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GENERALIZED KKM THEOREMS ON HYPERCONVEX METRIC SPACES WITH APPLICATIONS

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Abstract. In this work, we establish a generalized KKM theorem in a hyperconvex metric space, and then we use this theorem to get a fixed point theorem, the matching theorem, the coincidence theorem, minimax inequality theorems and the variational inequality theorems.

1. Introduction and Preliminaries

In 1929, Knaster, Kurnatoaski and Mazurkiewicz [10] had proved the well-known KKM theorem on n-simplex. In 1961, Ky Fan [5] had generalized the KKM theorem in the infinite dimensional topological vector space. Later, Chang and Yen [3] introduced the family $KKM(X, Y)$, and get some results about fixed point theorems, coincidence theorems, and minimax inequality theorems. And, the notion of hyperconvexity is due to Aronszajn and Panitchpakdi [2] who proved that a hyperconvex space is an absolute retract. In 1996 Khamsi [8] established an analogue of the famous KKM-maps principle due to Ky Fan for hyperconvex metric spaces. Besides, in [9], Kirk et al. proved a KKM theorem for the geheralized KKM mapping on hyperconvex metric space, and get some results about fixed point theorems, Fan-matching theorems and minimax inequality theorems under compact assumptiom.

Recently, Amini, Fakhar and Zafarani introduced the class $KKM(X, Y)$ in metric space [1], and get some results about fixed point theorems and matching theorem. In this work, we use the conception of J. C. Jeng, H. C. Hsu and Y. Y. Huang [7] to define the KKM family on metric space. We establish a generalized KKM theorem in a hyperconvex metric space, and then we use this theorem to

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get a fixed point theorem, the matching theorem, the coincidence theorem, minimax inequality theorems and the variational inequality theorems.

We digress briefly to list some notations and review some definitions. Let X and Y be two Hausdorff topological spaces and $T : X \to 2^Y$ be a set-valued mapping. Then T is said to be closed if its graph $\mathcal{G}_T = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed. If D is a nonempty subset of X, then $\langle D\rangle$ denotes the class of all nonempty finite subset of D.

We shall use the following notations in the sequel.

(i) $T(x) = \{y \in Y : y \in T(x)\}$,

(ii) $T(A) = \cup_{x\in A}T(x)$,

(iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$,

(iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \phi\}$, and

(v) the set-valued mapping $T^c : X \to 2^Y$ is defined by $T^c(x) = Y\backslash T(x)$, for $x \in X$.

A metric space (M, d) is called hyperconvex [2] if for any collection of points $\{x_\alpha\}$ of X and for r_α a collection of non-negative reals such that

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta,$$

then

$$\cap_\alpha B(x_\alpha, r_\alpha) \neq \phi.$$

Here $B(x, r)$ denotes the closed ball with center $x \in X$ and radius $r \geq 0$.

Suppose X is a bounded subset of a metric space (M, d). Then

(i) the admissible hull of X is defined by

$ad(X) = \cap\{B \subset M : B$ is a closed ball in M such that $X \subset B$ $\}$,

(ii) a subset X of M is called admissible if $X = ad(X)$,

$\mathcal{A}(M) = \{X \subset M : X = ad(X)\}$, that is; $X \in \mathcal{A}(M)$ iff X is the intersection of closed balls containing X, and

(iii) if A is a subset of M, then A is said to be subadmissible iff for each $D \subset \langle A\rangle$, $ad(D) \subset A$. We denote

$$\mathcal{SA}(M) = \{X \subset M : X \text{ is a closed subadmissible subset of } X\ \}$$

It is clear that if A_α is subadmissible for each $\alpha \in \Lambda$, then $\cap_{\alpha\in\Lambda}A_\alpha$ is subadmissible. Let B is a subset of M, the smallest subadmissible set containing B is called the subadmissible hull of B and we denote it by $sad(B)$. So $sad(B) = \cap\{A|B \subset A$ and A is subadmissible $\}$.

Remark 1. If A is an admissible subset of a metric space M, then A must be subadmissible. Moreover, every compact subadmissible subset of a hyperconvex metric space is admissible (see [14]).

Definition 1. Let X be a subset of a metric space (M, d), and let Y be a nonempty set. If $F : Y \to 2^X$ is a set-valued mapping satisfying that for each $\{y_1, y_2, ..., y_n\} \in \langle Y \rangle$, there exists $\{x_1, x_2, ..., x_n\} \in \langle X \rangle$ such that $ad\{x_{i_1}, x_{i_2}, ..., x_{i_k}\} \subset \cup_{j=1}^{k} F(y_{i_j})$, for all $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$, then F is called a generalized KKM mapping.

Definition 2. Let X be a metric space, Y be a nonempty set, and Z be a hyperconvex metric space. If $T : X \to 2^Z$, $F : Y \to 2^Z$ are two set-valued mappings satisfying that for each $\{y_1, y_2, ..., y_n\} \in \langle Y \rangle$, there exists $\{x_1, x_2, ..., x_n\} \in \langle X \rangle$ such that $T(ad(\{x_{i_1}, x_{i_2}, ..., x_{i_k}\})) \subset \cup_{j=1}^{k} F(y_{i_j})$, for all $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$, then F is called a generalized KKM mapping with respect to T. If the set-valued mapping $T : X \to 2^Z$ satisfies the requirement that for any generalized KKM mapping $F : Y \to 2^Z$ with respect to T, the family $\{\overline{F(y)} : y \in Y\}$ has the finite intersection property, then T is said to have the KKM property. We denote

$$KKM(X, Z) = \{T : X \to 2^Z | T \text{ has the } KKM \text{ property.}\}$$

Let X and Y be two topological spaces. Then we said that the set-valued mapping $T : X \to 2^Y$ has a continuous selection if there exists a continuous function $f : X \to Y$ such that $f(x) \in T(x)$ for each $x \in X$. And, we denote

$$\mathcal{Q}(X, Y) = \{T : X \to 2^Y | T \text{ has a continuous selection } \}, \quad \text{and}$$
$$\mathcal{C}(X, Y) = \{f : X \to Y | f \text{ is a continuous function } \}$$

Definition 3. Let X be a topological space, and let Y be a hyperconvex metric space. A set-valued mapping $T : X \to 2^Y$ is called a Φ-mapping if there exists a set-valued mapping $F : X \to 2^Y$ such that

(i) for each $x \in X$, $N \in \langle F(x) \rangle$, implies $ad(N) \subset T(x)$, and
(ii) $X = \cup_{y \in Y} int F^{-1}(y)$.

The mapping F is said to be a companion mapping of T.

Remark 2. It is easy to show that if $T : X \to 2^Y$ is a Φ-mapping, then for each nonempty subset X_1 of X, $T|_{X_1} : X_1 \to 2^Y$ is also a Φ-mapping.

2. Main Results

Definition 4. Let M be a metric space. A subset $A \subset M$ is called finitely closed if for every $x_1, x_2, ..., x_n \in M$, the set $ad(\{x_1, x_2, ..., x_n\}) \cap A$ is closed.

The following theorem will plays an important role for our main theorem.

Theorem 1. ([8]). *Let M be a hyperconvex metric space, X an arbitrary subset of M, and $G : X \to 2^M$ a KKM map such that each $G(x)$ is finitely closed. Then the family $\{G(x) : x \in X\}$ has the finite intersection property.*
Note that if in addition Gx_0 is compact for some $x_0 \in X$, then $\cap_{x \in X} Gx \neq \phi$.

By Theorem 1, we have the following KKM theorem:

Theorem 2. *Let M be a hyperconvex metric space. Suppose that $\{x_1, x_2, ...x_n\} \subset M$ and $A_1, A_2, ..., A_n$ be finitely closed subsets of M such that for each $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$, we have $ad\{x_{i_1}, x_{i_2}, ..., x_{i_k}\} \subset \cup_{j=1}^{k} A_{i_j}$, then $\cap_{i=1}^{n} A_i \neq \phi$.*

Proof. Define $G : M \to 2^M$ by

$$G(x_i) = A_i \text{ for each } \quad i = 1, 2, ..., n, \text{ and}$$
$$G(x) = M \quad \text{for all} \quad x \notin \{x_1, x_2, ..., x_n\}$$

Then it is easily to see that G is a KKM map with finitely closed values. Hence, by Theorem 1, the family $\{G(x) : x \in X\}$ has the finite intersection property, and hence $\cap_{i=1}^{n} A_i \neq \phi$.

■

The following KKM theorem concerning generalized KKM map were introduced by W. A. Kirk and B. Sims [9].

Theorem 3. ([9]). *Let Y be a nonempty set, and let X be a hyperconvex metric space. If $F : Y \to 2^X$ is a generalized KKM mapping such that each $F(y)$ is finitely closed, then the family $\{F(y) : y \in Y\}$ has the finite intersection property.*

The following Lemma 1 is the Penot's formulation for admissible sets (see page 406 of "Handbook of metric fixed point theory", Edited by W. A. Kirk, and B. Sims), and it will plays an important role for this paper.

Lemma 1. *Let (X, d) be a hyperconvex metric space, and let $\{A_\alpha\}_{\alpha \in \Lambda} \subset \mathcal{A}(X)$. If for each $\alpha, \beta \in \Lambda$, $A_\alpha \cap A_\beta \neq \phi$, then $\cap_{\alpha \in \Lambda} A_\alpha \neq \phi$.*

Follows from the above Lemma 1 and Theorem 3, we have the following KKM theorem.

Theorem 4. *Let Y be a nonempty set, and let X be a hyperconvex metric space. If $F : Y \to \mathcal{A}(X)$ is a generalized KKM mapping, then $\cap_{y\in Y} F(y) \neq \phi$.*

Follows from the above Lemma 1 and the definition of the KKM property, we have the following generalized KKM theorem.

Theorem 5. *Let X be a metric space, Y be a nonempty set, and let Z be a hyperconvex metric space. If $T \in KKM(X, Z)$ and if $F : Y \to \mathcal{A}(Z)$ is a generalized KKM mapping with respect to T, then $\cap_{y\in Y} F(y) \neq \phi$.*

The following propositions show the relations among the families $\mathcal{Q}(X, Y)$, $KKM(X, Y)$ and Φ-mapping.

It is well known that the continuous functions are generalized KKM mapping on hyperconvex metric spaces, since the identity map is KKM, so when a set-valued maping T has a continuous selection, then T trivially is a generalized KKM. So we immediate have the following Proposition.

Proposition 1. *Let X be a metric space, and let Z be a hyperconvex metric space. If $T \in \mathcal{Q}(X, Z)$, then $T \in KKM(X, Z)$.*

Proposition 2. ([15]). *Let X be a paracompact topological space, and let Y be a hyperconvex metric space. Suppose $F : X \to 2^Y$ is a set-valued mapping satisfying*

(i) *for each $x \in X$, $F(x)$ is subadmissible, and*

(ii) $X = \cup_{y\in Y} intF^{-1}(y)$.

Then F has continuous selection, that is; there exists a continuous function $f : X \to Y$ such taht $f(x) \in F(x)$ for all $x \in X$.

Follow from Proposition 1 and Proposition 2, we are easy to conclude the following proposition.

Proposition 3. *Let X be a paracompact topological space, and let Y be a hyperconvex metric space. Suppose $F : X \to 2^Y$ is a Φ-mapping. Then $F \in \mathcal{Q}(X, Y)$.*

Applying Proposition 1 and Proposition 3, we immediate get the following corollary.

Corollary 1. *Let X be a paracompact topological space, and let Y be a hyperconvex metric space. Suppose $F : X \to 2^Y$ is a Φ-mapping. Then $F \in KKM(X, Y)$.*

As an application of Lemma 1, we have the following Ky-Fan-type matching theorem.

Theorem 6. *Let X be a metric space, Z be a hyperconvex metric space, and let $T \in KKM(X, Z)$. Suppose $\{U_i\}_{i\in I}$ is a family of open subset of Zsuch that $\cup_{i\in I}U_i = Z$ and U_i^c is admissible for each $i \in I$, then for any subset $\{x_i\}_{i\in I}$ of X indexed by the same set I, there esists a finite subset $\{x_1, x_2, ..., x_n\} \subset \{x_i\}_{i\in I}$ such that*

$$T(ad(\{x_1, x_2, ..., x_n\})) \cap \cap_{i=1}^n U_i \neq \phi.$$

Proof. Suppsoe the conclusion is false. Then $T(ad(\{x_1, x_2, ..., x_n\})) \subset \cup_{i=1}^n U_i^c$ for each $\{x_1, x_2, ..., x_n\} \subset \{x_i\}_{i\in I}$. Define a set-valued mapping $F : X \to 2^Z$ by

$$F(x) = \begin{cases} U_i^c & x \in \{x_i\}_{i\in I}, \\ Z & x \notin \{x_i\}_{i\in I} \end{cases}$$

Then for each $\{y_1, y_2, ..., y_m\} \in \langle X\rangle$, we have $T(ad(\{y_1, y_2, ..., y_m\})) \subset \cup_{i=1}^m F(y_i)$. Since $T \in KKM(X, Z)$ and $F(x)$ is closed for each $x \in X$, the family $\{F(x_i) : i \in I\}$ has the finite intersection property, that is; the family $\{U_i^c : i \in I\}$ has the finite intersection property. By Lemma 1, we have $\cap_{i\in I}U_i^c \neq \phi$, which implies $\cup_{i\in I}U_i \neq Z$, so we get a contradiction. ∎

Apply Theorem 6, we get the following coincidence theorem.

Theorem 7. *Let X be a metric space, Z be a hyperconvex metric space, and let $T \in KKM(X, Z)$. Suppose $F : X \to 2^Z$ is a set-valued mapping with open valued such that $F^c(x)$ is admissible for each $x \in X$ and $\cup_{x\in X}F(x) = Z$. If for each $z \in Z$, $F^{-1}(z)$ is subadmissible, then there exists $x_0 \in X$ such that $T(x_0) \cap F(x_0) \neq \phi$.*

Proof. As the same case in Theorem 6, we consider the family $\{F(x) : x \in X\}$. Then all the assumptions of Therorem 6 are satisfied. Hence there exists $\{x_1, x_2, ..., x_n\} \in \langle X\rangle$ such that $T(ad(\{x_1, x_2, ..., x_n\})) \cap \cap_{i=1}^n F(x_i) \neq \phi$. Therefore, there exists $x_0 \in ad(\{x_1, x_2, ..., x_n\})$ and $z_0 \in T(x_0)$ such that $z_0 \in \cap_{i=1}^n F(x_i)$. That is; $z_0 \in F(x_i)$ for each $i = 1, 2, ..., n$, $x_i \in F^{-1}(z_0)$ for each $i = 1, 2, ..., n$. Since $F^{-1}(z_0)$ is subadmissible, hence $x_0 \in ad(\{x_1, x_2, ..., x_n\}) \subset F^{-1}(z_0)$, and so $z_0 \in F(x_0)$. This implies $T(x_0) \cap F(x_0) \neq \phi$. ∎

Remark 3. The assumption $\cup_{x\in X}F(x) = Z$ of Theorem 6 can replaced to be $T(X) \subset F(X)$, and then we can obtain the same conclusion.

If we assume $X = Z$ and $T = i_Z$ of Theorem 7, then we immediate have the following corollary.

Corollary 2. *Let Z be a hyperconvex metric space. Suppose $F : Z \to 2^Z$ is a set-valued mapping with open valued such that $F^c(z)$ is admissible for each $z \in Z$ and $\cup_{z\in Z}F(z) = Z$. If for each $z \in Z$, $F^{-1}(z)$ is subadmissible, then there exists $z_0 \in Z$ such that $z_0 \in F(z_0)$.*

Applying Proposition 3, Corollary 1 and Theorem 7, we also have the following corollary.

Corollary 3. *Let X be a metric space, Z be a hyperconvex metric space, and let $T : X \to 2^Z$ is a Φ-mapping. Suppose $F : X \to 2^Z$ is a set-valued mapping with open valued such that $F^c(x)$ is admissible for each $x \in X$ and $\cup_{x\in X}F(x) = Z$. If for each $z \in Z$, $F^{-1}(z)$ is subadmissible, then there exists $x_0 \in X$ such that $T(x_0) \cap F(x_0) \neq \phi$.*

3. Applications

The following notoin is introduced in [9].

Definition 5. Let X be a nonempty set, and let Y be a metric space. A function $f : X \times Y \to \Re$ is said to be metrically quasi-convex(resp. metrically quasi-concave) in y if for each $x \in X$ and $\lambda \in \Re$, the set $\{y \in Y : f(x, y) \leq \lambda\}$(resp. $\{y \in Y : f(x, y) \geq \lambda\}$) is admissible.

Definition 6. Let X be a metric space, Y be a nonempty set, Z be a hyperconvex metric space, and let $\psi : X \times Z \to \Re$, $\varphi : Y \times Z \to \Re$ be two real-valued functions. ψ is said to be φ-generalized-quasiconcave in x if for any finite subset $B = \{y_1, y_2, ..., y_n\}$ of Y, there exists $A = \{x_1, x_2, ..., x_n\}$ of X such that for each $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$ and for each $x \in ad(\{x_{i_1}, x_{i_2}, ..., x_{i_n}\})$, we have

$$\psi(x, z) \geq \min_{1\leq j\leq k} \varphi(y_i, z) \quad \text{for all} \quad z \in Z.$$

By using the above definition and Theorem 5, we have the following theorem.

Theorem 8. *Let X be a metric space, Y be a nonempty set, and let Z be a hyperconvex metric space If $T \in KKM(X, Y)$ and $\psi : X \times Z \to \Re$, $\varphi : Y \times Z \to \Re$ are two real-valued functions satisfying that*

(i) *for each* $y \in Y$*,* φ *is metrically quasi-convex in* z*, and*

(ii) ψ *is* φ*-generalized-quasiconcave in* x*,*

then for each $\lambda \in \Re$*, one of the following properties holds:*

(1) *there exists* $\overline{z} \in Z$ *such that*

$$\varphi(y, \overline{z}) \leq \lambda \text{ for all } y \in Y;$$

(2) $\psi(x_0, z_0) > \lambda$*, for some* $(x_0, z_0) \in \mathcal{G}_T$*.*

Proof. Define two set-valued mapppings $F : X \to 2^Z$, $S : Y \to 2^Z$by

$$F(x) = \{z \in Z : \psi(x, z) \leq \lambda\} \text{ for each } x \in X, \text{ and}$$
$$S(y) = \{z \in Z : \varphi(y, z) \leq \lambda\} \text{ for each } y \in Y.$$

Suppose the conclusion (2)is false. Then $\psi(x, z) \leq \lambda$, for all $(x, z) \in \mathcal{G}_T$, and so we have $\mathcal{G}_T \subset \mathcal{G}_F$. The assumption (ii) implies that S is a generalized KKM mapping with respect to F. Hence S is a generalized KKM mapping with respect to T. By (i), $S(y)$ is admissible for each $y \in Y$. Since $T \in KKM(X, Y)$, the family $S(y) : y \in Y$ has the finite intersection property. By Theorem 5, $\cap_{y \in Y} S(y) \neq \phi$. Take $\overline{z} \in \cap_{y \in Y} S(y)$. Then $\varphi(y, \overline{z}) \leq \lambda$ for all $y \in Y$. This completes the proof. ∎

Apply Theorem 8, we immediate the following theorem.

Theorem 9. *The conclusion of Theorem 8 implies the inequality*

$$\inf_{z \in Z} \sup_{y \in Y} \varphi(y, z) \leq \sup_{(x,z) \in \mathcal{G}_T} \psi(x, z).$$

Proof. Let $\lambda = \sup_{(x,z) \in \mathcal{G}_T} \psi(x, z)$. Then the conclusion (2) of Theorem 8 is false. So there exists $\overline{z}$ such that $\varphi(y, \overline{z}) \leq \lambda$ for all $y \in Y$. This implies $\sup_{y \in Y} \varphi(y, \overline{z}) \leq \lambda$. So we have $\inf_{z \in Z} \sup_{y \in Y} \varphi(y, z) \leq \sup_{(x,z) \in \mathcal{G}_T} \psi(x, z)$. ∎

Since $\mathcal{C}(X, Z) \subset KKM(X, Z)$, by Theorem 9, we have the following corollary.

Corollary 4. *Let* X *be a metric space,* Y *be a nonempty set, and let* Z *be a hyperconvex metric space If* $s \in C(X, Y)$ *and* $\psi : X \times Z \to \Re$*,* $\varphi : Y \times Z \to \Re$ *are two real-valued functions satisfying that*

(i) *for each* $y \in Y$*,* φ *is metrically quasi-convex in* z*, and*

(ii) ψ *is* φ*-generalized-quasiconcave in* x*,*

then for each $\lambda \in \Re$*, one of the following properties holds:*

(*1*) *there exists* $\overline{z} \in Z$ *such that*

$$\varphi(y, \overline{z}) \le \lambda \ \textit{for all} \ \ y \in Y;$$

(*2*) $\psi(x_0, s(x_0)) > \lambda$, *for some* $x_0 \in X$.

Applying the property of the family $KKM(X, Z)$ and Theorem 8, we have the following generalized variational inequality theorem.

Theorem 10. *Let* X, Y, Z *be three hyperconvex convex spaces, let* $\psi : X \times Z \to \Re$, $\varphi : Y \times Z \to \Re$ *are two real-valued functions satisfying the conditions* (i) *and* (ii) *of Theorem 8, and let* $\lambda \in \Re$. *If for any polytope* Δ *in* X *and for* $f \in \mathcal{C}(Z, \Delta)$, *there exists* $z_0 \in Z$ $($*depends on* $f)$ *such that* $\psi(f(z_0), z_0) \le \lambda$, *then there exists* $\overline{z} \in Z$ *such that*

$$\varphi(y, \overline{z}) \le \lambda, \ \ \text{for all} \ \ y \in Y.$$

Proof. Define two set-valued mapppings $F : X \to 2^Z$, $S : Y \to 2^Z$by

$$F(x) = \{z \in Z : \psi(x, z) \le \lambda\} \ \text{ for each } \ x \in X, \ \text{ and}$$

$$S(y) = \{z \in Z : \varphi(y, z) \le \lambda\} \ \text{ for each } \ y \in Y.$$

By the assumption, for any polytope Δ in X and for $f \in \mathcal{C}(Z, \Delta)$, there exists $z_0 \in Z$ such that $\psi(f(z_0), z_0) \le \lambda$, it follows that $z_0 \in F(f(z_0))$, and so we have $f(z_0) \in f(F(f(z_0)))$. This shows that $fF|_\Delta$ has a fixed point in Δ, and so we have $F \in KKM(X, Z)$. By the definition of F, the conclusion (2) is false. Hence there exists $\overline{z} \in Z$ such that $\varphi(y, \overline{z}) \le \lambda$ for all $y \in Y$. ■

References

1. A. Amini, M. Fakhar and J. Zafarani, KKM mappings in metric spaces, *Nonlinear Analysis.*, **60** (2005), 1045-1052.
2. N. Aronszajn and P. Panitchpakdi, Extensions of uniformly continuous transformations and hyperconvex metric spaces, *Pacific J. Math.*, **6** (1956), 405-439.
3. T. H. Chang and C. L. Yen, KKM property and fixed point theorems, *J. Math. Anal. Appl.*, **203** (1996), 224-235.
4. X. P. Ding, Y. C. Liou and J. C. Yao, Generalized R-KKM type theorems topological spaces with applications, *Appl. Math. Letters*, **18** (2005), 1345-1350.
5. Ky Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.*, **142** (1961), 305-310.
6. Ky Fan, Some properties of convex sets related to fixed point theorem, *Math. Ann.*, **266** (1984), 519-537.

7. J. C. Jeng, H. C. Hsu and Y. Y. Huang, Fixed point theorem for multifunctions having KKM property on almost convex sets, *J. Math. Anal. Appl.*, **319** (2006), 187-198.

8. M. A. Khamsi, KKM and Ky Fan Theorems in Hyperconvex Metric Spaces, *J. Math. Anal. Appl.*, **204** (1996), 298-306.

9. W. A. Kirk, B. Sims and G. X. Z. Yuan, The Knaster-Kurnatoaski and Mazurkiewicz Theory in Hyperconvex Metric Spaces and some of its applications, *Nonlinear Anal.*, **39** (2000), 611-627.

10. B. Knaster, C. Kurnatoaski and S. Mazurkiewicz, Ein Beweis des Fixpunksatzes fur n-dimensionale simplexe, *Fund. Math.*, **14** (1929), 132-137.

11. J. T. Markin, A selection theorem for quasi-lower semicontinuous mappings in hyperconvex metric spaces, *J. Math. Anal. Appl.*, **321** (2006), 862-866.

12. S. Park, Fixed point theorems in hyperconvex metric spaces, *Nonlinear Anal.*, **37** (1999), 467-472.

13. N. T. Vinh, Matching theorems, fiexed point theorems and minimax inequalities in topological ordered spaces, *Acta. Math. Vietnamica*, **30** (2005), 211-224.

14. X. Wu, B. Thompson and G. X. Yuan Fixed point theorems of upper semicontinuous multivalued with applications in Hyperconvex Metric Spaces, *J. Math. Anal. Appl.*, **276** (2002), 80-89.

15. X. Wu, B. Thompson and G. X. Yuan, Approximate selection theorems on H-spaces with applications, *J. Appl. Anal.*, **9** (2003), 249-260.

16. G. X. Z. Yuan, The charactreization of generalized metric KKM mappings with open values in hyperconvex metric spaces and some applications, *J. Math. Anal. Appl.*, **235** (1984), 315-325.

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