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MULTIPLE POSITIVE SOLUTIONS FOR p-LAPLACIAN FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper we consider the following boundary value problems for p-Laplacian functional dynamic equations on time scales

$$\left[\Phi_p(u^{\triangle}(t))\right]^{\nabla} + a(t)f(u(t), u(\mu(t))) = 0, t \in (0,T)_{\mathbf{T}}\,,$$
$$u_0(t) = \varphi(t),\ t \in [-r,0]_{\mathbf{T}}\,,\ u(0) - B_0(u^{\triangle}(\eta)) = 0,\ u^{\triangle}(T) = 0,\ \text{or}$$
$$u_0(t) = \varphi(t),\ t \in [-r,0]_{\mathbf{T}}\,,\ u^{\triangle}(0) = 0, u(T) + B_1(u^{\triangle}(\eta)) = 0.$$

Some existence criteria of at least three positive solutions are established by using the well-known Leggett-Williams fixed-point theorem. An example is also given to illustrate the main results.

1. INTRODUCTION

Let $\mathbf{T}$ be a time scale, i.e., $\mathbf{T}$ is a nonempty closed subset of R. Let 0, T be points in $\mathbf{T}$, an interval $[0,T]_{\mathbf{T}}$ denoting time scales interval, that is, $[0,T]_{\mathbf{T}} := [0,T] \cap \mathbf{T}$. Other types of intervals are defined similarly.

The theory of dynamic equations on time scales has been a new important mathematical branch (see, for example, [1, 2, 9, 10, 17]) since it was initiated by Hilger [16]. At the same time, boundary value problems (BVPs) for dynamic equation on time scales have received considerable attention [3-7, 11-15, 18, 20-25]. However, to the best of our knowledge, there is not much concerning for BVPs of p-Laplacian dynamic equations on time scales [5, 14, 15, 21, 24, 25], especially for p-Laplacian functional dynamic equations on time scales [21].

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For convenience, throughout this paper we denote $\Phi_p(s)$ as the p-Laplacian operator, i.e., $\Phi_p(s) = |s|^{p-2} s$, $p > 1$, $(\Phi_p)^{-1} = \Phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

In [5], Anderson, Avery and Henderson considered the following BVP on time scales

$$\left[\Phi_p(u^{\triangle}(t))\right]^{\nabla} + c(t)f(u) = 0, t \in (a, b)_{\mathbf{T}},$$

$$u(a) - B_0(u^{\triangle}(v)) = 0, \ u^{\triangle}(b) = 0,$$

where $v \in (a, b)_{\mathbf{T}}$, $f \in C_{\mathbf{ld}}([0, +\infty), [0, +\infty))$, $c \in C_{\mathbf{ld}}([a, b], [0, +\infty))$ and $K_m x \leq B_0(x) \leq K_M x$ for some positive constants K_m, K_M. They established the existence result of at least one positive solution by a fixed point theorem of cone expansion and compression of functional type.

In [21], by using a double fixed-point theorem due to Avery et al.[8], Song and Xiao considered the existence of at least twin positive solutions to the following p-Laplacian functional dynamic equations on time scales

$$\left[\Phi_p(u^{\triangle}(t))\right]^{\nabla} + a(t)f(u(t), u(\mu(t))) = 0, \ t \in (0, T)_{\mathbf{T}}, \tag{1.1}$$

satisfying the boundary value conditions

$$u_0(t) = \varphi(t), \ t \in [-r, 0]_{\mathbf{T}}, \ u(0) - B_0(u^{\triangle}(\eta)) = 0, \ u^{\triangle}(T) = 0, \tag{1.2}$$

where $\eta \in (0, \rho(T))_{\mathbf{T}}$.

Very recently, Zhao, Wang and Ge [26] considered the existence of at least three positive solutions to the following p-Laplacian problem

$$\left[\Phi_p(u'(t))\right]' + a(t)f(u, u') = 0, \ t \in [0, 1],$$

$$u'(0) = u(1) = 0.$$

The main tool used in [26] is Leggett-Williams fixed-point theorem.

Motivated by the results mentioned above, in this paper, let $\mathbf{T}$ be a time scale such that $-r$, 0, $T \in \mathbf{T}$, we shall show that the BVP (1.1) with the boundary value conditions (1.2) or boundary value conditions

$$u_0(t) = \varphi(t), \ t \in [-r, 0]_{\mathbf{T}}, \ u^{\triangle}(0) = 0, u(T) + B_1(u^{\triangle}(\eta)) = 0, \tag{1.3}$$

has at least three positive solutions by using Leggett-Williams fixed-point theorem [19].

In this article, we always assume that:

(C_1) $f : [0, +\infty)^2 \to (0, +\infty)$ is continuous ;

(C_2) $a : \mathbf{T} \to (0, +\infty)$ is left dense continuous (i.e., $a \in C_{\mathbf{ld}}(\mathbf{T}, (0, +\infty))$ and dose not vanish identically on any closed subinterval of $[0, T]_{\mathbf{T}}$, where $C_{\mathbf{ld}}(\mathbf{T}, (0, +\infty))$ denotes the set of all left dense continuous functions from $\mathbf{T}$ to $(0, +\infty)$, $\min_{t \in [0,T]_{\mathbf{T}}} a(t) = \Phi_p(m)$, $\max_{t \in [0,T]_{\mathbf{T}}} a(t) = \Phi_p(M)$, and $m < M$;

(C_3) $\varphi : [-r, 0]_{\mathbf{T}} \to [0, +\infty)$ is continuous and $r > 0$;

(C_4) $\mu : [0, T]_{\mathbf{T}} \to [-r, T]_{\mathbf{T}}$ is continuous, $\mu(t) \leq t$ for all t;

(C_5) $B_0(v)$ and $B_1(v)$ are both continuous functions defined on R and satisfy that there exist $B \geq 0$ and $A \geq 1$ such that

$$Bx \leq B_j(x) \leq Ax, \text{ for all } x \geq 0, \ j = 0, 1.$$

In the remainder of this section we list the following well known definitions which can be found in [2, 7, 9, 10].

Definition 1.1. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and the backward jump operator ρ, respectively,

$$\sigma(t) = \inf\{\tau \in \mathbf{T} | \tau > t\} \in \mathbf{T}, \ \rho(r) = \sup\{\tau \in \mathbf{T} | \tau < r\} \in \mathbf{T}$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is sad to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 1.2. For $x : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$, we define the delta derivative of $x(t)$, $x^{\triangle}(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| [x(\sigma(t)) - x(s)] - x^{\triangle}(t) [\sigma(t) - s] \right| < \varepsilon \left| \sigma(t) - s \right|,$$

for all $s \in U$. For $x : \mathbf{T} \to R$ and $t \in \mathbf{T}_k$, we define the nabla derivative of $x(t)$, $x^{\nabla}(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood V of t such that

$$\left| [x(\rho(t)) - x(s)] - x^{\nabla}(t) [\rho(t) - s] \right| < \varepsilon \left| \rho(t) - s \right|,$$

for all $s \in V$.

If $\mathbf{T} = R$, then $x^{\triangle}(t) = x^{\nabla}(t) = x'(t)$. If $\mathbf{T} = Z$, then $x^{\triangle}(t) = x(t+1) - x(t)$ is the forward difference operator while $x^{\nabla}(t) = x(t) - x(t-1)$ is the backward difference operator.

Definition 1.3. If $F^{\triangle}(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\triangle s = F(t) - F(a).$$

If $\Phi^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^t f(s)\nabla s = \Phi(t) - \Phi(a).$$

Throughout this papers, we assume $\mathbf{T}$ is closed subset of $\mathbf{R}$ with $0 \in \mathbf{T}_k$ and $T \in \mathbf{T}^k$.

Lemma 1.1. ([15]). *The following formulas hold:*

(i) $\left(\int_a^t f(s)\triangle s\right)^{\triangle} = f(t),$

(ii) $\left(\int_a^t f(s)\triangle s\right)^{\nabla} = f(\rho(t)),$

(iii) $\left(\int_a^t f(s)\nabla s\right)^{\triangle} = f(\sigma(t)),$

(iv) $\left(\int_a^t f(s)\nabla s\right)^{\nabla} = f(t).$

2. Preliminaries

In this section, we provide some background materials from the theory of cones in Banach spaces and we then state the Leggett-Williams fixed-point theorem.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:

(i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;
(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2.2. Let E be a real Banach space and P$\subset E$ be a cone. A function $\alpha : P \to [0, \infty)$ is called a nonnegative continuous concave functional if α is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x,\, y \in P$ and $t \in [0,1]$.

Let $a, b, c > 0$ be constants, $P_c = \{x \in P : \|x\| < c\}$, $P(\alpha, a, b) = \{x \in P : a \leq \alpha(x), \|x\| \leq b\}$.

To prove our main results, we need the following theorem [19].

Theorem 2.1. (Leggett-Williams). *Let $A : \overline{P}_c \to \overline{P}_c$ be a completely continuous map and α be a nonnegative continuous concave functional on P such that $\alpha(x) \leq \|x\|$, $\forall x \in \overline{P}_c$. Suppose there exist a, b, d with $0 < a < b < d \leq c$, such that:*

(i) $\{x \in P(\alpha, b, d) : \alpha(x) > b\} \neq \phi$ *and* $\alpha(Ax) > b$ *for all* $x \in P(\alpha, b, d)$;

(ii) $\|Ax\| < a$ *for all* $x \in \overline{P}_a$;

(iii) $\alpha(Ax) > b$, *for all* $x \in P(\alpha, b, c)$ *with* $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2, x_3 satisfying

$$\|x_1\| < a,\ b < \alpha(x_2),\ \|x_3\| > a \text{ and } \alpha(x_3) < b.$$

3. Positive Solutions of the BVP (1.1), (1.2)

In this section we consider the existence of three positive solutions for the BVP (1.1), (1.2).

We say u is concave on $[0,T]_{\mathbf{T}}$ if $u^{\triangle\nabla}(t) \leq 0$ for $t \in [0,T]_{\mathbf{T}^k \cap \mathbf{T}_k}$.

We note that $u(t)$ is a solution of the BVP (1.1), (1.2) if and only if

$$u(t) = \begin{cases} B_0\left(\Phi_q\left(\int_\eta^T a(r) f(u(r), u(\mu(r)))\nabla r\right)\right) \\ \quad + \int_0^t \Phi_q\left(\int_s^T a(r) f(u(r), u(\mu(r)))\nabla r\right)\triangle s, & t \in [0,T]_{\mathbf{T}}, \\ \varphi(t), & t \in [-r,0]_{\mathbf{T}}. \end{cases}$$

Let $E = C^{\triangle}_{\mathbf{ld}}([0,T]_{\mathbf{T}}, R)$ with $\|u\| = \max\left\{\max_{t\in[0,T]_{\mathbf{T}}} |u(t)|,\ \max_{t\in[0,T]_{\mathbf{T}^k}} \left|u^{\triangle}(t)\right|\right\}$, $P = \{u \in E :\ u$ is nonnegative, increasing and concave on $[0,T]_{\mathbf{T}}\}$. So E is a Banach space with the norm $\|u\|$ and P is a cone in E. For each $u \in E$, extend $u(t)$ to $[-r,T]_{\mathbf{T}}$ with $u(t) = \varphi(t)$ for $t \in [-r,0]_{\mathbf{T}}$.

Define $F : P \to E$ by

$$\begin{aligned}(Fu)(t) = \ & B_0\left(\Phi_q\left(\int_\eta^T a(r) f(u(r), u(\mu(r)))\nabla r\right)\right) \\ & + \int_0^t \Phi_q\left(\int_s^T a(r) f(u(r), u(\mu(r)))\nabla r\right)\triangle s, t \in [0,T]_{\mathbf{T}}.\end{aligned}$$

It is well known that this operator F is completely continuous.

We seek a fixed point, u_1, of F in the cone P. Define

$$u(t) = \begin{cases} \varphi(t), & t \in [-r, 0]_{\mathbf{T}}, \\ u_1(t), & t \in [0, T]_{\mathbf{T}}. \end{cases}$$

Then $u(t)$ denotes a positive solution of the BVP (1.1), (1.2).

Lemma 3.1. $F : P \to P$.

Proof. The proof of the lemma is similar to that of [25, Lemma 3.1]. For the sake of convenience, we list it here.

$\forall u \in P$, $Fu \in E$ and $(Fu)(t) \geq 0$, $\forall t \in [0, T]_{\mathbf{T}}$. It follows from Lemma 1.1 we have

$$(Fu)^{\triangle}(t) = \Phi_q\left(\int_t^T a(r) f(u(r), u(\mu(r)))\nabla r\right).$$

Obviously $(Fu)^{\triangle}(t)$ is a continuous function and $(Fu)^{\triangle}(t) \geq 0$, that is $(Fu)(t)$ is increasing on $[0, T]_{\mathbf{T}}$. Note that Φ_q is increasing, we have that $(Fu)^{\triangle}(t)$ is decreasing.

If $t \in [0, T]_{\mathbf{T}^k \cap \mathbf{T}_k}$, then from [7, Theorem 2.3] it follows that $(Fu)^{\triangle\nabla}(t) \leq 0$, i.e., Fu is concave on $[0, T]_{\mathbf{T}}$. This implies that $Fu \in P$ and $F : P \to P$.

Let $l \in \mathbf{T}$ be fixed such that $0 < \eta < l < T$, and set

$$Y_1 = \{t \in [0, T]_{\mathbf{T}} : \mu(t) \leq 0\};\ Y_2 = \{t \in [0, T]_{\mathbf{T}} : \mu(t) > 0\};\ Y_3 = Y_1 \cap [\eta, T]_{\mathbf{T}}.$$

Throughout this section, we assume $Y_3 \neq \phi$ and $\int_{Y_3} a(r)\nabla r > 0$.

Now we define the nonnegative continuous concave functional $\alpha : P \to [0, \infty)$ by

$$\alpha(u) = \min_{t \in [\eta, l]_{\mathbf{T}}} u(t),\ \forall u \in P.$$

It is easy to see that $\alpha(u) = u(\eta) \leq \max_{t \in [0,T]_{\mathbf{T}}} |u(t)| \leq \|u\|$ if $u \in P$ and $\alpha(Fu) = (Fu)(\eta)$.

For convenience, we denote

$$\rho = (A + T)\Phi_q\left(\int_0^T a(r)\nabla r\right),\ \delta = (B + \eta)\Phi_q\left(\int_{Y_3} a(r)\nabla r\right).$$

We now state growth conditions on f so that the BVP (1.1), (1.2) has at least three positive solutions.

Theorem 3.1. *Let* $0 < a < b \leq \frac{m(B+\eta)}{M(A+T)} d < d \leq c$, *and suppose that* f *satisfies the following conditions:*

(H_1) $f(x,\varphi(s)) < \Phi_p(\frac{a}{\rho})$, *for all* $0 \le x \le a$, *uniformly in* $s \in [-r,0]_{\mathbf{T}}$;
$f(x_1,x_2) < \Phi_p(\frac{a}{\rho})$, *for all* $0 \le x_i \le a$, $i = 1,2$,

(H_2) $f(x,\varphi(s)) \le \Phi_p(\frac{c}{\rho})$, *for all* $0 \le x \le c$, *uniformly in* $s \in [-r,0]_{\mathbf{T}}$;
$f(x_1,x_2) \le \Phi_p(\frac{c}{\rho})$, *for all* $0 \le x_i \le c$, $i = 1,2$,

(H_3) $f(x,\varphi(s)) > \Phi_p(\frac{b}{\delta})$, *for all* $b \le x \le d$, *uniformly in* $s \in [-r,0]_{\mathbf{T}}$,

(H_4) $\min_{x\in[0,c]} f(x,\varphi(s)) \cdot \Phi_p\left(\frac{M}{m}\right)\int_{Y_3} a(r)\nabla r \ge \max_{x_1,x_2\in[0,c]} f(x_1,x_2) \cdot \int_0^T a(r)\nabla r$, *uniformly in* $s \in [-r,0]_{\mathbf{T}}$.

Then the BVP (1.1), (1.2) has at least three positive solutions of the form

$$u(t) = \begin{cases} \varphi(t), & t \in [-r,0]_{\mathbf{T}}, \\ u_i(t), & t \in [0,T]_{\mathbf{T}}, \quad i = 1,2,3, \end{cases}$$

where $\|u_1\| < a$, $b < \alpha(u_2)$, $\|u_3\| > a$ and $\alpha(u_3) < b$.

Proof. We first assert that $F : \overline{P}_c \to \overline{P}_c$.

Indeed, if $u \in \overline{P}_c$, then, in view of lemma 3.1, we have $F\overline{P}_c \subset P$. Furthermore, $\forall u \in \overline{P}_c$, we have $0 \le u \le c$, and then from (H$_2$), we have

$$\begin{aligned}
|Fu(t)| &= \left| B_0\left(\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\right) \right. \\
&\quad \left. + \int_0^t \Phi_q\left(\int_s^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\triangle s \right| \\
&\le A\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right) \\
&\quad + T\Phi_q\left(\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r\right) \\
&\le (A+T)\Phi_q\left(\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r\right) \\
&= (A+T)\Phi_q\left(\int_{Y_1} a(r)f(u(r),\varphi(\mu(r)))\nabla r \right. \\
&\quad \left. + \int_{Y_2} a(r)f(u(r),u(\mu(r)))\nabla r\right) \\
&\le (A+T)\Phi_q\left(\int_0^T a(r)\nabla r\right)\frac{c}{\rho} \\
&= c,
\end{aligned}$$

$$\begin{aligned}
\left|(Fu)^{\triangle}(t)\right| &= \left|\Phi_q\left(\int_t^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\right| \\
&\le \Phi_q\left(\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r\right) \\
&= \Phi_q\left(\int_{Y_1} a(r)f(u(r),\varphi(\mu(r)))\nabla r\right. \\
&\quad \left. + \int_{Y_2} a(r)f(u(r),u(\mu(r)))\nabla r\right) \\
&\le \Phi_q\left(\int_0^T a(r)\nabla r\right)\frac{c}{\rho} \\
&= \frac{c}{A+T} \\
&\le c.
\end{aligned}$$

Therefore, $\|Fu\| \le c$, i.e., $F : \overline{P}_c \to \overline{P}_c$.

By (H_1) and in a way similar to above, we arrive that $F : \overline{P}_a \to P_a$.

Next, we assert that $\{u \in P(\alpha, b, d) : \alpha(u) > b\} \neq \phi$ and $\alpha(Au) > b$ for all $u \in P(\alpha, b, d)$.

Let $u = \frac{b+d}{2}$, then $u \in P$, $\|u\| = \frac{b+d}{2} \le d$ and $\alpha(u) = \frac{b+d}{2} > b$. That is, $\{u \in P(\alpha, b, d) : \alpha(u) > b\} \neq \phi$.

Moreover, $\forall u \in P(\alpha, b, d)$, we have $b \le u(t) \le d$, $t \in [\eta, T]_{\mathbf{T}}$, then from (H_3), we see that

$$\begin{aligned}
\alpha(Fu) &= (Fu)(\eta) \\
&= B_0\left(\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\right) \\
&\quad + \int_0^\eta \Phi_q\left(\int_s^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\triangle s \\
&\ge B\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right) \\
&\quad +\eta\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right) \\
&\ge (B+\eta)\Phi_q\left(\int_{Y_3} a(r)f(u(r),\varphi(\mu(r)))\nabla r\right) \\
&> (B+\eta)\Phi_q\left(\int_{Y_3} a(r)\nabla r\right)\frac{b}{\delta} \\
&= b,
\end{aligned}$$

as required.

Finally, we assert that $\alpha(Fu) > b$, for all $u \in P(\alpha, b, c)$ and $\|Fu\| > d$.

To see this, $\forall u \in P(\alpha, b, c)$ and $\|Fu\| > d$, then $0 \le u(t) \le c$, $t \in [0, T]_{\mathbf{T}}$, then from (H$_4$), we have

$$\Phi_p\left(\frac{M}{m}\right)\int_{Y_3} a(r)f(u(r),\varphi(\mu(r)))\nabla r \ge \int_0^T a(r)f(u(r),u(\mu(r)))\nabla r,$$

i.e.

$$\int_{Y_3} a(r)f(u(r),\varphi(\mu(r)))\nabla r \ge \frac{\displaystyle\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r}{\Phi_p\left(\frac{M}{m}\right)}$$

holds.

So,

$$\begin{aligned}
\alpha(Fu) &= (Fu)(\eta)\\
&= B_0\left(\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\right)\\
&\quad +\int_0^\eta \Phi_q\left(\int_s^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\triangle s\\
&\ge B\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\\
&\quad +\eta\Phi_q\left(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\\
&\ge (B+\eta)\Phi_q\left(\int_{Y_3} a(r)f(u(r),\varphi(\mu(r)))\nabla r\right)\\
&\ge (B+\eta)\Phi_q\left(\frac{\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r}{\Phi_p\left(\frac{M}{m}\right)}\right)\\
&= \frac{m\,(B+\eta)}{M}\Phi_q\left(\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\\
&= \frac{m\,(B+\eta)}{M\,(A+T)}\,(A+T)\,\Phi_q\left(\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r\right)\\
&\ge \frac{m\,(B+\eta)}{M\,(A+T)}\,\|Fu\|\\
&> \frac{m\,(B+\eta)}{M\,(A+T)}d\\
&\ge b.
\end{aligned}$$

To sum up, all the hypotheses of Theorem 2.1 are satisfied. Hence F has at least three fixed points, i.e., the BVP (1.1), (1.2) has at least three positive solutions of the form

$$u(t) = \begin{cases} \varphi(t), & t \in [-r, 0]_{\mathbf{T}}, \\ u_i(t), & t \in [0, T]_{\mathbf{T}}, \quad i = 1, 2, 3, \end{cases}$$

where $\|u_1\| < a$, $b < \alpha(u_2)$, $\|u_3\| > a$ and $\alpha(u_3) < b$.

4. Positive Solutions of the BVP (1.1), (1.3)

In this section we deal with the BVP (1.1), (1.3) .

We note that $u(t)$ is a solution of the BVP (1.1), (1.3) if and only if

$$u(t) = \begin{cases} B_1\left(\Phi_q\left(\int_0^\eta a(r)f(u(r), u(\mu(r)))\nabla r\right)\right) & \\ \quad + \int_t^T \Phi_q\left(\int_0^s a(r)f(u(r), u(\mu(r)))\nabla r\right)\triangle s, & t \in [0, T]_{\mathbf{T}}, \\ \varphi(t), & t \in [-r, 0]_{\mathbf{T}}. \end{cases}$$

Let $E = C_{\mathbf{ld}}^{\triangle}([0, T]_{\mathbf{T}}, R)$ with $\|u\| = \max\left\{\max_{t\in[0,T]_{\mathbf{T}}} |u(t)|,\ \max_{t\in[0,T]_{\mathbf{T}^k}} \left|u^{\triangle}(t)\right|\right\}$, $P_1 = \{u \in E :\ u$ is nonnegative, decreasing and concave on $[0, T]_{\mathbf{T}}\}$. So E is a Banach space with the norm $\|u\|$ and P_1 is a cone in E. For each $u \in E$, extend $u(t)$ to $[-r, T]_{\mathbf{T}}$ with $u(t) = \varphi(t)$ for $t \in [-r, 0]_{\mathbf{T}}$.

Define completely continuous operator $G : P_1 \to E$ by

$$\begin{aligned}(Gu)(t) = \ & B_1\left(\Phi_q\left(\int_0^\eta a(r)f(u(r), u(\mu(r)))\nabla r\right)\right) \\ & + \int_t^T \Phi_q\left(\int_0^s a(r)f(u(r), u(\mu(r)))\nabla r\right)\triangle s,\ t \in [0, T]_{\mathbf{T}}.\end{aligned}$$

We seek a fixed point, u_1, of G in the cone P_1. Define

$$u(t) = \begin{cases} \varphi(t), & t \in [-r, 0]_{\mathbf{T}}, \\ u_1(t), & t \in [0, T]_{\mathbf{T}}. \end{cases}$$

Then $u(t)$ denotes a positive solution of the BVP (1.1), (1.3).

Lemma 4.1. $G : P_1 \to P_1$.

Proof. The proof is similar to Lemma 3.1, so we omit here.

Let $l \in \mathbf{T}$ be fixed such that $0 < \eta < l < T$, and set

$$Y_1 = \{t \in [0, T]_{\mathbf{T}} : \mu(t) \le 0\};\ Y_2 = \{t \in [0, T]_{\mathbf{T}} : \mu(t) > 0\};\ Y_3 = Y_1 \cap [0, \eta]_{\mathbf{T}}.$$

Throughout this section, we assume $Y_3 \neq \phi$ and $\int_{Y_3} a(r)\nabla r > 0$.

Define the nonnegative continuous concave functional $\alpha : P_1 \to [0, \infty)$ by

$$\alpha(u) = \min_{t \in [\eta, l]_{\mathbf{T}}} u(t), \ \forall u \in P_1.$$

It is easy to see that $\alpha(u) = u(l) \leq \max_{t\in[0,T]_{\mathbf{T}}} |u(t)| \leq \|u\|$ if $u \in P$ and $\alpha(Fu) = (Fu)(l)$.

Let ρ remains unchanged and we denotes

$$\delta_* = (B + T - l)\Phi_q\left(\int_{Y_3} a(r)\nabla r\right).$$

Similarly to Theorem 3.1, we have

Theorem 4.1. *Let* $0 < a < b \leq \frac{m(B+T-l)}{M(A+T)}d < d \leq c$, *and suppose that* f *satisfies the following conditions:*

(H_1) $f(x, \varphi(s)) < \Phi_p(\frac{a}{\rho})$, *for all* $0 \leq x \leq a$, *uniformly in* $s \in [-r, 0]_{\mathbf{T}}$;
$f(x_1, x_2) < \Phi_p(\frac{a}{\rho})$, *for all* $0 \leq x_i \leq a$, $i = 1, 2$,

(H_2) $f(x, \varphi(s)) \leq \Phi_p(\frac{c}{\rho})$, *for all* $0 \leq x \leq c$, *uniformly in* $s \in [-r, 0]_{\mathbf{T}}$;
$f(x_1, x_2) \leq \Phi_p(\frac{c}{\rho})$, *for all* $0 \leq x_i \leq c$, $i = 1, 2$,

(H_3) $f(x, \varphi(s)) > \Phi_p(\frac{b}{\delta_*})$, *for all* $b \leq x \leq d$, *uniformly in* $s \in [-r, 0]_{\mathbf{T}}$,

(H_4) $\min_{x\in[0,c]} f(x, \varphi(s))\cdot\Phi_p\left(\frac{M}{m}\right)\int_{Y_3} a(r)\nabla r \geq \max_{x_1,x_2\in[0,c]} f(x_1, x_2)\cdot\int_0^T a(r)\nabla r$, *uniformly in* $s \in [-r, 0]_{\mathbf{T}}$.

Then the BVP (1.1), (1.3) *has at least three positive solutions of the form*

$$u(t) = \begin{cases} \varphi(t), & t \in [-r, 0]_{\mathbf{T}}, \\ u_i(t), & t \in [0, T]_{\mathbf{T}}, \quad i = 1, 2, 3, \end{cases}$$

where $\|u_1\| < a$, $b < \alpha(u_2)$, $\|u_3\| > a$ *and* $\alpha(u_3) < b$.

5. Example

Let $\mathbf{T} = \left[-\frac{3}{4}, -\frac{1}{4}\right] \cup \left\{0, \ \frac{3}{4}\right\} \cup \left\{\left(\frac{1}{2}\right)^{\mathbb{N}_0}\right\}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers.

Consider the following p-Laplacian functional dynamic equation on time scale $\mathbf{T}$

$$(5.1) \quad \begin{cases} \left[\Phi_p(u^{\triangle}(t))\right]^{\nabla} + a(t)\left[\dfrac{8u^3(t)}{u^3(t)+u^3(t-\dfrac{3}{4}) + 1} + \frac{1}{5}\right] = 0, \ t \in (0, 1)_{\mathbf{T}}, \\ u_0(t) = \varphi(t) \equiv 0, \ t \in \left[-\frac{3}{4}, 0\right]_{\mathbf{T}}, \ u(0) - B_0(u^{\triangle}(\frac{1}{4})) = 0, u^{\triangle}(1) = 0, \end{cases}$$

where $T = 1$, $p = \frac{3}{2}$, $B = \frac{1}{2}$, $A = 2$, $\mu : [0,1]_{\mathbf{T}} \to \left[-\frac{3}{4}, 1\right]_{\mathbf{T}}$ and $\mu(t) = t - \frac{3}{4}$, $r = \frac{3}{4}$, $\eta = \frac{1}{4}$, $l = \frac{1}{2}$, $f(u, \varphi(s)) = \frac{8u^3}{u^3+1} + \frac{1}{5}$, $f(u_1, u_2) = \frac{8u_1^3}{u_1^3+u_2^3+1} + \frac{1}{5}$ and

$$a(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}]_{\mathbf{T}}, \\ -\frac{99}{50}t + \frac{199}{100}, & t \in [\frac{1}{2}, 1]_{\mathbf{T}}. \end{cases}$$

We deduce that $Y_1 = \left[0, \frac{3}{4}\right]_{\mathbf{T}}$, $Y_2 = \left(\frac{3}{4}, 1\right]_{\mathbf{T}}$, $Y_3 = \left[\frac{1}{4}, \frac{3}{4}\right]_{\mathbf{T}}$. Then by [7, Theorem 2.8] we have $\int_{Y_3} a(r)\nabla r = \int_{\frac{1}{4}}^{\frac{3}{4}} a(r)\nabla r = \frac{301}{800}$, $\int_0^T a(r)\nabla r = \int_0^1 a(r)\nabla r = \frac{503}{800}$.

Thus it is easy to see by calculation that $\rho = 3\left(\frac{503}{800}\right)^2$, $\delta = \frac{3}{4}\left(\frac{301}{800}\right)^2$.

Choose $a = \frac{1}{10}$, $b = 1$, $d = 42000, c = 45000$ then by $M = 1, m = \frac{1}{10000}$ we have $0 < a < b < \frac{m(B+\eta)}{M(A+T)}d < d < c$, then

$f(u, \varphi(s)) \leq \frac{8}{1001} + \frac{1}{5} \approx 0.2080 < \Phi_p(\frac{a}{\rho}) = \sqrt{\frac{\frac{1}{10}}{3\left(\frac{503}{800}\right)^2}} \approx 0.2904$, $0 \leq u \leq \frac{1}{10}$, uniformly in $s \in \left[-\frac{3}{4}, 0\right]_{\mathbf{T}}$;

$f(u_1, u_2) \leq \frac{8}{1002} + \frac{1}{5} \approx 0.2080 < \Phi_p(\frac{a}{\rho}) = \sqrt{\frac{\frac{1}{10}}{3\left(\frac{503}{800}\right)^2}} \approx 0.2904, 0 \leq u_i \leq \frac{1}{10}, i = 1, 2,$

$f(u, \varphi(s)) < 8.2 < \Phi_p(\frac{c}{\rho}) = \sqrt{\frac{45000}{3\left(\frac{503}{800}\right)^2}} \approx 195$, $0 \leq u \leq 45000$, uniformly in $s \in \left[-\frac{3}{4}, 0\right]_{\mathbf{T}}$;

$f(u_1, u_2) < 8.2 < \Phi_p(\frac{c}{\rho}) = \sqrt{\frac{45000}{3\left(\frac{503}{800}\right)^2}} \approx 195, 0 \leq u_i \leq 45000, i = 1, 2,$

$f(u, \varphi(s)) \geq 4.2 > \Phi_p(\frac{b}{\delta}) = \sqrt{\frac{1}{\frac{3}{4}\left(\frac{301}{800}\right)^2}} \approx 3.0690$, $1 \leq u \leq 42000$, uniformly in $s \in \left[-\frac{3}{4}, 0\right]_{\mathbf{T}}$

$\min_{u\in[0,c]} f(u, \varphi(s)) \cdot \Phi_p\left(\frac{M}{m}\right) \int_{Y_3} a(r)\nabla r = 7.5250 > 5.1558 \approx \frac{41}{5} \cdot \frac{503}{800} > \max_{u_i\in[0,c]} f(u_1, u_2) \cdot \int_0^T a(r)\nabla r$, uniformly in $s \in \left[-\frac{3}{4}, 0\right]_{\mathbf{T}}$.

Thus by Theorem 3.1, the BVP (5.1) has at least three positive solutions of the form

$$u(t) = \begin{cases} u_i(t), & t \in [0, T], \ i = 1, 2, 3, \\ \varphi(t), & t \in [-r, 0], \end{cases}$$

where $\|u_1\| < \frac{1}{10}$, $1 < \alpha(u_2)$, $\|u_3\| > \frac{1}{10}$ and $\alpha(u_3) < 1$.

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