}(\mathbf{\Delta})\|\right)$$

In general, similar bounds to those in §2 can then obviously be derived. Notice that the left-hand-sides of the perturbation results equal the chordal metric used in §2, when the eigenvalues are scaled appropriately. Notice also that $S_p(\Delta, r)$ is monotonically increasing in $\Delta$, $r$ and $p$, and various perturbation bounds can be obtained.

Finally, clusters of eigenvalues can be considered, similar to Case II in Theorem 2.4. For simple eigenvalues under asymptotic perturbations, simpler forms of perturbation results can be obtained by ignoring higher order terms when considering $\|\text{Cliff}(\mathbf{\Delta})\|$, similar to the approach in §2.

## 5. Concluding Remarks

Applying the Bauer-Fike and joint spectrum techniques techniques, new perturbation results of periodic eigenvalue problems are obtained. Although it is difficult to compare perturbation results (see comments at the end of §1), the perturbation results from this paper provide new tools in the investigation of periodic eigenvalues, linking the theoretical tool of joint spectra with the engineering problem involving periodic systems. More work has to be done, in terms of the efficient numerical computation of the perturbation bounds and their comparison to other existing bounds.

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