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IMPROVEMENTS OF SOME INEQUALITIES OF OSTROWSKI TYPE AND THEIR APPLICATIONS

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Abstract. In this paper, we establish some inequalities which improve some Ostrowski type inequalities. Applications for Euler's Beta mapping and special means are also given.

1. Introduction

Throughout, let $V_c^b(f)$ be the total variation of f on the interval $[c,d]$ and

$$\left\|f'\right\|_{[c,d],1} = \int_c^d \left|f'(t)\right| dt$$

and let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$, $h_i := x_{i+1} - x_i$ $(i = 0, 1, \cdots, n-1)$ and $v(h) := \max_{i=0,1,\cdots,n-1} h_i$.

The *Ostrowski's inequality* [9, p.469] (see also $[10,$ p. $933])$, states that if f' exists and is bounded on (a,b), then, for all $x \in (a,b)$, we have the inequality

$$\left|\int_a^b f(t)dt - f(x)(b-a)\right| \le L\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \tag{1.1}$$

where

$$\sup_{t\in(a,b)} \left|f'(t)\right| \le L.$$

In (1.1), the constant $\frac{1}{4}$ is the best possible.

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Now if f is as above, then we can approximate the integral $\int_a^b f(t)dt$ by the *Ostrowski quadrature formula* $A_n(f, I_n, \xi)$, having an error given by $R_n(f, I_n, \xi)$, where

$$A_n(f, I_n, \xi) := \sum_{i=0}^{n-1} f(\xi_i)\, h_i,$$

and the remainder satisfies the estimation

$$|R_n(f, I_n, \xi)| \le \sum_{i=0}^{n-1} \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_{i-1} + x_i}{2} \right)^2 \right] \|f'\|_\infty . \tag{1.2}$$

For some recent results which generalize, improve and extend the inequalities (1.1) and (1.2), see the papers $[2-8, 10]$.

In this paper, we establish some Ostrowski type inequalities which improve some inequalities in $[5, 7]$. Applications for Euler's Beta mapping and special means are also given.

2. Some Integral Inequalities

We may state the following results.

Theorem 1. *Let $f : [a, b] \to R$ be a mapping with bounded variation on $[a, b]$. Then, for all $x \in [a, b]$, we have the inequality*

$$\begin{aligned} &\left| \int_a^b f(t)dt - f(x)(b-a) \right| \\ &\le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] V_a^b(f) \\ &\quad -2 \left| x - \frac{a+b}{2} \right| \left[V_a^m(f) + V_{a+b-m}^b(f) \right] \end{aligned} \tag{2.1}$$

where $m = \min\{x, a+b-x\}$.

The constant $\frac{1}{2}$ is the best possible in (2.1).

Proof. Let $x \in [a, b]$. Define

$$s(t) := \begin{cases} t-a, & t \in [a, x] \\ t-b, & t \in (x, b] \end{cases} . \tag{2.2}$$

Using the integration by parts formula, we have the following identity

$$
\begin{aligned}
&\int_a^b s(t)\, df(t) \\
&= (t-a) f(t) \,|_{t=a}^{t=x} - \int_a^x f(t)dt + (t-b) f(t) \,|_{t=x}^{t=b} - \int_x^b f(t)dt \qquad (2.3) \\
&= f(x)(b-a) - \int_a^b f(t)dt.
\end{aligned}
$$

It is well known [1, p.159] that if $\mu, \nu : [c,d] \to R$ are such that μ is continuous on $[c,d]$ and ν is of bounded variation on $[c,d]$, then $\int_c^d \mu(t)\, d\nu(t)$ exists and [1, p.177]

$$
\left| \int_c^d \mu(t)\, d\nu(t) \right| \le \sup_{t \in [c,d]} |\mu(t)|\, V_c^b(\nu). \qquad (2.4)
$$

In the case $a \le x \le \frac{a+b}{2}$, using (2.3) and (2.4), we have $m = x$ and

$$
\begin{aligned}
&\left| \int_a^b f(t)dt - f(x)(b-a) \right| \\
&= \left| \int_a^x (t-a)\, df(t) + \int_x^{a+b-x} (t-b)\, df(t) + \int_{a+b-x}^b (t-b)\, df(t) \right| \\
&\le \left| \int_a^x (t-a)\, df(t) \right| + \left| \int_x^{a+b-x} (t-b)\, df(t) \right| + \left| \int_{a+b-x}^b (t-b)\, df(t) \right| \\
&\le (x-a)\, V_a^x(f) + (b-x)\, V_x^{a+b-x}(f) + (x-a)\, V_{a+b-x}^b(f) \qquad (2.5) \\
&= (b-x)\, V_a^b(f) - (a+b-2x) \Big(V_a^x(f) + V_{a+b-x}^b(f) \Big) \\
&= \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] V_a^b(f) - 2 \left| x - \frac{a+b}{2} \right| \left[V_a^x(f) + V_{a+b-x}^b(f) \right] \\
&= \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] V_a^b(f) - 2 \left| x - \frac{a+b}{2} \right| \left[V_a^m(f) + V_{a+b-m}^b(f) \right].
\end{aligned}
$$

In the case $\frac{a+b}{2} < x \le b$, using (2.3) and (2.4), we have $m = a+b-x$ and

$$
\begin{aligned}
&\left| \int_a^b f(t)dt - f(x)(b-a) \right| \\
&= \left| \int_a^{a+b-x} (t-a)\, df(t) + \int_{a+b-x}^x (t-a)\, df(t) + \int_x^b (t-b)\, df(t) \right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \left|\int_a^{a+b-x}(t-a)\,df\,(t)\right|+\left|\int_{a+b-x}^{x}(t-a)\,df\,(t)\right|+\left|\int_x^b(t-b)\,df\,(t)\right|\\
&\leq (b-x)\,V_a^{a+b-x}(f)+(x-a)\,V_{a+b-x}^{x}(f)+(b-x)\,V_x^b\,(f)\\
&=(x-a)\,V_a^b(f)-(2x-a-b)\left(V_a^{a+b-x}(f)+V_x^b(f)\right)\\
&=\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]V_a^b(f)\\
&-2\left|x-\frac{a+b}{2}\right|\left[V_a^{a+b-x}(f)+V_x^b(f)\right]\\
&=\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]V_a^b(f)\\
&-2\left|x-\frac{a+b}{2}\right|\left[V_a^{m}(f)+V_{a+b-m}^b(f)\right].
\end{aligned}\tag{2.6}$$

Thus, by (2.5) and (2.6), we obtain (2.1).

We assume that the inequality (2.1) holds with a constant $C>0$, i.e.,

$$\begin{aligned}
&\left|\int_a^b f(t)dt-f(x)(b-a)\right|\\
&\leq\left[C\,(b-a)+\left|x-\frac{a+b}{2}\right|\right]V_a^b(f)\\
&\quad-2\left|x-\frac{a+b}{2}\right|\left[V_a^{m}(f)+V_{a+b-m}^b(f)\right].
\end{aligned}\tag{2.7}$$

Let

$$f(x)=\begin{cases}0, & if\quad x\in[a,b]\setminus\left\{\dfrac{a+b}{2}\right\}\\[2mm] 1, & if\quad x=\dfrac{a+b}{2}\end{cases}.$$

Then f is with bounded variation on $[a,b]$, and

$$V_a^b(f)=2,\qquad \int_a^b f(t)dt=0$$

and for x=$\frac{a+b}{2}$, we get in (2.7)

$$b-a\leq 2C\,(b-a)$$

which implies the constant $\frac{1}{2}$ is the best possible.

This completes the proof.

Under the conditions of Theorem 1, we have the following remarks and corollaries.

Remark 1. In Theorem 1, we get an improvement of Theorem 2.1 in [5, p. 59].

Corollary 1. *In Theorem 1, let $f : [a, b] \to R$ be a monotonic mapping. Then we have the inequality*

$$\begin{aligned}
&\left|\int_a^b f(t)dt - f(x)(b-a)\right| \\
&\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] |f(b) - f(a)| \\
&-2\left|x - \frac{a+b}{2}\right| \left[|f(m) - f(a)| + |f(b) - f(a+b-m)|\right].
\end{aligned}$$

Remark 2. Corollary 1 is an improvement of Corollary 2.2 in [5, p. 61].

Corollary 2. *In Theorem 1, let $f : [a, b] \to R$ be an L-Lipschitzian mapping on $[a, b]$, i.e., we recall*

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in [a, b]$. Then we have the inequality

$$\begin{aligned}
&\left|\int_a^b f(t)dt - f(x)(b-a)\right| \\
&\leq L\left\{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right](b-a)\right. \\
&\left. - 4\left|x - \frac{a+b}{2}\right|(m-a)\right\}.
\end{aligned} \tag{2.8}$$

Proof. Let $x \in [a, b]$. In the case $a \leq x \leq \frac{a+b}{2}$, using (2.5), we have $m = x$ and

$$\begin{aligned}
&\left|\int_a^b f(t)dt - f(x)(b-a)\right| \\
&\leq (x-a)V_a^x(f) + (b-x)V_x^{a+b-x}(f) + (x-a)V_{a+b-x}^b(f) \\
&\leq L\left[(x-a)(x-a) + (b-x)(a+b-2x) + (x-a)(x-a)\right] \\
&= L\left[(b-x)(b-a) - 4\left(\frac{a+b}{2} - x\right)(x-a)\right] \qquad (2.9) \\
&= L\left\{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right](b-a) - 4\left|x - \frac{a+b}{2}\right|(x-a)\right\} \\
&= L\left\{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right](b-a) - 4\left|x - \frac{a+b}{2}\right|(m-a)\right\}.
\end{aligned}$$

In the case $\frac{a+b}{2} < x \leq b$, using (2.6), we have $m = a + b - x$ and

$$
\begin{aligned}
&\left| \int_a^b f(t)dt - f(x)(b-a) \right| \\
&\leq L\left[(b-x)(b-x) + (x-a)(2x-a-b) + (b-x)(b-x)\right] \\
&\leq (b-x)\, V_a^{a+b-x}(f) + (x-a)\, V_{a+b-x}^{x}(f) + (b-x)\, V_x^b(f) \\
&= L\left[(x-a)(b-a) - 4\left(x - \frac{a+b}{2}\right)(b-x)\right] \qquad (2.10) \\
&= L\left\{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right](b-a) - 4\left|x - \frac{a+b}{2}\right|(b-x)\right\} \\
&= L\left\{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right](b-a) - 4\left|x - \frac{a+b}{2}\right|(m-a)\right\}.
\end{aligned}
$$

Thus, by (2.9) and (2.10), we obtain (2.8). This completes the proof.

Remark 3. Corollary 2 is an improvement of Corollary 2.3 in [5, p. 61].

Theorem 2. *Let $f : I \subset R \to R$ be a differentiable mapping in $Int\,(I)$ and $a, b \in Int\,(I)$ with $a < b$. If $f' \in L_1[a, b]$, then, for all $x \in [a, b]$, we have the inequality*

$$
\begin{aligned}
&\left| \int_a^b f(t)dt - f(x)(b-a) \right| \\
&\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \|f'\|_{[a,b],1} \qquad (2.11) \\
&\quad -2\left|x - \frac{a+b}{2}\right| \left[\|f'\|_{[a,m],1} + \|f'\|_{[a+b-m,b],1}\right]
\end{aligned}
$$

where $m = \min\{x, a+b-x\}$.

Proof. Let $x \in [a, b]$ and let $s(t)$ $(t \in [a, b])$ be defined as in (2.2). Using the integration by parts formula, we have the following identity

$$
\begin{aligned}
&\int_a^b s(t)\, f'(t)\, dt \\
&= (t-a)\, f(t)\, |_{t=a}^{t=x} - \int_a^x f(t)dt + (t-b)\, f(t)\, |_{t=x}^{t=b} - \int_x^b f(t)dt \qquad (2,12) \\
&= f(x)(b-a) - \int_a^b f(t)dt.
\end{aligned}
$$

In the case $a \leq x \leq \frac{a+b}{2}$, using (2.12), we have $m = x$ and

$$\begin{aligned}
&\left|\int_a^b f(t)dt - f(x)(b-a)\right| \\
&= \left|\int_a^x (t-a)\, f'(t)\, dt + \int_x^{a+b-x} (t-b)\, f'(t)\, dt + \int_{a+b-x}^b (t-b)\, f'(t)\, dt\right| \\
&\leq \left|\int_a^x (t-a)\, f'(t)\, dt\right| + \left|\int_x^{a+b-x} (t-b)\, f'(t)\, dt\right| + \left|\int_{a+b-x}^b (t-b)\, f'(t)\, dt\right| \\
&\leq (x-a)\left\|f'\right\|_{[a,x],1} + (b-x)\left\|f'\right\|_{[x,a+b-x],1} + (x-a)\left\|f'\right\|_{[a+b-x,b],1} \\
&= (b-x)\left\|f'\right\|_{[a,b],1} - (a+b-2x)\left(\left\|f'\right\|_{[a,x],1} + \left\|f'\right\|_{[a+b-x,b],1}\right) \qquad (2.13) \\
&= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left\|f'\right\|_{[a,b],1} \\
&\quad -2\left|x - \frac{a+b}{2}\right|\left(\left\|f'\right\|_{[a,x],1} + \left\|f'\right\|_{[a+b-x,b],1}\right). \\
&= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left\|f'\right\|_{[a,b],1} \\
&\quad -2\left|x - \frac{a+b}{2}\right|\left(\left\|f'\right\|_{[a,m],1} + \left\|f'\right\|_{[a+b-m,b],1}\right).
\end{aligned}$$

In the case $\frac{a+b}{2} < x \leq b$, using (2.12), we have $m = a+b-x$ and

$$\begin{aligned}
&\left|\int_a^b f(t)dt - f(x)(b-a)\right| \\
&= \left|\int_a^{a+b-x} (t-a)\, f'(t)\, dt + \int_{a+b-x}^{x} (t-a)\, f'(t)\, dt + \int_x^b (t-b)\, f'(t)\, dt\right| \\
&\leq \left|\int_a^{a+b-x} (t-a)\, f'(t)\, dt\right| + \left|\int_{a+b-x}^{x} (t-a)\, f'(t)\, dt\right| + \left|\int_x^b (t-b)\, f'(t)\, dt\right| \\
&\leq (b-x)\left\|f'\right\|_{[a,a+b-x],1} + (x-a)\left\|f'\right\|_{[a+b-x,x],1} + (b-x)\left\|f'\right\|_{[x,b],1} \\
&= (x-a)\left\|f'\right\|_{[a,b],1} - (2x-a-b)\left(\left\|f'\right\|_{[a,a+b-x],1} + \left\|f'\right\|_{[x,b],1}\right) \qquad (2.14) \\
&= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left\|f'\right\|_{[a,b],1} \\
&\quad -2\left|x - \frac{a+b}{2}\right|\left[\left\|f'\right\|_{[a,a+b-x],1} + \left\|f'\right\|_{[x,b],1}\right]. \\
&= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left\|f'\right\|_{[a,b],1} \\
&\quad -2\left|x - \frac{a+b}{2}\right|\left[\left\|f'\right\|_{[a,m],1} + \left\|f'\right\|_{[a+b-m,b],1}\right].
\end{aligned}$$

Thus, by (2.13) and (2.14), we obtain (2.11).

This completes the proof. ■

Remark 4. In Theorem 2, we get an improvement of Theorem 2.1 in [7, p. 240].

3. Applications for Quadrature Rules

We have the following quadrature formula.

Theorem 3. *Let f be defined as in Theorem 1. Then, for ξ_i, h_i $(i = 0, 1, \cdots, n-1)$, $A_n(f, I_n, \xi)$ and $v(h)$ as above, we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(t)dt - A_n(f, I_n, \xi)\right| \\ &\le \max_{i=0,1,\cdots,n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] V_a^b(f) - M \\ &\le \left[\frac{1}{2}v(h) + \max_{i=0,1,\cdots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] V_a^b(f) - M \\ &\le v(h)V_a^b(f) - M \end{aligned} \tag{3.1}$$

where $m_i = \min\{\xi_i, x_i + x_{i+1} - \xi_i\}$ $(i = 0, 1, \cdots, n-1)$ *and*

$$M = \sum_{i=0}^{n-1} 2\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\left[V_{x_i}^{m_i}(f) + V_{x_i+x_{i+1}-m_i}^{x_{i+1}}(f)\right].$$

The constant $\frac{1}{2}$ is the best possible in (3.1).

Proof. Using Theorem 1 on the interval $[x_i, x_{i+1}]$, we have the inequality

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)\, h_i\right| \\ &\le \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] V_{x_i}^{x_{i+1}}(f) \\ &\quad - 2\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\left[V_{x_i}^{m_i}(f) + V_{x_i+x_{i+1}-m_i}^{x_{i+1}}(f)\right] \end{aligned}$$

for all $i = 0, 1, \cdots, n-1$. Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$\left|\int_a^b f(t)dt - A_n(f, I_n, \xi)\right|$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\xi_i\right)h_i\right| \\
&\leq \sum_{i=0}^{n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right] V_{x_i}^{x_{i+1}}(f) - M \\
&\leq \max_{i=0,1,\cdots,n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right] \sum_{i=0}^{n-1} V_{x_i}^{x_{i+1}}(f) - M \\
&= \max_{i=0,1,\cdots,n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right] V_a^b(f) - M.
\end{aligned}
$$

The second inequality follows by the properties of sup$(\cdot)$.

Now, as

$$
\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right| \leq \frac{1}{2}h_i \tag{3.2}
$$

for all $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$ the last part of (3.1) is also proved.

Under the conditions of Theorem 3, we have the following remarks and corollaries.

Remark 5. In Theorem 3, we get an improvement of Theorem 3.1 in $[5,\ \text{p. } 63]$.

Corollary 3. *In Theorem 3, let $f : [a, b] \to R$ be a monotonic mapping. Then we have the inequality*

$$
\begin{aligned}
&\left|\int_a^b f(t)dt - A_n(f, I_n, \xi)\right| \\
&\leq \max_{i=0,1,\cdots,n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right] |f(b) - f(a)| - M \\
&\leq \left[\frac{1}{2}v(h) + \max_{i=0,1,\cdots,n-1}\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right] |f(b) - f(a)| - M \\
&\leq v(h)\left|f\left(b\right) - f\left(a\right)\right| - M
\end{aligned}
$$

where

$$
M = \sum_{i=0}^{n-1} 2\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right| \left[\left|f\left(m_i\right) - f\left(x_i\right)\right| + \left|f\left(x_{i+1}\right) - f\left(x_i + x_{i+1} - m_i\right)\right|\right].
$$

Remark 6. Corollary 3 is an improvement of Corollary 3.2 in $[5,\ \text{p. } 64]$.

Using Corollary 2 the generalized triangle inequality and (3.2), we have the following corollary:

Corollary 4. *In Theorem 3, let $f : [a,b] \to R$ be a L-Lipschitzian mapping. Then we have the inequality*

$$\begin{aligned}&\left|\int_a^b f(t)dt - A_n(f, I_n, \xi)\right| \\ &\leq L\left\{\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]h_i - M\right\} \\ &\leq L\left\{\sum_{i=0}^{n-1} h_i^2 - M\right\}\end{aligned}$$

where

$$M = \sum_{i=0}^{n-1} 4\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|(m_i - x_i).$$

Remark 7. The Corollary 4 is an improvement of Corollary 3.3 in [5, p. 64].

Theorem 4. *Let f be defined as in Theorem 2. Then, for ξ_i, h_i $(i = 0, 1, \cdots, n-1)$, $A_n(f, I_n, \xi)$ and $v(h)$ as above, we have the inequality*

$$\begin{aligned}&\left|\int_a^b f(t)dt - A_n(f, I_n, \xi)\right| \\ &\leq \max_{i=0,1,\cdots,n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\left\|f'\right\|_{[a,b],1} - M \\ &\leq \left[\frac{1}{2}v(h) + \max_{i=0,1,\cdots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\left\|f'\right\|_{[a,b],1} - M \\ &\leq v(h)\left\|f'\right\|_{[a,b],1} - M\end{aligned} \tag{3.3}$$

where $m_i = \min\{\xi_i, x_i + x_{i+1} - \xi_i\}$ $(i = 0, \cdots, n-1)$ and

$$M = \sum_{i=0}^{n-1} 2\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\left[\left\|f'\right\|_{[x_i, m_i],1} + \left\|f'\right\|_{[x_i + x_{i+1} - m_i, x_{i+1}],1}\right].$$

Proof. *The proof is obvious by applying Theorem 2 to the intervals $[x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$ and using the generalized triangle inequality . We shall omit the details.*

Remark 8. In Theorem 4, we get an improvement of Theorem 4.1 in [7, p. 243].

4. Applications for Euler's Beta Mapping

Consider the mapping Beta for real numbers

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad p,q>0$$

and the mapping

$$e_{p,q}(t) := t^{p-1}(1-t)^{q-1}, t \in [0,1]\,.$$

In [5, p. 65], Dragomir get the following results:
We have for $p,q>1$ that

$$e'_{p,q}(t) = e_{p-1,q-1}(t)\,[p-1-(p+q-2)t] \tag{4.1}$$

and as

$$|p-1-(p+q-2)t| \le \max\{p-1,q-1\} \tag{4.2}$$

for all $t \in [0,1]$, then

$$\begin{aligned} \left\|e'_{p,q}\right\|_{[0,1],1} &\le \max\{p-1,q-1\}\,\|e_{p-2,q-2}\|_{[0,1],1} \\ &= \max\{p-1,q-1\}\,B\,(p-1,q-1)\,. \end{aligned} \tag{4.3}$$

Using Theorem 2, Theorem 4 and $(4.1)-(4.3)$, we have the following corollaries:

Corollary 5. *Let $p,q>1$. Then, for all $x\in[0,1]$, we have the inequality*

$$\begin{aligned} &\left|B\,(p,q) - x^{p-1}(1-x)^{q-1}\right| \\ &\le \max\{p-1,q-1\}\,B\,(p-1,q-1)\left[\frac{1}{2}+\left|x-\frac{1}{2}\right|\right] \\ &\quad -2\left|x-\frac{1}{2}\right|\left[\left\|e'_{p,q}\right\|_{[0,m],1}+\left\|e'_{p,q}\right\|_{[1-m,1],1}\right] \end{aligned} \tag{4.4}$$

where $m=\min\{x,1-x\}$.

Remark 9. Corollary 5 is an improvement of Proposition 4.1 in [5, p. 65].

Corollary 6. *Let ξ_i, h_i $(i = 0, 1, \cdots, n-1)$ and $v(h)$ be as above. Then, for $p, q > 1$ we have the inequality*

$$\begin{aligned}
&\left| B(p,q) - \sum_{i=0}^{n-1} \xi_i^{p-1} (1-\xi_i)^{q-1} h_i \right| \\
&\le \max\{p-1, q-1\} \left[\frac{1}{2} v(h) + \max_{i=0,1,\cdots,n} \left| \xi_i - \frac{x_i + x_{i1}}{2} \right| \right] \\
&\quad B(p-1, q-1) - M \\
&\le \max\{p-1, q-1\} v(h) B(p-1, q-1) - M
\end{aligned} \tag{4.5}$$

where $m_i = \min\{\xi_i, x_i + x_{i+1} - \xi_i\}$ $(i = 0, 1, \cdots, n-1)$ *and*

$$M = \sum_{i=0}^{n-1} 2 \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \left[\left\| e'_{p,q} \right\|_{[x_i, m_i], 1} + \left\| e'_{p,q} \right\|_{[x_i + x_{i+1} - m_i, x_{i+1}], 1} \right].$$

Remark 10. Corollary 6 is an improvement of Proposition 4.3 in [5, p. 65].

5. Applications for the Special Means

Let us recall the following means of the two nonnegative number a and b:

1. The arithematic mean

$$A = A(a,b) := \frac{a+b}{2}, \quad a, b \ge 0;$$

2. The geometric mean

$$G = G(a,b) := \sqrt{ab}, \quad a, b > 0;$$

3. The harmonic mean

$$H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0;$$

4. The logarithmic mean

$$L = L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a} & \text{if } a \ne b \\ a & \text{if } a = b \end{cases}, \ a, b > 0;$$

5. The identric mean

$$I = I(a,b) := \begin{cases} \dfrac{1}{e} \left(\dfrac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \ne b \\ a & \text{if } a = b \end{cases}, \ a, b > 0;$$

6. The p-logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, p \in R \setminus \{-1, 0\}, \ a, b > 0.$$

It is well known that L_p is monotonically increasing in $p \in R$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality

$$H \leq G \leq L \leq I \leq A.$$

In what follows, by the use of Theorem 2, we point out some inequalities for the above means.

Case 1. $f(x) = x^p \ (p \in R \setminus \{-1, 0\})$.
Using the inequality (2.11), we get

$$\begin{aligned} &\left|L_p^p - x^p\right| \\ &\leq \left[\frac{b-a}{2} + |x - A|\right] |p| \, L_{p-1}^{p-1} \\ &-2\,|x - A|\,|p| \left[L_{p-1}^{p-1}(a,m) - L_{p-1}^{p-1}(a+b-m, b)\right] \end{aligned} \tag{5.1}$$

for all $x \in [a,b]$ and $p \neq 1$ where $m = \min\{x, a+b-x\}$.
Let $x = I$ in (5.1). We have

$$\begin{aligned} &\left|L_p^p - I^p\right| \\ &\leq \left[\frac{b-a}{2} + A - I\right] |p| \, L_{p-1}^{p-1} \\ &-2\,(A - I)\,|p| \left[L_{p-1}^{p-1}(a,m) - L_{p-1}^{p-1}(a+b-m, b)\right]. \end{aligned} \tag{5.2}$$

Case 2. $f(x) = \frac{1}{x}$.
Using the inequality (2.11), we get

$$\begin{aligned} &|L - x| \\ &\leq xL \left[\frac{b-a}{2} + |x - A|\right] L_{-2}^{-2} \\ &-2xL\,|x - A| \left[L_{-2}^{-2}(a,m) - L_{-2}^{-2}(a+b-m, b)\right] \end{aligned} \tag{5.3}$$

for all $x \in [a,b]$ where $m = \min\{x, a+b-x\}$.

Let $x = I$ in (5.3). We have

$$\begin{aligned} &0 \leq I - L \\ &\leq xL\left[\frac{b-a}{2} + A - I\right] L_{-2}^{-2} \\ &-2xL\left(A - I\right)\left[L_{-2}^{-2}\left(a, m\right) - L_{-2}^{-2}\left(a + b - m, b\right)\right]. \end{aligned} \tag{5.4}$$

Case 3. $f(x) = -\ln x$.
Using the inequality (2.11), we get

$$\begin{aligned} &\left|\ln I - \ln x\right| \\ &\leq \left[\frac{b-a}{2} + \left|x - A\right|\right] L^{-1} \\ &-2\left|x - A\right|\left[L^{-1}\left(a, m\right) - L^{-1}\left(a + b - m, b\right)\right] \end{aligned} \tag{5.5}$$

for all $x \in [a, b]$ where $m = \min\{x, a + b - x\}$.
Let $x = L$ in (5.5). We have

$$\begin{aligned} &1 \leq \frac{I}{L} \\ &\leq \exp\left(\left[\frac{b-a}{2} + A - L\right] L^{-1}\right. \\ &\left. -\, 2\left(A - L\right)\left[L^{-1}\left(a, m\right) - L^{-1}\left(a + b - m, b\right)\right]\right). \end{aligned} \tag{5.6}$$

Remark 11. The inequalities are an improvements of the inequalities (3.1) – (3.3) in [7, p. 242].

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