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SOME FURTHER RESULTS ON ENTIRE FUNCTIONS SHARING A POLYNOMIAL WITH THEIR LINEAR DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we study the growth of all solutions of a linear differential equation. From this we obtain some uniqueness theorems of a nonconstant entire function and its linear differential polynomials having the same fixed points. The results in this paper also improve some known results. Two example are provided to show that the results in this paper are best possible.

1. Introduction and Main Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [5],[7],[9]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$, we denote by $S(r,h)$ any quantity satisfying

$$S(r,h)=o(T(r,h)) \quad (r\to\infty, r\notin E).$$

Let h be a nonconstant meromorphic function, and k a positive integer. We use $\overline{N}_{k)}(r,h)$ to denote the reduced counting function of poles of h, whose multiplicities are not greater than k. Likewise, we use $\overline{N}_{(k}(r,h)$ to denote the reduced counting function of poles of h, whose multiplicities are not less than k. When multiplicities are duly counted in the above notations, we use $N_{k)}(r,h)$ and $N_{(k}(r,h)$ to indicate them (see [11]).

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Let f and g be two nonconstant meromorphic functions, and let P be a polynomial. We say that f and g share P CM, provided that $f-P$ and $g-P$ have the same zeros with the same multiplicities. Similarly, we say that f and g share P IM, provided that $f-P$ and $g-P$ have the same zeros ignoring multiplicities (see[11]). In this paper, we also need the following two definitions.

Definition 1.1. Let f be a nonconstant entire function, the order of f, denoted $\sigma(f)$, is defined by

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r},$$

where and in what follows, $M(r,f)=\max\limits_{|z|=r}\{|f(z)|\}$.

Definition 1.2. Let f be a nonconstant meromorphic function, the hyper-order of f, denoted $\sigma_2(f)$, is defined by

$$\sigma_2(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}==\limsup_{r\to\infty}\frac{\log\log\log M(r,f)}{\log r}.$$

In 1976, L. A. Rubel and C. C. Yang proved the following theorem.

Theorem A. (see [8]). *Let f be a nonconstant entire function. If f and f' share two finite distinct values CM, then $f\equiv f'$.*

In 1996, R.Brück proved the following theorems.

Theorem B. (see [1]). *Let f be a nonconstant entire function satisfying $\sigma_2(f)<\infty$, and $\sigma_2(f)$ is not a positive integer. If f and f' share the value 0 CM, then $f\equiv cf'$ for some constant $c\neq 0$.*

Theorem C. (see [1]). *Let f be a nonconstant entire function. If f and f' share 1 CM, and if $N(r,\frac{1}{f'})=S(r,f)$, then $f-1\equiv c(f'-1)$ for some constant $c\neq 0$.*

In the same paper, Brück made the following conjecture.

Conjecture 1.1. (see [1]). Let f be a nonconstant entire function satisfying $\sigma_2(f)<\infty$, and $\sigma_2(f)$ is not a positive integer. If f and f' share one finite value a CM, then $f-a\equiv c(f'-a)$ for some constant $c\neq 0$.

Consider the differential equation

$$f'-e^{Q(z)}f=1, \tag{1.1}$$

where $Q(z)$ is an entire function.

In 1998, G. G. Gundersen and L. Z. Yang proved that the conjecture is true for $a \neq 0$, provided that f satisfies the additional assumption $\sigma(f) < \infty$. In fact, they proved the following results.

Theorem D. (see [4, Lemma 1]). *Let $Q(z)$ be a nonconstant polynomial. Then every solution of (1.1) is an entire function of infinite order.*

Theorem E. (see [4, Theorem 1]). *Let f be a nonconstant entire function of finite order. If f and f' share one finite value a CM, then $f - a \equiv c(f' - a)$ for some constant $c \neq 0$.*

Let

$$L[f] = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f, \tag{1.2}$$

where k is a positive integer, and $a_0, a_1, \cdots a_{k-1}, a_k$ are k finite complex numbers.

In this paper, we will prove the following results, which improve Theorem D and Theorem E.

Theorem 1.1. *Let $P(z)$ and $Q_j(z)\,(j = 1, 2)$ be polynomials. If f is a nonconstant solution of the equation*

$$L[f] - Q_1 = (f - Q_2) \cdot e^{P(z)}, \tag{1.3}$$

where $L[f]$ is defined by (1.2), then one of the following three cases will occur.

- (i) *If f is a polynomial or $P(z)$ is a constant, then $L[f] - Q_1 = c(f - Q_2)$, where c is a finite nonzero complex number;*
- (ii) *If $P(z)$ is not a constant and $\mu(f) > 1$, then $\sigma(f) = \infty$ and $\sigma_2(f) = \gamma_P$, where and in what follows, $\mu(f)$ denotes the lower order of f, γ_P denotes the degree of $P(z)$;*
- (iii) *If $P(z)$ is not a constant and $\mu(f) \leq 1$, then $\mu(f) = 1$ and $P(z) = p_1 z + p_0$, where $p_1(\neq= 0)$ and p_0 are two finite complex numbers, moreover, $a_0, a_1, \cdots a_{k-2}$ and $a_{k-1}(k \geq 2)$ are not all equal to zero.*

From Theorem 1.1 we get the following three corollaries, of which Corollary 1.1 improves Theorem D, Corollary 1.2 improves Theorem E.

Corollary 1.1. *Let $P(z)$ be a nonconstant polynomial such that $\gamma_p \geq 2$. Then every solution of (1.3) is an entire function of infinite order, and $\sigma_2(f) = \gamma_P$, where γ_P is the degree of $P(z)$.*

Corollary 1.2. *Let f be a nonconstant entire function of finite order, and let $Q_j(z)\,(j = 1, 2)$ be two polynomials. If $f - Q_2$ and $L[f] - Q_1$ share 0 CM, then $\mu(f) = \sigma(f) = 1$ and one of the following two cases will occur.*

(*i*) $L[f] - Q_1 = c(f - Q_2)$, *where c is a finite nonzero complex number;*

(*ii*) $L[f] - Q_1 = (f - Q_2) \cdot e^{p_1 z + p_0}$, *where $p_1(\neq= 0)$ and p_0 are two finite complex numbers, $a_0, a_1, \cdots a_{k-2}$ and $a_{k-1}(k \geq 2)$ are not all equal to zero.*

Corollary 1.3. *Let $P(z)$ and $Q_j(z)\,(j = 1, 2)$ be polynomials. If f is a solution of (1.3) such that $\sigma_2(f)$ is not a positive integer, then the conclusions (i) and (ii) of Corollary 1.2 hold.*

Proceeding as in the proof of Theorem 1.1 in Section 3 of this paper, we get the following theorem.

Theorem 1.2. *Let $P(z)$ and $Q_j(z)(j = 1, 2)$ be polynomials. If f is a nonconstant solution of the equation $f^{(k)} - Q_1 = (f - Q_2) \cdot e^{P(z)}$, where $k(\geq 1)$ is a positive integer, then $\sigma_2(f) = \gamma_p$.*

Example 1.1. Let f be a solution of the differential equation

$$f' - z = (f - z) \cdot e^{z^n}, \tag{1.4}$$

where n is a positive integer. Since (1.4) can be rewritten by

$$f' - f \cdot e^{z^n} = z(1 - e^{z^n}), \tag{1.5}$$

from (1.5) and Lemma 2.3 in Section 2 of this paper we can see that every solution of (1.5) is a nonconstant entire function. Moreover, it follows from (1.4) that $f - z$ and $f' - z$ share 0 CM. From Lemma 1.1.2 in [7] and in the same manner as in the proof of (3.16) in the proof of Theorem 1.1 in Section 3 of this paper, we get $\mu_2(f) = n$, where and in what follows, $\mu_2(f)$ denotes the lower hyper order of f. This example shows that the condition "$a_0, a_1, ...a_{k-2}$ and $a_{k-1}(k \geq 2)$ are not all equal to zero" in (iii) of Theorem 1.1 and (ii) of Corollaries 1.2-1.3 is best possible.

Example 1.2. Let $f = (e^z - 1)^2$ and $L[f] = f^{(3)} - 3f" + \frac{5}{3}f' - f$. Then we verify that $\mu(f) = \sigma(f) = 1$ and $L[f] - 1 = (f - 1) \cdot e^{-z}$. This example shows that the conclusion (iii) of Theorem 1.1 and (ii) of Corollaries 1.2-1.3 can occur.

Corollary 1.4. *Let $P(z)$ be a polynomial such that $\gamma_P \neq 1$, and let $a\,(\neq 0)$ be a finite complex number. Suppose that f is a nonconstant solution of the differential equation*

$$\frac{L[f] - z}{f - z} = e^{P(z)}, \tag{1.6}$$

where $L[f]$ is defined as in (1.2), and that $\sigma_2(f)$ is not a positive integer. If f and $L[f]$ share the value a IM, then $f \equiv L[f]$.

Proof of Corollary 1.4. First, from Corollary 1.3 and the condition $\gamma_P \neq 1$ we get $L[f]-z = c(f-z)$. On the other hand, from the condition that f and $L[f]$ share a IM and Milloux<¦s inequality (see [5, Theorem 3.2]) we see that there exists one point z_0 such that $L[f](z_0) = f(z_0) = a \neq z_0$. From this and $L[f] - z = c(f - z)$ we get the conclusion of Corollary 1.4.

Corollary 1.5. *Let $P(z)$ be a polynomial, such that $\gamma_P \neq 1$, and a_0 be a constant. Suppose that f is a nonconstant solution of the differential equation (1.6), such that $\sigma_2(f)$ is not a positive integer, where*

$$L[f] = f' + a_0 f. \tag{1.7}$$

If f and $L[f]$ share 0 IM, then $f \equiv L[f]$.

Proof. First, from Corollary 1.3 and the assumptions of Corollary 1.5 we get

$$\frac{f'(z) + a_0 f(z) - z}{f(z) - z} \equiv c, \tag{1.8}$$

where c is a nonzero constant. If $c = 1$, then from (1.8) we can get the conclusion of Corollary 1.5. Next we assume that $c \neq 1$. Since (1.8) can be rewritten as

$$f' + (a_0 - c)f = (1 - c)z, \tag{1.9}$$

which is a linear ODE of order 1. Suppose that there exists a finite complex number z_0 such that $f(z_0) = 0$, then from the condition that f and $L[f]$ share 0 IM we have $L[f](z_0) = f'(z_0) + a_0 f(z_0) = 0$, and so $f'(z_0) = 0$. Combining (1.9) we deduce $z_0 = 0$. That is, f and $L[f]$ have at most one zero $z = 0$. We discuss the following two cases.

Case 1. Suppose that $a_0 = c$. Then from (1.9) and the condition that f has at most one zero $z = 0$, we deduce

$$\begin{aligned} f(z) &= \frac{1}{2}(1-c)z^2, \\ L[f] &= f' + cf = \frac{1}{2}c(1-c)z^2 + (1-c)z. \end{aligned} \tag{1.10}$$

Noting that $c \neq 1$ and that $L[f]$ has at most one zero $z = 0$, from (1.10) we get a contradiction.

Case 2. uppose that $a_0 \neq c$. Then the general solution (1.9) is

$$f = c_1 e^{(c-a_0)z} + \frac{(1-c)z}{a_0 - c} + \frac{c-1}{(a_0-c)^2}, \tag{1.11}$$

where c_1 is a finite complex number. Noting that $c \neq 1$ and that f has at most one zero $z = 0$, from (1.11) we get a contradiction.

In 1995, H. X. Yi and C. C. Yang posed the following question.

Question 1.1. (see [11, pp. 398]). Let f be a nonconstant meromorphic function, and let a be a nonzero constant. If f, $f^{(n)}$ and $f^{(m)}$ share the value a CM, where n and m $(n < m)$ are distinct positive integers not all even or odd, then can we get the result $f \equiv f^{(n)}$?

Regarding Question 1.1, G. G. Gundersen and L. Z. Yang proved the following result in 1998.

Theorem F. (see [4, Theorem 2]). *Let f be a nonconstant entire function of finite order, let a be a nonzero constant, and let n be a positive integer. If the value a is shared by f, $f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f \equiv f'$.*

In this paper, we will prove the following result, which supplements Theorem F.

Theorem 1.3. *Let f be a nonconstant solution of the differential equation*

$$\frac{L'[f] - z}{L[f] - z} = e^P, \tag{1.12}$$

where $L[f]$ is defined as in (1.7), and $P(z)$ is a polynomial. If $\sigma_2(f)$ is not a positive integer, and if $f(z)$ and $L[f]$ share z IM, then e^P is a constant, and f is given by one of the following two expressions.

- (i) *$f = c_1 z + a_0 c_1(1 - c_1)$ and $a_0^2 - a_0 + 1 = 0$, where $c_1\,(\neq 0, 1, 1/a_0)$ is a finite complex number, and $e^P \equiv 1/(1 - a_0 c_1)$.*
- (ii) *$f = d_1 e^z$ and $a_0 = 0$, where $d_1\,(\neq 0)$ is a finite complex constant, and $e^P \equiv 1$.*

2. Some Lemmas

Lemma 2.1. (see [6, pp36-37] or [7, Theorem 3.1]). *If f is an entire function of order $\sigma(f)$, then*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\nu(r,f)}{\log r},$$

where, and in the sequel, $\nu(r,f)$ denotes the central-index of $f(z)$.

Lemma 2.2. (see [2, Lemma 2] or [3, Lemma 4]). *If f is a transcendental entire function of hyper-order $\sigma_2(f)$, then*

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log\nu(r,f)}{\log r}.$$

Lemma 2.3. (see[7, Proposition 8.1]). *Let*

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f = F(z), \tag{2.1}$$

where $a_0\ (\not\equiv 0)$, $a_1, \cdots, a_{n-1}$ *and* $F\ (\not\equiv 0)$ *are entire functions. Then all solutions of (2.1) are entire functions.*

Lemma 2.4. (see [11, Theorem 1.49]). *Suppose that* $f_1, f_2, \cdots, f_n$ *are linearly independent meromorphic functions satisfying the following identity*

$$\sum_{i=1}^{n} f_i \equiv 1.$$

If

$$\sum_{i=1}^{n} \overline{N}(r, f_i) = S(r),$$

where

$$S(r) = o(T(r))\ (r \to \infty, r \notin E),\ T(r) = \max_{1\le i\le n} \{T(r, f_i)\},$$

then

$$T(r) \le \sum_{i=1}^{n} N(r, \frac{1}{f_i}) + S(r).$$

Lemma 2.5. *Let* $f_j\ (j = 1, 2, \cdots, n)$ *be nonconstant meromorphic functions satisfying*

$$N(r, \frac{1}{f_j}) + \overline{N}(r, f_j) = S(r, f_j) \quad (j = 1, 2, \cdots, n), \tag{2.2}$$

and let

$$F \equiv a + \sum_{j=1}^{n} f_j, \tag{2.3}$$

where a *is a meromorphic function satisfying* $a \not\equiv 0$. *If* F *is not constant, and* $T(r, a) = S(r, F)$, *then*

$$T(r, F) = N(r, \frac{1}{F}) + S(r, F). \tag{2.4}$$

Proof. Obviously, $\sum\limits_{j=1}^{n} f_j \not\equiv 0$. Without loss of generality, let

$$\sum_{j=1}^{n} f_j \equiv \sum_{j=1}^{k} c_j f_j, \tag{2.5}$$

where $f_1, f_2, \cdots, f_k$ are linearly independent, and $c_1, c_2, \cdots, c_k$ are nonzero constants. Let

$$g_j = -\frac{c_j f_j}{a} \ (j = 1, 2, \cdots, k) \quad \text{and} \quad g_{k+1} = \frac{F}{a}. \tag{2.6}$$

From (2.3), (2.5) and (2.6) we have

$$\sum_{j=1}^{k+1} g_j \equiv 1. \tag{2.7}$$

It is easy to see that $g_1, g_2, \cdots, g_{k+1}$ are linearly independent. By Lemma 2.4, (2.2), (2.3), (2.6) and (2.7) we obtain

$$T(r) \leq \sum_{i=1}^{k+1} N(r, \frac{1}{f_i}) + S(r) \leq N(r, \frac{1}{F}) + S(r) \leq T(r, F) + S(r), \tag{2.8}$$

where

$$T(r) = \max_{1 \leq j \leq k+1} \{T(r, g_j)\} \quad \text{and} \quad S(r) = o(T(r)) \ (r \to \infty, r \notin E).$$

From (2.8) we can obtain (2.4).

Lemma 2.6. (see [11, Theorem 1.57]). *Suppose that f_1, f_2, f_3 are meromorphic functions satisfying*

$$f_1 + f_2 + f_3 \equiv 1.$$

If f_1 is not a constant and

$$\sum_{i=1}^{3} N(r, \frac{1}{f_i}) + 2 \sum_{i=1}^{3} \overline{N}(r, f_i) < \lambda T(r, f_1) + S(r, f_1),$$

where $\lambda < 1$, then $f_2 \equiv 1$ or $f_3 \equiv 1$.

Lemma 2.7. *Suppose that α and β are nonconstant entire functions, and that a_1, a_2, b_1 and b_2 are meromorphic functions satisfying $T(r, a_1) + T(r, a_2) = S(r, e^{\alpha})$, $T(r, b_1) + T(r, b_2) = S(r, e^{\beta})$ and $a_1 a_2 b_1 b_2 \not\equiv 0$. If $a_1 e^{\alpha} - a_2$ and $b_1 e^{\beta} - b_2$ share 0 IM, then one of the following relations holds:*

(i) $a_1 b_2 e^{\alpha} \equiv a_2 b_1 e^{\beta}$,

(ii) $a_1 b_1 e^{\alpha+\beta} \equiv a_2 b_2$.

Proof. By the second fundamental theorem, we have

$$T(r,e^{\alpha})=\overline{N}(r,\frac{1}{a_1e^{\alpha}-a_2})+S(r,e^{\alpha})=N_{1)}(r,\frac{1}{a_1e^{\alpha}-a_2})+S(r,e^{\alpha}) \tag{2.9}$$

and

$$T(r,e^{\beta})=\overline{N}(r,\frac{1}{b_1e^{\beta}-b_2})+S(r,e^{\beta})=N_{1)}(r,\frac{1}{b_1e^{\beta}-b_2})+S(r,e^{\beta}). \tag{2.10}$$

Let

$$H=\frac{a_1e^{\alpha}-a_2}{b_1e^{\beta}-b_2}. \tag{2.11}$$

Noting that $a_1e^{\alpha}-a_2$ and $b_1e^{\beta}-b_2$ share 0 IM, from (2.9)-(2.11) we obtain

$$N(r,H)=S(r,e^{\alpha}) \quad \text{and} \quad N(r,\frac{1}{H})=S(r,e^{\alpha}). \tag{2.12}$$

By (2.11) we get

$$\frac{a_1}{a_2}e^{\alpha}-\frac{b_1}{a_2}He^{\beta}+\frac{b_2}{a_2}H=1. \tag{2.13}$$

From (2.12) and (2.13), by Lemma 2.6 we obtain $\frac{b_2}{a_2}H=1$ or $-\frac{b_2}{a_2}He^{\beta}=1$.

If $\frac{b_2}{a_2}H=1$, from (2.13) we have $\frac{a_1}{a_2}e^{\alpha}=\frac{b_1}{a_2}He^{\beta}$. From this we have the relation (i) of Lemma 2.7.

If $-\frac{b_1}{a_2}He^{\beta}=1$, from (2.13) we have $\frac{a_1}{a_2}e^{\alpha}=-\frac{b_2}{a_2}H$. From this we have the relation (ii) of Lemma 2.7.

Lemma 2.8. *Suppose that R_1 and R_2 are rational functions, and that a_1 and a_2 are two constants satisfying $0<|a_1|\le|a_2|$ and $a_1\ne a_2$. Then there exists a constant A (>1) such that*

$$AT(r,e^{a_1z})\le T(r,R_1e^{a_1z}+R_2e^{a_2z})+O(\log r). \tag{2.14}$$

Proof. It is easy to see that

$$T(r,e^{a_1z})=\frac{|a_1|r}{\pi},\quad T(r,e^{a_2z})=\frac{|a_2|r}{\pi}. \tag{2.15}$$

Let $a_j=|a_j|e^{i\theta_j}$ $(j=1,2)$, where $0\le\theta_j<2\pi$ $(j=1,2)$. We have

$$\begin{aligned} &T(r,R_1e^{a_1z}+R_2e^{a_2z})\\ &=\frac{r}{2\pi}\int_0^{2\pi}\max\{|a_1|\cos(\theta+\theta_1),|a_2|\cos(\theta+\theta_2),0\}d\theta+O(\log r)\\ &=\frac{r}{2\pi}\int_0^{2\pi}\max\{|a_1|\cos(\theta+\theta_1-\theta_2),|a_2|\cos\theta,0\}d\theta+O(\log r). \end{aligned} \tag{2.16}$$

Suppose that $|a_1| < |a_2|$. From (2.16) we have

$$
\begin{aligned}
T(r, R_1e^{a_1z} + R_2e^{a_2z}) &\geq \frac{r}{2\pi}\int_0^{2\pi} \max\{|a_2|\cos\theta, 0\}d\theta + O(\log r) \\
&= T(r, e^{a_2z}) + O(\log r).
\end{aligned} \tag{2.17}
$$

From (2.15) and (2.17) we can obtain (2.14).

Suppose that $|a_1| = |a_2|$. Noting $a_1 \neq a_2$, we may assume, without loss of generality, $0 \leq \theta_2 < \theta_1 < 2\pi$. If $\theta_1 - \theta_2 \leq \pi$, then $\frac{\pi}{2} \leq \frac{3\pi}{2} - \theta_1 + \theta_2 < \frac{3\pi}{2}$. From (2.16) we have

$$
\begin{aligned}
&T(r, e^{R_1a_1z} + R_2e^{a_2z}) \\
&= \frac{|a_1|r}{2\pi}\int_0^{2\pi} \max\{\cos(\theta + \theta_1 - \theta_2), \cos\theta, 0\}d\theta + O(\log r) \\
&\geq \frac{|a_1|r}{2\pi}\left\{\int_0^{\frac{\pi}{2}} \cos\theta d\theta + \int_{\frac{3\pi}{2}-\theta_1+\theta_2}^{\frac{3\pi}{2}} \cos(\theta + \theta_1 - \theta_2)d\theta \right. \\
&\quad \left. + \int_{\frac{3\pi}{2}}^{2\pi} \cos\theta d\theta\right\} + O(\log r) \\
&= \frac{|a_1|r}{2\pi}(3 - \cos(\theta_1 - \theta_2)) + O(\log r).
\end{aligned} \tag{2.18}
$$

From (2.15) and (2.18) we can obtain (2.14). If $\pi < \theta_1 - \theta_2$, then $\frac{\pi}{2} < \frac{5\pi}{2} - \theta_1 + \theta_2 < \frac{3\pi}{2}$. From (2.16) we have

$$
\begin{aligned}
&T(r, R_1e^{a_1z} + R_2e^{a_2z}) \\
&= \frac{|a_1|r}{2\pi}\int_0^{2\pi} \max\{\cos(\theta + \theta_1 - \theta_2), \cos\theta, 0\}d\theta + O(\log r) \\
&\geq \frac{|a_1|r}{2\pi}\left\{\int_0^{\frac{\pi}{2}} \cos\theta d\theta + \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}-\theta_1+\theta_2} \cos(\theta + \theta_1 - \theta_2)d\theta \right. \\
&\quad \left. + \int_{\frac{3\pi}{2}}^{2\pi} \cos\theta d\theta\right\} + O(\log r) \\
&= \frac{|a_1|r}{2\pi}(3 - \cos(\theta_1 - \theta_2)) + O(\log r).
\end{aligned} \tag{2.19}
$$

From (2.15) and (2.19) we can obtain (2.14).

This completes the proof of Lemma 2.8.

3. Proof of Theorems

Proof of Theorem 1.1. Suppose that f is a polynomial, then from (1.3) we see that there exists a nonzero constant c such that $e^{P(z)} \equiv c$. So $\sigma_2(f) = \gamma_P = 0$, thus the conclusion (i) of Theorem 1.1 is valid. Next we suppose that f is a transcendental entire function. We discuss the following two cases.

Case 1. Suppose that

$$\sigma(f) = \infty. \tag{3.1}$$

From (3.1) and Lemma 2.1 we see that

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\nu(r,f)}{\log r} = \infty, \tag{3.2}$$

where $\nu(r, f)$ denotes the central-index of $f(z)$. If $P(z)$ is a constant, by (1.3) and Theorem 4.1 in [7] we deduce that all solutions of

$$f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_2f'' + a_1f' + (a_0 - e^P)f = Q_1 - Q_2e^P$$

have finite order, this is a contradiction. Thus, $P(z)$ is a nonconstant polynomial. Let

$$P(z) = p_nz^n + p_{n-1}z^{n-1} + \cdots + p_1z + p_0, \tag{3.3}$$

where $p_n(\neq 0), p_{n-1}, \cdots, p_1$ and p_0 are complex constants. It follows from (3.3) that

$$\lim_{|z|\to+\infty} \frac{|P(z)|}{|p_nz^n|} = 1. \tag{3.4}$$

From (3.4) we see that there exists a sufficiently large positive number r_0, such that

$$\frac{|P(z)|}{|p_nz^n|} > \frac{1}{e} \quad (|z| > r_0). \tag{3.5}$$

From (1.3) and (3.5) we deduce

$$\begin{aligned} &n\log r + \log|p_n| - 1 \\ &= \log\frac{|p_nz^n|}{e} \le \log|P(z)| = \log|\log e^{P(z)}| \le |\log\log e^{P(z)}| \\ &= |\log\log\frac{L[f]-Q_1}{f-Q_2}| \quad (|z| > r_0), \end{aligned} \tag{3.6}$$

On the other hand, since f is a nonconstant entire function, thus

$$M(r,f)\to+\infty, \tag{3.7}$$

as $r\to+\infty$, where $M(r,f)=\max\limits_{|z|=r}|f(z)|$. Again let

$$M(r,f)=|f(z_r)|, \tag{3.8}$$

where $z_r=re^{i\theta(r)}$, and $\theta(r)\in[0,2\pi)$. From (3.8) and the Wiman-Valiron theory (see [7, Theorem 3.2]), we see that there exists a subset $E_j\subset(1,\infty)$ $(1\le j\le n)$ with finite logarithmic measure, i.e., $\int_{E_j}\frac{dt}{t}<\infty$, such that for some point $z_r=re^{i\theta(r)}$ $(\theta(r)\in[0,2\pi))$ satisfying $|z_r|=r\notin E_j$ and $M(r,f)=|f(z_r)|$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)}=(\frac{\nu(r,f)}{z_r})^j(1+o(1))\quad(1\le j\le n), \tag{3.9}$$

as $r\to+\infty$. Noting that f is a transcendental entire function, and $Q_i\,(i=1,2)$ are polynomials, from (3.1) and (3.8) we deduce

$$\lim_{r\to\infty}\frac{|Q_i(z_r)|}{|f(z_r)|}=\lim_{r\to\infty}\frac{|Q_i(z_r)|}{M(r,f)}=0\quad(i=1,2). \tag{3.10}$$

Since

$$\frac{L[f]-Q_1}{f-Q_2}=\frac{\frac{L[f]}{f}-\frac{Q_1}{f}}{1-\frac{Q_2}{f}}, \tag{3.11}$$

from (1.2), (1.3), (3.2) and (3.6)-(3.11) we deduce

$$n\log|z_r|+\log|p_n|-1\le|\log\log((\frac{\nu(r,f)}{z_r})^k(1+o(1)))| \tag{3.12}$$

and

$$\begin{aligned}&\log\left(\left(\frac{\nu(r,f)}{z_r}\right)^k(1+o(1))\right)\\&=k\left(\log\nu(r,f)-\log re^{i\theta(r)}\right)+o(1)\\&=k\left(\log\nu(r,f)-\log r-i\theta(r)\right)+o(1)\\&=k\left(1-\frac{\log r}{\log\nu(r,f)}-\frac{i\theta(r)}{\log\nu(r,f)}\right)\log\nu(r,f)+o(1),\end{aligned} \tag{3.13}$$

as $r \to +\infty$. Noting that $\theta(r) \in [0, 2\pi)$, from (3.2), (3.13) and Lemma 2.2 we deduce

$$
\begin{aligned}
&\limsup_{r\to\infty} \frac{|\log\log((\frac{\nu(r,f)}{z_r})^k(1+o(1)))|}{\log r}. \\
&\le \limsup_{r\to\infty} \frac{\log\log\nu(r,f)}{\log r} + \limsup_{r\to\infty} \frac{|\log(1-\frac{\log r}{\log\nu(r,f)}-\frac{i\theta(r)}{\log\nu(r,f)})|}{\log r} \\
&\quad + \lim_{r\to\infty}\frac{\log 2}{\log r} + \lim_{r\to\infty}\frac{2k_1\pi}{\log r} \\
&= \limsup_{r\to\infty} \frac{\log\log\nu(r,f)}{\log r} = \sigma_2(f),
\end{aligned}
\tag{3.14}
$$

where k_1 is some nonnegative integer. Noting that $|z_r| = r$, from (3.12) and (3.14) we deduce

$$
n \le \limsup_{r\to\infty} \frac{\log\log\nu(r,f)}{\log r} = \sigma_2(f). \tag{3.15}
$$

From (3.3) we obtain

$$
\sigma(e^P) = \gamma_{P(z)} = n. \tag{3.16}
$$

From (3.15) and (3.16) we get

$$
\sigma(e^P) \le \sigma_2(f). \tag{3.17}
$$

If $\liminf\limits_{r\to\infty}(\log\nu(r,f))/(\log r) > 1$, from (1.3), (3.2) and (3.9)-(3.11) we deduce

$$
(\frac{\nu(r,f)}{z_r})^k(1+o(1)) = e^{P(z_r)}, \tag{3.18}
$$

as $r \to \infty$, and so it follows from (3.18) that

$$
\begin{aligned}
\limsup_{r\to\infty} \frac{\log\log\nu(r,f)}{\log r} &= \limsup_{r\to\infty} \frac{\log\log(\frac{\nu(r,f)}{2r})^k}{\log r} \\
&\le \limsup_{r\to\infty} \frac{\log\log M(r,e^P)}{\log r}.
\end{aligned}
\tag{3.19}
$$

From (3.19) and Lemma 2.2 we get

$$
\sigma_2(f) \le \sigma(e^P). \tag{3.20}
$$

From (3.16), (3.17) and (3.20) we get the conclusion (ii) of Theorem 1.1.

Case 2. Suppose that

$$\sigma(f) < \infty. \tag{3.21}$$

First, from (3.21) we can deduce

$$\sigma_2(f) = 0. \tag{3.22}$$

On the other hand, from (1.2), (1.3), (3.7)-(3.11), (3.21) and Lemma 2.1 we deduce

$$\begin{aligned} |P(z_r)|^2 &= |\log \frac{L[f(z_r)] - Q_1(z_r)}{f(z_r) - Q_2(z_r)}|^2 \\ &= (\log|\frac{L[f(z_r)] - Q_1(z_r)}{f(z_r) - Q_2(z_r)}|)^2 + O(1) \\ &\le (\log((\frac{\nu(r,f)}{r})^k + |a_{k-1}|(\frac{\nu(r,f)}{r})^{k-1} + \cdots \\ &\quad + |a_2|(\frac{\nu(r,f)}{r})^2 + |a_1|(\frac{\nu(r,f)}{r})))^2 + O(1) \\ &\le O((\log r)^2), \end{aligned} \tag{3.23}$$

as $r \to +\infty$. Since $P(z)$ is a polynomial, from (3.23) we deduce that $P(z)$ is a constant, and so $r_P = 0$. From this and (3.22) we see that the conclusion of Theorem 1.1 is valid. If $\liminf_{r\to\infty}(\log\nu(r,f))/(\log r) \le 1$, then from $\mu(f) = \liminf_{r\to\infty}$ $(\log\nu(r,f))/(\log r)$ which can be verified in the same manner as in the proof of Lemma 2.2 in [2], we have $\mu(f) \le 1$. On the other hand, from (1.2) and (1.3) we get $T(r, e^P) \le O(T(r,f) + \log T(r,f) + \log r)(r \notin E)$. From this and Lemma 1.1.1 in [7] we see that for sufficiently large positive number r_0, we have $T(r, e^P) \le O(T(2r,f) + \log T(2r,f) + \log r + \log 2)(r \ge r_0)$. From this we get $1 \le n = r_P = \sigma(e^P) = \mu(e^P) \le \mu(f)$. Combining $\mu(f) \le 1$, we get $\mu(f) = n = 1$. Combining (1.3) and (3.3) we get $P(z) = p_1 z + p_0$. If $a_j = 0 (0 \le j \le k-1)$, then (1.3) can be rewritten by $f^{(k)} - Q_1 = (f - Q_2) \cdot e^{p_1 z + p_0}$. From this, Lemma 1.1.2 in [7] and in the same manner as in the proof of (3.16) we get $\mu_2(f) = r_P = 1$. This contradicts $\mu(f) = 1$. Thus $a_0, a_1, \cdots a_{k-2}$ and a_{k-1} are not all equal to zero. From the above analysis we get (iii) of Theorem 1.1.

Theorem 1.1 is thus completely proved.

Proof of Theorem 1.3. Suppose that f is a nonconstant polynomial. If $a_0 = 0$, it follows by (1.7) and (1.12) that f is a polynomial of degree 2. Let

$$f(z) = b_2 z^2 + b_1 z + b_0, \tag{3.24}$$

where $b_2\,(\neq 0)$, b_1 and b_0 are three finite complex numbers. From (1.7), (1.12) and (3.24) we deduce

$$\frac{z-2b_2}{(1-2b_2)z-b_1}=e^P,$$

where e^p is a constant. From this we deduce

$$2b_2-1\neq 0 \tag{3.25}$$

and

$$b_1=2b_2(1-2b_2). \tag{3.26}$$

Noting that

$$f(z)-z=b_2z^2+(b_1-1)z+b_0 \tag{3.27}$$

and

$$L[f]-z=f'-z=(2b_2-1)z+b_1, \tag{3.28}$$

from (3.27), (3.28) and the condition that $f(z)-z$ and $L[f](z)-z$ share 0 IM we can get

$$(b_1-1)^2-4b_0b_2=0$$

and

$$b_1+2b_2=1. \tag{3.29}$$

From (3.25), (3.26) and (3.29) we get a contradiction. Thus,

$$a_0\neq 0. \tag{3.30}$$

Then it follows from (1.7), (1.12) and (3.30) that f is a polynomial of degree 1. Let

$$f(z)=c_1z+c_0, \tag{3.31}$$

where $c_1\,(\neq 0)$ and c_0 are two finite complex numbers. From (1.7), (1.12) and (3.31) we deduce

$$L'[f](z)-z=a_0c_1-z \tag{3.32}$$

and

$$\frac{a_0c_1-z}{(a_0c_1-1)z+a_0c_0+c_1}=e^P,$$

which implies that $1 - a_0c_1 \neq 0$,

$$e^P \equiv \frac{1}{1 - a_0c_1} \tag{3.33}$$

and

$$\frac{a_0c_0 + c_1}{1 - a_0c_1} = a_0c_1. \tag{3.34}$$

Noting that $f(z) - z$ and $L'[f](z) - z$ share 0 IM, from (3.31) and (3.32) we deduce

$$c_1 - 1 \neq 0 \tag{3.35}$$

and

$$c_0 = a_0c_1(1 - c_1). \tag{3.36}$$

Substituting (3.36) into (3.34) we deduce

$$a_0^2 - a_0 + 1 = 0. \tag{3.37}$$

From (3.31), (3.33) and (3.35)-(3.37) we can get the conclusion (i) of Theorem 1.3.

Next we suppose that f is a transcendental entire function. First, by Milloux's inequality (see [5, Theorem 3.2]) we have

$$T(r, f) < N(r, \frac{1}{f(z) - z}) + \overline{N}(r, \frac{1}{L'[f](z) - z}) + S(r, f). \tag{3.38}$$

Let z_0 be a zero of $f(z) - z$ with multiplicity≥ 2. Then $f(z_0) = z_0$ and $f'(z_0) = 1$. Since f and $L[f]$ share z IM, we have $L[f](z_0) = z_0$. Thus $1 + a_0z_0 = z_0$, and hence $f(z) - z$ has at most one zero with multiplicity≥ 2. From this we obtain

$$N(r, \frac{1}{f(z) - z}) = \overline{N}(r, \frac{1}{f(z) - z}) + O(\log r) \tag{3.39}$$

$$= \overline{N}(r, \frac{1}{L[f](z) - z}) + O(\log r). \tag{3.40}$$

From (1.12) we have

$$\overline{N}(r, \frac{1}{L'[f](z) - z}) = \overline{N}(r, \frac{1}{L[f](z) - z}). \tag{3.41}$$

From (3.38), (3.40) and (3.41) we obtain

$$\begin{aligned} T(r, f) &\leq 2\overline{N}(r, \frac{1}{L[f](z) - z}) + S(r, f) \\ &\leq 2T(r, L[f]) + S(r, f) \leq 2T(r, f) + S(r, f). \end{aligned} \tag{3.42}$$

From (3.42) we deduce

$$\sigma_2(f) = \sigma_2(L[f]). \tag{3.43}$$

From (3.43) and the condition that $\sigma_2(f)$ is not a positive integer we know that $\sigma_2(L[f])$ is not a positive integer, and so it follows from (1.12) and Theorem 1.2 that there exists a finite nonzero complex number d such that

$$\frac{L'[f] - z}{L[f] - z} \equiv d. \tag{3.44}$$

We discuss the following three cases.

Case 1. Suppose that $a_0 = 0$. Then it follows from (1.7) that (3.44) can be rewritten by

$$f'' - df' = (1 - d)z. \tag{3.45}$$

From (3.45) we deduce

$$f = d_1 e^{dz} + \frac{d-1}{2d} z^2 + \frac{d-1}{d^2} z + d_2, \tag{3.46}$$

where $d_1 (\neq 0)$ and d_2 are constants. Thus,

$$f - z = d_1 e^{dz} + \frac{d-1}{2d} z^2 + \frac{d - 1 - d^2}{d^2} z + d_2, \tag{3.47}$$

$$L[f] - z = d_1 d e^{dz} - \frac{1}{d} z + \frac{d-1}{d^2}. \tag{3.48}$$

Assume that $d \neq 1$. Since $f(z) - z$ and $L[f] - z$ share 0 IM, by Lemma 2.7, (3.47) and (3.48) we get a contradiction. Thus $d = 1$, and so it follows from (3.47) and (3.48) that $f - z = d_1 e^z - z + d_2$ and $L[f] - z = d_1 e^z - z$. Combining the condition that $f - z$ and $L[f] - z$ share 0 IM we deduce $d_2 = 0$, and so it follows that $f = d_1 e^z$, which reveals the conclusion (ii) of Theorem 1.3.

Case 2. Suppose that $a_0 \neq 0$ and $a_0 = -d$. Then it follows from (1.7) and (3.44) that

$$f'' - 2df' + d^2 f = (1 - d)z. \tag{3.49}$$

From (3.49) we deduce

$$f(z) = (d_3 z + d_4) e^{dz} + \frac{1-d}{d^2} z + \frac{2(1-d)}{d^3},$$

where d_3 and d_4 are constants satisfying $d_3z + d_4 \not\equiv 0$. Thus,

$$f(z) - z = (d_3z + d_4)e^{dz} + \frac{1 - d - d^2}{d^2}z + \frac{2(1-d)}{d^3}, \tag{3.50}$$

$$L[f] - z = d_3e^{dz} - \frac{1}{d}z + \frac{d-1}{d^2}. \tag{3.51}$$

By Lemma 2.7, (3.50) and (3.51) we get a contradiction.

Case 3. Suppose that $a_0 \neq 0$ and $a_0 \neq -d$. Then it follows from (1.7) and (3.44) that

$$f'' + (a_0 - d)f' - a_0df = (1-d)z. \tag{3.52}$$

From (3.52) we deduce

$$f = d_5e^{-a_0z} + d_6e^{dz} + \frac{d-1}{a_0d}z + \frac{(a_0-d)(d-1)}{a_0^2d^2},$$

where d_5 and d_6 are constants satisfying $d_5e^{-a_0z} + d_6e^{dz} \not\equiv 0$. Thus,

$$f - z = d_5e^{-a_0z} + d_6e^{dz} + P_1(z), \tag{3.53}$$

$$L[f] - z = d_6(d + a_0)e^{dz} + P_2(z), \tag{3.54}$$

where

$$P_1(z) = \frac{d - 1 - a_0d}{a_0d}z + \frac{(a_0-d)(d-1)}{a_0^2d^2},$$

$$P_2(z) = -\frac{1}{d}z + \frac{d-1}{d^2}.$$

If $d_5 = 0$, then $d_6 \neq 0$. By Lemma 2.7, (3.53) and (3.54) we get a contradiction. If $d_6 = 0$, then $d_5 \neq 0$. From (3.53) and (3.54) we obtain a contradiction. Next, we suppose that $d_5 \neq 0$ and $d_6 \neq 0$.

Let z_0 be a zero of $L[f] - z$. From (3.54) we obtain

$$d_6(d + a_0)e^{dz_0} + P_2(z_0) = 0. \tag{3.55}$$

Since $f - z$ and $L[f] - z$ share 0 IM, from (3.53) we deduce

$$d_5e^{-a_0z_0} + d_6e^{dz_0} + P_1(z_0) = 0. \tag{3.56}$$

From (3.55) and (3.56) we have

$$d_5(d + a_0)e^{-a_0z_0} + (d + a_0)P_1(z_0) - P_2(z_0) = 0. \tag{3.57}$$

Noting that z_0 is a zero of $L[f]-z$, from (3.55) and (3.57) we obtain

$$\begin{aligned}&\overline{N}(r,\frac{1}{d_6(d+a_0)e^{dz}+P_2(z)})\\ &\leq \overline{N}(r,\frac{1}{d_5(d+a_0)e^{-a_0z}+(d+a_0)P_1(z)-P_2(z)}).\end{aligned} \tag{3.58}$$

It is easy to see that

$$T(r,e^{dz})=\overline{N}(r,\frac{1}{d_6(d+a_0)e^{dz}+P_2(z)})+O(\log r), \tag{3.59}$$

$$\begin{aligned}&T(r,e^{-a_0z})\\ &=\overline{N}(r,\frac{1}{d_5(d+a_0)e^{-a_0z}+(d+a_0)P_1(z)-P_2(z)})+O(\log r).\end{aligned} \tag{3.60}$$

From (3.58)-(3.60) we deduce

$$T(r,e^{dz})\leq T(r,e^{-a_0z})+O(\log r). \tag{3.61}$$

Since

$$T(r,e^{dz})=\frac{|d|r}{\pi}\quad\text{and}\quad T(r,e^{-a_0z})=\frac{|a_0|r}{\pi},$$

from (3.61) we get $|d|\leq|a_0|$. Noting that $d\neq -a_0$, by Lemma 2.8, (3.53) and (3.54) we know that there exists a constant A (>1) such that

$$AT(r,L[f])\leq T(r,f)+O(\log r). \tag{3.62}$$

On the other hand, from (3.54) we have

$$T(r,L[f])=\overline{N}(r,\frac{1}{L[f]-z})+O(\log r). \tag{3.63}$$

By Lemma 2.7 and the condition that $f-z$ and $L[f]-z$ share 0 IM, we deduce $P_1(z)\not\equiv 0$. Combining (3.53), (3.54) and Lemma 2.5 we deduce

$$T(r,f)=N(r,\frac{1}{f-z})+O(\log r). \tag{3.64}$$

Again from (3.39) and (3.64) we obtain

$$T(r,f)=\overline{N}(r,\frac{1}{f-z})+O(\log r). \tag{3.65}$$

Since $f-z$ and $L[f]-z$ share 0 IM, we have

$$\overline{N}(r,\frac{1}{f-z})=\overline{N}(r,\frac{1}{L[f]-z}). \tag{3.66}$$

From (3.63), (3.65) and (3.66) we obtain

$$T(r, L[f]) = T(r, f) + O(\log r). \tag{3.67}$$

Noting that f is a transcendental entire function, from (3.62) and (3.67) we get a contradiction.

Theorem 1.3 is thus completely proved. ∎

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