NONLINEAR SIMULTANEOUS APPROXIMATION IN COMPLETE LATTICE BANACH SPACES

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Abstract. This paper is concerned with the problem of nonlinear best simultaneous approximations in conditional complete lattice Banach spaces with a strong unit. Characterization results of the best simultaneous approximation from simultaneous suns and suns are established. A counterexample, to which the characterization theorem for convex sets due to Mohebi (*Numer. Funct. Anal. Optim.*, **25** (2004), 685-705) fails, is provided and a corrected version of the theorem is presented.

1. INTRODUCTION

The problem of best simultaneous approximations from convex sets (in particular, subspaces) in normed linear spaces has been studied extensively, see, for example, [1, 6, 9, 15, 19] and references herein. Extensions of the study to real locally convex spaces are done in [8, 14]. These works are mainly on the characterization and/or uniqueness of best simultaneous approximations.

Recent interests are focused on the study of the best simultaneous approximation in conditional complete lattice Banach spaces with strong unit **1**, see [16-18]. In particular, the following characterization result, which plays key roles in [18], was proved by Mohebi in [16, Theorem 3.1].

Theorem M. *Let W be a closed convex subset of X, S be a bounded set in X with $S \cap W = \emptyset$, and $w_0 \in W$. Then $w_0 \in P_W(S)$ if and only if there*

exists $f \in X^*$ *with* $\|f\| = 1$ *such that* $f(\sup S - w_0) = \sup_{s \in S} \|s - w_0\|$ *and* $f(w_0 - w) \geq 0$ *for each* $w \in W$.

Unfortunately, Theorem M is not true in general. In the present paper, we first present an example in section 2 to which Theorem M fails; and then continue to study the problem of the simultaneous approximation in conditional complete lattice Banach spaces with strong unit **1** but from nonlinear sets (not necessary convex). Characterization results of the best simultaneous approximation from simultaneous suns and suns are established. As a corollary, a corrected version of Theorem M is obtained. In addition, Some examples of nonconvex simultaneous suns are provided.

2. PRELIMINARIES

Let X be a normed linear space with the dual X^*. For a nonempty subset W of X and a nonempty bounded set S in X, Define

$$d(S, W) = \inf_{w \in W} \sup_{s \in S} \|s - w\|$$

and
$$P_W(S) = \{w \in W : \sup_{s \in S} \|s - w\| = d(S, W)\}.$$

Each element in $P_W(S)$ (if $P_W(S) \neq \emptyset$) is called a best simultaneous approximation to S from W.

For a set A in X^*, let $\mathrm{ext}A$ stand for the set of all extreme points of A. Furthermore, we use $\mathbf{B}(X)$ to denote the closed unit ball of X. Recall that the supporting mapping $\sigma(\cdot)$ on X is defined by

$$\sigma(x) = \{f \in \mathbf{B}(X^*) : f(x) = \|x\|\} \quad \text{for each } x \in X.$$

It is known (cf. [2]) that $\sigma(x)$ is a nonempty weak* compact convex subset of $\mathbf{B}(X^*)$ for each $x \in X$. We then recall some notions concerning vector lattices.

Definition 2.1. A lattice $(L, \leq)$ is said to be conditionally complete if it satisfies one of the following equivalent conditions:

(1) Every non-empty lower bounded set admits an infimum.

(2) Every non-empty upper bounded set admits an supremum.

(3) There exists a complete lattice $\bar{L} := L \cup \{\bot, \top\}$, which we call the minimal completion of L, with bottom $\bot$ and top element $\top$, such that L is a sublattice of $\bar{L}$, $\inf L = \bot$ and $\sup L = \top$.

Definition 2.2. A conditionally complete lattice Banach space (resp. normed

linear lattice) X is a (real) Banach space (resp. normed linear space) which is also a conditionally complete lattice such that

(2.1) $$|x| \leq |y| \Longrightarrow \|x\| \leq \|y\| \quad \text{for any } x, y \in X,$$

where $|x| := \sup\{x, -x\}$ for each $x \in X$.

Clearly, a conditional complete lattice Banach space is a normed linear lattice. Recall that an element, denoted by $\mathbf{1}$, of a normed linear lattice X is called a strong unit if $\|\mathbf{1}\| = 1$ and $x \leq \mathbf{1}$ for each $x \in \mathbf{B}(X)$. Throughout the remainder of the paper, we always assume that X is a conditionally complete lattice Banach space with strong unit $\mathbf{1}$. Then by [7, Lemma 1, p. 18] one has that

(2.2) $$\|\,|x|\,\| = \|x\| \quad \text{for each } x \in X,$$

and the norm on X is monotonic, that is,

(2.3) $$-y \leq x \leq y \Longrightarrow \|x\| \leq \|y\| \quad \text{for all } x, y \in X.$$

In particular,

(2.4) $$x \in \mathbf{B}(X) \Longleftrightarrow -\mathbf{1} \leq x \leq \mathbf{1}.$$

We end this section with an example to which Theorem M fails.

Example 2.1. Let $X = l_\infty^2$ be defined by

$$l_\infty^2 = \{x = (t_1, t_2) : t_1, t_2 \in \mathbb{R}\}$$

with the natural order and the norm defined by

$$\|x\| = \max\{|t_1|, |t_2|\} \quad \text{for each } x = (t_1, t_2) \in l_\infty^2.$$

Then X is a conditional complete lattice Banach space with the strong unit $\mathbf{1} = (1, 1)$. Let $W = \{(t, t - 1) : t \in [0, 1]\}$. Then W is a closed convex subset of X. Take $S = \{(t, -t) : t \in [-1, 0]\}$ and $w_0 = (\frac{1}{2}, -\frac{1}{2})$. Then S is a bounded subset of X and $S \cap W = \emptyset$. Furthermore, one has that $w_0 \in P_W(S)$ and $\sup S = (0, 1)$. Let $f = (f_1, f_2) \in X^*$ be such that $\|f\| = 1$ and $f(\sup S - w_0) = \sup_{s \in S} \|s - w_0\|$. Then $-\frac{1}{2}f_1 + \frac{3}{2}f_2 = \frac{3}{2}$ and $|f_1| + |f_2| = 1$. It follows that $f = (0, 1)$. However, $f(w_0 - \bar{w}) = -\frac{1}{2} < 0$, where $\bar{w} = (1, 0)$. Therefore, Theorem M fails.

3. CHARACTERIZATIONS OF SIMULTANEOUS APPROXIMATIONS

We begin with the following key lemma.

Lemma 3.1. *Let S be a bounded subset of X. Then*

$$(3.1) \qquad \max\{\|\sup S\|, \|\inf S\|\} = \sup_{s \in S} \|s\|.$$

Proof. Set $\|S\| = \sup_{s \in S} \|s\|$. Since $\left\|\frac{s}{\|S\|}\right\| \leq 1$ for each $s \in S$, it follows from (2.4) that $-\|S\|\mathbf{1} \leq s \leq \|S\|\mathbf{1}$ and

$$-\|S\|\mathbf{1} \leq \inf S \leq \sup S \leq \|S\|\mathbf{1}.$$

This together with (2.3) implies that

$$(3.2) \qquad \max\{\|\sup S\|, \|\inf S\|\} \leq \|S\|.$$

For the opposite inequality, we note that, for each $s \in S$,

$$(3.3) \qquad \begin{aligned} -\sup\{|\sup S|, |\inf S|\} &\leq -|\inf S| \leq \inf S \leq s \\ &\leq \sup S \leq |\sup S| \leq \sup\{|\sup S|, |\inf S|\}. \end{aligned}$$

Applying (3.2) (with $\{|\sup S|, |\inf S|\}$ in place of S) and (2.2), we have that

$$\|\sup\{|\sup S|, |\inf S|\}\| \leq \max\{\|\sup S\|, \|\inf S\|\}.$$

It follows from (2.3) and (3.3) that

$$\|s\| \leq \|\sup\{|\sup S|, |\inf S|\}\| \leq \max\{\|\sup S\|, \|\inf S\|\}.$$

Consequently,

$$\|S\| = \sup_{s \in S} \|s\| \leq \max\{\|\sup S\|, \|\inf S\|\}.$$

The proof is complete. ∎

The following proposition describes the equivalence of the best simultaneous approximations to a bounded set S and to $\{\sup S, \inf S\}$.

Proposition 3.1. *Let W be a subset of X and S a bounded set in X. Then $w_0 \in W$ is a best simultaneous approximation to S from W if and only if w_0 is a best simultaneous approximation to $\{\sup S, \inf S\}$ from W.*

Proof. Let $w \in W$. Then, by Lemma 3.1, we have that

$$\sup_{s \in S} \|s - w\| = \max\{\|\sup(S - w)\|, \inf\|(S - w)\|\} = \max\{\|\sup S - w\|, \|\inf S - w\|\}.$$

Thus the conclusion follows. ∎

Below we define the notions of suns and simultaneous suns. For a bounded set S in X, $w_0 \in X$ and $\lambda \geq 0$, we set

(3.4) $$S_\lambda = w_0 + \lambda(S - w_0).$$

Definition 3.1. Let W be a nonempty subset of X. W is called

(1) a sun if, for each $w_0 \in W$ and $x \in X$, $w_0 \in P_W(x)$ implies that $w_0 \in P_W(w_0 + \lambda(x - w_0))$ for all $\lambda > 0$.

(2) a simultaneous sun if, for each $w_0 \in W$ and each bounded set S in X, $w_0 \in P_W(S)$ implies $w_0 \in P_W(S_\lambda)$ for all $\lambda > 0$.

The notion of suns introduced by Efimove and Stechkin in [5] plays an important role in nonlinear approximation theory and have been investigated extensively, see, for example, [3-5, 23]; while the notion of simultaneous suns was introduced in [21, 22]. Extensions to the case of other various simultaneous approximation problems can be found in [10-14, 23]. Clearly, a simultaneous sun is a sun. As is well known (cf. [21-23]), any convex subset is a simultaneous sun. Below we present examples of nonconvex simultaneous suns.

Example 3.1. Recall from [23] that a subset W of X is called a

(a) quasi-convex set if for each pair of $w_1, w_2 \in W$, $[w_1, w_2] \cap W$ is dense in $[w_1, w_2]$.

(b) pseudo-convex set if there exist a convex set D and a closed set C in X such that $W = D \setminus C$.

Let W be a quasi-convex set or pseudo-convex subset of X. Then W is a simultaneous sun.

In fact, let S be a bounded subset of X. Note that, if $w_0 \in P_W(S)$, then, for each $\lambda \in [0, 1]$ and $w \in W$,

$$\sup_{s \in S_\lambda} \|s - w_0\| = \sup_{s \in S} \|s - w_0\| - (1 - \lambda) \sup_{s \in S} \|s - w_0\|$$

$$\leq \sup_{s \in S} \|s - w\| - (1 - \lambda) \sup_{s \in S} \|s - w_0\|$$

$$\leq \sup_{s \in S_\lambda} \|s - w\|.$$

Hence

(3.5) $$w_0 \in P_W(S) \Longrightarrow w_0 \in P_W(S_\lambda) \quad \text{for each } \lambda \in [0, 1].$$

Thus we need only prove that

$$w_0 \in P_W(S) \Longrightarrow w_0 \in P_W(S_\lambda) \quad \text{for each } \lambda > 1.$$

For this end, let $\lambda > 1$, $w_0 \in P_W(S)$ and $w \in W \backslash \{w_0\}$. Set $v_\lambda := (1 - \frac{1}{\lambda})w_0 + \frac{1}{\lambda}w$. It suffices to show that

$$(3.6) \qquad \sup_{s \in S_\lambda} \|s - w_0\| \leq \sup_{s \in S_\lambda} \|s - w\|;$$

or equivalently,

$$(3.7) \qquad \sup_{s \in S} \|s - w_0\| \leq \sup_{s \in S} \|s - v_\lambda\|.$$

In the case when W is a quasi-convex set, there exists a sequence $\{w_n\} \subseteq [w_0, w] \cap W$ such that $\lim_{n \to \infty} \|w_n - v_\lambda\| = 0$ because $v_\lambda \in [w_0, w]$ and $[w_0, w] \cap W$ is dense in $[w_0, w]$ by definition. Consequently, $\sup_{s \in S} \|s - w_0\| \leq \sup_{s \in S} \|s - w_n\|$ for each n. Passing to the limit, one sees that (3.7) holds. In the case when W is a pseudo-convex set, since $W = D \backslash C$ for some convex set D and closed set C in X, one has that $w_0 \in D$ and $w_0 \notin C$. By the closedness of C, there exists $\delta > 0$ such that $\mathbf{B}(w_0, \delta) \cap C = \emptyset$. Hence, $(w_0, w) \cap \mathbf{B}(w_0, \delta) \subseteq W$ because D is convex. Set $\lambda_0 = \max\{1, \|w - w_0\|/\delta\}$. If $\lambda > \lambda_0$, then $\|v_\lambda - w_0\| = \frac{1}{\lambda}\|w - w_0\| < \delta$, and so $v_\lambda \in (w_0, w) \cap \mathbf{B}(w_0, \delta) \subseteq W$. This together with the fact that $w_0 \in P_W(S)$ implies that (3.7) holds. Consequently, (3.6) holds in either cases. From this we get that

$$(3.8) \qquad w_0 \in P_{[w_0, w]}(S_\lambda) \quad \text{if } \lambda > \lambda_0.$$

If $\lambda \in (1, \lambda_0]$, we take $\lambda_1 > \lambda_0$. Then $\frac{\lambda}{\lambda_1} \in (0, 1)$ and $w_0 \in P_{[w_0, w]}(S_{\lambda_1})$ by (3.8). Since

$$S_\lambda = w_0 + \lambda(S - w_0) = w_0 + \frac{\lambda}{\lambda_1}(S_{\lambda_1} - w_0),$$

we apply (3.5) to S_{λ_1}, $[w_0, w]$ and $\frac{\lambda}{\lambda_1}$ in place of S, W and λ to get that $w_0 \in P_{[w_0, w]}(S_\lambda)$. This implies that (3.6) holds.

The following proposition regarding characterizations of a sun in Banach spaces is well known, see for example [3, 4, 23]. Recall that, for a subset W and $x \in X$, $w_0 \in W$ is called a local best approximation to x from W if there exists $\delta > 0$ such that $w_0 \in P_{W \cap \mathbf{B}(w_0, \delta)}(x)$.

Proposition 3.2. *Let W be a subset of X. Then the following statements are equivalent.*

(i) *W is a sun in X.*

(ii) *For each x in X and $w_0 \in W$, $w_0 \in P_W(x)$ if and only if*

$$\max_{f \in \sigma(x - w_0)} f(w_0 - w) \geq 0 \quad \text{for each } w \in W.$$

(iii) *For each x in X and $w_0 \in W$, $w_0 \in P_W(x)$ if and only if*

$$\max_{f \in \text{ext}(\sigma(x-w_0))} f(w_0 - w) \geq 0 \quad \textit{for each } w \in W.$$

(iv) *For each x in X and $w_0 \in W$, $w_0 \in P_W(x)$ if and only if w_0 is a local best approximation to x from W.*

For convenience, we define for a bounded subset S of X

(3.9) $M_S = \{f \in \mathbf{B}(X^*) : f(\sup S) = \sup_{s \in S} \|s\| \text{ or } f(\inf S) = \sup_{s \in S} \|s\|\}$

and

$$E_S = M_S \cap \text{ext}\mathbf{B}(X^*).$$

Note that M_S is weak* compact in $\mathbf{B}(X^*)$. Since, by [20, Corollary 1.8, p. 59], $\sigma(x)$ is an extremal subset of $\mathbf{B}(X^*)$, it follows from [20, Lemma 1.7, p. 58] that $\text{ext } \sigma(x) = \sigma(x) \cap \text{ext}\mathbf{B}(X^*)$ for each $x \in X$. Consequently, in view of (3.1), one has that

(3.10) $E_S = \begin{cases} \text{ext } \sigma(\sup S) \cup \text{ext } \sigma(\inf S) & \text{if } \|\sup S\| = \|\inf S\|, \\ \text{ext } \sigma(\sup S) & \text{if } \|\sup S\| > \|\inf S\|, \\ \text{ext } \sigma(\inf S) & \text{if } \|\sup S\| < \|\inf S\|. \end{cases}$

Theorem 3.1 below provides the characterization results of a simultaneous sun in X and of the best simultaneous approximation from a simultaneous sun. To prove this theorem, we need two lemmas, which will also be used in the proof of Theorem 3.2.

Lemma 3.2. *Let $w_0, w \in X$ and S be a bounded subset of X. Then*

(3.11) $\max_{f \in M_{S-w_0}} f(w_0 - w) \geq 0 \iff \max_{f \in E_{S-w_0}} f(w_0 - w) \geq 0.$

Proof. The right-hand side of (3.11) implies clearly the left-hand one of (3.11) since $E_{S-w_0} \subseteq M_{S-w_0}$. To verify the opposite implication, suppose that the left-hand side of (3.11) holds. Then there exists $f \in M_{S-w_0}$ such that $f(w_0-w) \geq 0$. In view of (3.9), $f(\sup S-w_0) = \sup_{s \in S} \|s-w_0\|$ or $f(\inf S-w_0) = \sup_{s \in S} \|s-w_0\|$. Without loss of generality, we may assume that $f(\sup S - w_0) = \sup_{s \in S} \|s - w_0\|$. It follows from (3.1) that $\|\sup S - w_0\| = \sup_{s \in S} \|s - w_0\|$ and

$$f(\sup S - w_0) = \|\sup S - w_0\|.$$

Hence, $f \in \sigma(\sup S - w_0)$ and Krein-Milman Theorem is applicable to concluding that there exists $f' \in \text{ext}\, \sigma(\sup S - w_0)$ such that

$$f'(w_0 - w) = \max\{h(w_0 - w) : h \in \sigma(\sup S - w_0)\} \geq f(w_0 - w) \geq 0.$$

By (3.10), one sees that the right-hand side of (3.11) holds. ∎

Lemma 3.3. *Let W be a subset of X and S a bounded subset of X. Suppose that $w_0 \in W$ and that*

(3.12) $$\max_{f \in M_{S-w_0}} f(w_0 - w) \geq 0 \text{ for each } w \in W.$$

Then $w_0 \in P_W(S)$.

Proof. Let $w \in W$ be arbitrary. Then there is $f \in M_{S-w_0}$ such that $f(w_0 - w) \geq 0$. Without loss of generality, we may assume that $f(\sup S - w_0) = \sup_{s \in S} \|s - w_0\|$. Thus,

$$\sup_{s \in S} \|s - w_0\| = f(\sup S - w) + f(w - w_0) \leq f(\sup S - w) \leq \|\sup S - w\| \leq \sup_{s \in S} \|s - w\|,$$

where the last inequality is because of (3.1). This shows that $w_0 \in P_W(S)$. ∎

Theorem 3.1. *Let W be a subset of X. Then the following statements are equivalent.*

(i) *W is a simultaneous sun in X.*

(ii) *For each bounded set S in X and $w_0 \in W$, $w_0 \in P_W(S)$ if and only if (3.12) holds.*

(iii) *For each bounded set S in X and $w_0 \in W$, $w_0 \in P_W(S)$ if and only if*

(3.13) $$\max_{f \in E_{S-w_0}} f(w_0 - w) \geq 0 \quad \text{for each } w \in W.$$

Proof. (i)$\Longrightarrow$(ii). Suppose that (i) holds. By Lemma 3.3, if (3.12) holds, then $w_0 \in P_W(S)$. Hence we only need to show that (3.12) holds if $w_0 \in P_W(S)$. To do this, suppose on the contrary that (3.12) doesn't hold. Then there exists $w' \in W$ such that

$$\max\{f(w_0 - w') : f \in M_{S-w_0}\} = -\epsilon < 0.$$

Let $U := \{f \in \mathbf{B}(X^*) : f(w_0 - w') < -\frac{\epsilon}{2}\}$. Then U is an open set containing M_{S-w_0}. Set $\hat{S} := \{\sup S, \inf S\}$ and let $\hat{S}_\lambda$ be defined by (3.4) with $\hat{S}$ in place of S, that is,

$$\hat{S}_\lambda = \{w_0 + \lambda(\hat{s} - w_0) : \hat{s} \in \hat{S}\} \quad \text{for each } \lambda > 0.$$

Then, for each $\hat{s}_\lambda = w_0 + \lambda(\hat{s} - w_0) \in \hat{S}_\lambda$,

(3.14)
$$\sup_{f \in U} f(\hat{s}_\lambda - w') = \sup_{f \in U} f(\lambda(\hat{s} - w_0) + w_0 - w')$$
$$\leq \lambda \|\hat{s} - w_0\| + \sup_{f \in U} f(w_0 - w')$$
$$\leq \sup_{s \in \hat{S}_\lambda} \|s - w_0\| - \frac{\epsilon}{2}.$$

On the other hand, since $\mathbf{B}(X^*) \setminus U$ is a compact set disjointing with M_{S-w_0}, there is $\epsilon_1 > 0$ such that

$$\sup_{f \in \mathbf{B}(X^*) \setminus U} f(\hat{s} - w_0) = \sup_{s \in S} \|s - w_0\| - \epsilon_1 \quad \text{for each } \hat{s} \in \hat{S}.$$

Let $\lambda > \frac{\|w_0 - w'\|}{\epsilon_1} + \frac{\epsilon}{2\epsilon_1}$. Then, for each $\hat{s}_\lambda = w_0 + \lambda(\hat{s} - w_0) \in \hat{S}_\lambda$,

(3.15)
$$\sup_{f \in \mathbf{B}(X^*) \setminus U} f(\hat{s}_\lambda - w') = \sup_{f \in \mathbf{B}(X^*) \setminus U} [\lambda f(\hat{s} - w_0) + f(w_0 - w')]$$
$$\leq \lambda(\sup_{s \in S} \|s - w_0\| - \epsilon_1) + \|w_0 - w'\|$$
$$< \sup_{s \in \hat{S}_\lambda} \|s - w_0\| - \frac{\epsilon}{2}$$

because

$$\sup_{s \in \hat{S}_\lambda} \|s - w_0\| = \lambda \sup_{s \in \hat{S}} \|s - w_0\| = \lambda \sup_{s \in S} \|s - w_0\|$$

thanks to (3.1). Combining (3.14) and (3.15) yields that

$$\sup_{s \in \hat{S}_\lambda} \|s - w'\| = \sup_{\hat{s}_\lambda \in \hat{S}_\lambda} \sup_{f \in \mathbf{B}(X^*)} f(s_\lambda - w') \leq \sup_{s \in \hat{S}_\lambda} \|s - w_0\| - \frac{\epsilon}{2}.$$

This means that $w_0 \notin P_W(\hat{S}_\lambda)$. Since W is a simultaneous sun by (i), it follows that $w_0 \notin P_W(\hat{S})$; hence $w_0 \notin P_W(S)$ by Proposition 3.1. Hence the implication that $w_0 \in P_W(S)$ implies (3.12) is proved and completes the proof of $(i) \Longrightarrow (ii)$.

$(ii) \Longrightarrow (iii)$. This results from Lemma 3.2.

$(iii) \Longrightarrow (i)$. Suppose that (iii) holds and S is a bounded set in X. Let $w_0 \in W$ be such that $w_0 \in P_W(S)$. Then (3.13) holds by (iii). Let $\lambda > 0$ be arbitrary. It suffices to verify that $w_0 \in P_W(S_\lambda)$. For this end, note that

$$\sup S_\lambda - w_0 = \lambda(\sup S - w_0) \quad \text{and} \quad \inf S_\lambda - w_0 = \lambda(\inf S - w_0).$$

We have that $E_{S_\lambda - w_0} = E_{S - w_0}$. This together with (3.13) implies that

$$\max_{f \in E_{S_\lambda - w_0}} f(w_0 - w) \geq 0 \quad \text{for each } w \in W.$$

Therefore, $w_0 \in P_W(S_\lambda)$ by (iii). The proof is complete.   ∎

The following theorem gives characterizations of the best simultaneous approximation from a sun.

Theorem 3.2. *Let W be a sun in X and $w_0 \in W$. Let S be a bounded subset of X such that $\| \sup S - w_0 \| \neq \| \inf S - w_0 \|$. Then $w_0 \in P_W(S)$ if and only if (3.12) holds.*

Proof. By Lemma 3.3, we need only verify that $w_0 \in P_W(S)$ implies that (3.12) holds. To do this, we assume that $w_0 \in P_W(S)$ and, without loss of generality, that

(3.16) $$\| \sup S - w_0 \| > \| \inf S - w_0 \|.$$

Then

$$\| \sup S - w_0 \| \leq \max\{\| \sup S - w \|, \| \inf S - w \|\} \quad \text{for each } w \in W.$$

Furthermore, there is $\delta > 0$ such that

$$\| \sup S - w \| > \| \inf S - w \| \quad \text{for each } w \in \mathbf{B}(w_0, \delta).$$

Consequently,

$$\| \sup S - w_0 \| \leq \| \sup S - w \| \quad \text{for each } w \in \mathbf{B}(w_0, \delta) \cap W.$$

This means that w_0 is a local best approximation to $\sup S$ from W. Since W is a sun in X, it follows from Proposition 3.2 that w_0 is a best approximation to $\sup S$ from W and

(3.17) $$\max\{f(w_0 - w) : f \in \sigma(\sup S - w_0)\} \geq 0 \quad \text{for each } w \in W.$$

In view of the definition of M_{S-w_0} and (3.16), one has that

$$M_{S-w_0} = \sigma(\sup S - w_0).$$

This together with (3.17) implies that (3.12) holds.   ∎

The following corollaries are direct consequences of Theorem 3.2.

Corollary 3.1. *Let W be a sun in X and $w_0 \in W$. Let S be a bounded subset of X such that*

(3.18) $$\| \sup S - w_0 \| > \| \inf S - w_0 \|.$$

Then the following statements are equivalent.

   (i) $w_0 \in P_W(S)$.
  (ii) $w_0 \in P_W(\sup S)$.
 (iii) $\max\{f(w_0 - w) : f \in \sigma(\sup S - w_0)\} \geq 0$ *for each $w \in W$.*
 (iv) $\max\{f(w_0 - w) : f \in \mathrm{ext}(\sigma(\sup S - w_0))\} \geq 0$ *for each $w \in W$.*

Corollary 3.2. *Let W be a sun in X and $w_0 \in W$. Let S be a bounded subset of X such that*

(3.19) $\| \sup S - w_0 \| < \| \inf S - w_0 \|.$

Then the following statements are equivalent.

   (i) $w_0 \in P_W(S)$.
  (ii) $w_0 \in P_W(\inf S)$.
 (iii) $\max\{f(w_0 - w) : f \in \sigma(\inf S - w_0)\} \geq 0$ *for each $w \in W$.*
 (iv) $\max\{f(w_0 - w) : f \in \mathrm{ext}(\sigma(\inf S - w_0))\} \geq 0$ *for each $w \in W$.*

The following theorem is a corrected version of Theorem M.

Theorem 3.3. *Let W be a convex subset of X and $w_0 \in W$. Let S be a bounded subset of X such that* (3.18) *(resp.* (3.19)*) holds. Then $w_0 \in P_W(S)$ if and only if there exists $f \in \mathbf{B}(X^*)$ such that $f(\sup S - w_0) = \| \sup S - w_0 \|$ (resp. $f(\inf S - w_0) = \| \inf S - w_0 \|$) and*

(3.20) $f(w_0 - w) \geq 0$ *for each $w \in W$.*

Proof. Without loss of generality, we may assume that (3.18) holds. Since W is a sun in X, Corollary 3.1 is applicable and so $w_0 \in P_W(S) \iff w_0 \in P_W(\sup S)$. It follows from [3, Theorem 2.1, P.6] (see also [23, Theorem 4.3, P.14]) that $w_0 \in P_W(\sup S)$ if and only if there exists $f \in \mathbf{B}(X^*)$ such that $f(\sup S - w_0) = \| \sup S - w_0 \|$ and (3.20) hold. This completes the proof. ∎

Corollary 3.3. *Let W be a convex subset of X and $W_0 \subseteq W$. Let S be a bounded subset of X. Suppose that there exists $w_0 \in P_W(S)$ such that* (3.18) *(resp.* (3.19)*) holds. Then $W_0 \subseteq P_W(S)$ if and only if there exists $f \in \mathbf{B}(X^*)$ such that $f(\sup S - \bar{w}) = \| \sup S - \bar{w} \|$ (resp. $f(\inf S - \bar{w}) = \| \inf S - \bar{w} \|$) for each $\bar{w} \in W_0$ and*

$$f(\bar{w} - w) \geq 0 \quad \text{for each } \bar{w} \in W_0 \text{ and } w \in W.$$

Proof. As in the proof of Theorem 3.3, we may assume that $w_0 \in P_W(S)$ is such that (3.18) holds. By Theorem 3.3, the sufficiency part is clear. Below we

shall verify the necessity part. To do this, let $W_0 \subseteq P_W(S)$ and $\bar{w} \in W_0$. Since $w_0 \in P_W(S)$, it follows from Theorem 3.3 that there exists $f \in \mathbf{B}(X^*)$ such that

$$(3.21) \qquad f(\sup S - w_0) = \|\sup S - w_0\|$$

and (3.20) hold. Then

$$f(\sup S - \bar{w}) = f(\sup S - w_0) + f(w_0 - \bar{w}) \geq f(\sup S - w_0) = \|\sup S - w_0\|.$$

Since $\bar{w} \in P_W(S)$, it follows from (3.18) and Lemma 3.1 that

$$f(\sup S - \bar{w}) \leq \|\sup S - \bar{w}\| \leq \sup_{s \in S} \|s - \bar{w}\| = \sup_{s \in S} \|s - w_0\| = \|\sup S - w_0\|.$$

Hence

$$f(\sup S - \bar{w}) = \|\sup S - \bar{w}\| = \|\sup S - w_0\|.$$

Combining this with (3.21) and (3.20) implies that

$$f(\bar{w} - w) = f(\bar{w} - \sup S) + f(\sup S - w_0) + f(w_0 - w) = f(w_0 - w) \geq 0 \quad \text{for each } w \in W,$$

and completes the proof of the necessity part. ∎

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