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SENSITIVITY ANALYSIS OF SOLUTION MAPPINGS OF PARAMETRIC GENERALIZED QUASI VECTOR EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we study the parametric generalized quasi vector equilibrium problem (PGQVEP). We investigate existence of solution for PGQVEP and continuities of the solution mappings of PGQVEP. In particular, resulta concerning the lower semicontinuity of the solution mapping of PGQVEP are presented.

1. Introduction

Let X be nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector space. A set $\mathcal{C} \subset \mathbb{Z}$ is said to be a cone if $\lambda x \in \mathcal{C}$ for any $\lambda \geq 0$ and for any $x \in \mathcal{C}$. The cone $\mathcal{C}$ is called proper if it is not whole space, i.e., $\mathcal{C} \neq \mathbb{Z}$. A cone $\mathcal{C}$ is said to be solid if it has nonempty interior, i.e., $\operatorname{int} \mathcal{C} \neq \emptyset$. Let $C : \mathcal{X} \to 2^{\mathbb{Z}}$ which has proper convex cone values. For any set $A \subset \mathbb{Z}$, we let $\operatorname{bd} A$ and $\operatorname{cl} A$ denote the boundary and closure of A, respectively. Also, we denote A^{c} the complement of the set A. For any set A of a real vector space, the convex hull of A, denoted by $\operatorname{co} A$, is the smallest convex set containing A. Furthermore, we denote zero vector of $\mathbb{Z}$ by $\theta_{\mathbb{Z}}$.

Let $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ and $K : \mathbb{P} \times X \to 2^{X} \setminus \{\emptyset\}$. For fixed $p \in \mathbb{P}$, the parametric generalized quasi vector equilibrium problem (PGQVEP) is to find $x \in K(p, x)$ such that

$$\text{(PGQVEP)} \qquad F(p, x, y) \not\subset -\operatorname{int} C(p, x), \text{ for all } y \in K(p, x).$$

Let $\Omega : \mathbb{P} \to 2^{X}$ be the set-valued mapping such that $\Omega(p)$ is the solutions set of

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PGQVEP for $p \in \mathbb{P}$, i.e.,

$$\Omega(p) = \{x \in K(p,x) : F(p,x,y) \not\subset -\mathrm{int}\, C(p,x), \text{ for all } y \in K(p,x)\}.$$

For fixed $p \in \mathbb{P}$, the parametric extended quasi vector equilibrium problem (PEQVEP) is to find $x \in K(p,x)$ such that

$$\text{(PEQVEP)} \qquad F(p,x,y) \cap (-\mathrm{int}\, C(p,x)) = \emptyset, \text{ for all } y \in K(p,x).$$

Let $\Phi : \mathbb{P} \to 2^X$ be the set-valued mapping such that $\Phi(p)$ is the solutions set of PEQVEP for $p \in \mathbb{P}$, i.e.,

$$\Phi(p) = \{x \in K(p,x) : F(p,x,y) \cap (-\mathrm{int}\, C(p,x)) = \emptyset, \ \forall y \in K(p,x)\}.$$

In the literature, existence results for a (generalized) vector quasi equilibrium problems has been investigated. See, e.g., [3, 9]. If for each fixed $p \in \mathbb{P}$, K and C have constant values for every $x \in X$, respectively, PGQVEP reduce to a parametric vector equilibrium problem (PVEP). Existence of solution and closedness of solution mapping for PVEP has been studied in [6]. Continuity of solution mapping for PVEP has been studied in [7].

We observe that our results in this paper can be employed to study the behavior of solution maps of parametric vector optimization, parametric vector variational inequalities, parametric vector equilibrium problems and those generalized problems and so on.

2. Preliminaries

Definition 2.1. (C-continuity, [8]). Let X be a topological space and $\mathbb{Z}$ a topological vector space with a partial ordering defined by a proper solid convex cone C. Suppose that f is a vector-valued function from X to $\mathbb{Z}$. Then, f is said to be *C-continuous at* $x \in X$, if for any neighbourhood $V_{f(x)} \subset \mathbb{Z}$ of $f(x)$, there exists a neighbourhood $U_x \subset X$ of x such that $f(u) \in V_{f(x)} + C$ for all $u \in U_x$. Moreover a vector-valued function f is said to be *C-continuous in* X if f is C-continuous at every x on X.

Definition 2.2. (Continuity for Set-valued mapping, See also [1]). Let X and Y be two topological spaces, $T : X \to 2^Y$ a set-valued mapping.

(i) T is said to be *upper semicontinuous* (u.s.c. for short) at $x \in X$ if for each open set V containing $T(x)$, there is an open set U containing x such that for each $z \in U$, $T(z) \subset V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(ii) T is said to be *lower semicontinuous* (l.s.c. for short) at $x \in X$ if for each open set V with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $z \in U$, $T(z) \cap V \neq \emptyset$; T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iii) T is said to be *continuous* at $x \in X$ if $T(x)$ is both u.s.c. and l.s.c.; T is said to be *continuous* on X if it is both u.s.c. and l.s.c. at each $x \in X$.

Proposition 2.1. *Let X be a topological space and $\mathbb{Z}$ a real topological vector space. Suppose that $C : X \to 2^{\mathbb{Z}}$ has proper solid convex cone values and that $W : X \to 2^{\mathbb{Z}}$ is defined by $Z \setminus (-\operatorname{int} C(x))$. Then we have the following two statements:*

(i) *if C is u.s.c. at x, then there exists a neighborhood U of x such that*

$$\operatorname{cl} C(x) \supset C(u), \text{ for all } u \in U;$$

(ii) *if W is u.s.c. at x, then there exists a neighborhood U of x such that*

$$W(x) \supset W(u), \text{ for all } u \in U.$$

Proof. First we prove (i). Let $x \in X$ and C is u.s.c. at x. Suppose $c \in \operatorname{int} C(x)$. Then $-c + \operatorname{int} C(x)$ is a neighborhood of $C(x)$. Since C is u.s.c. at x, there exists a neighborhood U of x such that

$$C(u) \subset -c + \operatorname{int} C(x), \text{ for all } u \in U.$$

Suppose $\operatorname{cl} C(x) \not\supset C(v)$ for some $v \in X$. Then there exist $z \in (\operatorname{cl} C(x))^{\mathrm{c}} \cap C(v)$ and a positive number $t > 0$ such that $z + tc \notin \operatorname{cl} C(x)$. Hence $\frac{1}{t} z \notin -c + \operatorname{cl} C(x)$, i.e., $\frac{1}{t} z \notin -c + \operatorname{int} C(x)$. Thus $\frac{1}{t} z \notin C(u)$ for all $u \in U$. Since $C(v)$ is cone, $\frac{1}{t} z \in C(v)$. Therefore $v \notin U$. Accordingly we have statement (i).

Second we prove (ii). Let $x \in X$ and W is u.s.c. at x. Suppose $c \in \operatorname{int} C(x)$. Then $-c - \operatorname{cl} C(x) \subset -\operatorname{int} C(x)$. Hence $(-c - \operatorname{cl} C(x))^{\mathrm{c}} = Z \setminus \big(-c - \operatorname{cl} C(x)\big)$ is a neighborhood of $W(x)$. Since W is u.s.c. at x, there exists a neighborhood U of x such that

$$W(u) \subset Z \setminus \big(-c - \operatorname{cl} C(x)\big), \text{ for all } u \in U.$$

Let $z \in W(x)^{\mathrm{c}}$, i.e., $z \in -\operatorname{int} C(x)$. Then there exists a positive number $t > 0$ such that $z + tc \in -\operatorname{int} C(x)$. Hence $\frac{1}{t} z \in -c - \operatorname{cl} C(x)$. Therefore for some $v \in X$, if $W(x) \not\supset W(v)$ then $v \notin U$. Accordingly we have statement (ii). ■

Proposition 2.2. *Let E be a nonempty subset of a topological space and $\mathbb{Z}$ a real topological vector space. Let $C : E \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $W : E \to 2^{\mathbb{Z}}$ defined by $W(x) = \mathbb{Z} \setminus \big(-\operatorname{int} C(x)\big)$. Suppose $F : E \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. If W is u.s.c. on E and for each $x \in E$, F is $(-C(x))$-u.s.c. at x, then the set $\mathfrak{S} = \{x \in E : F(x) \subset -\operatorname{int} C(x)\}$ is open.*

Proof. Let $x \in \mathfrak{S}$ and $c \in \operatorname{int} C(x)$. Then for each $z \in F(x)$ there exists a positive number $t_z > 0$ such that

$$z + t_z c \in -\operatorname{int} C(x).$$

Note that $\mathcal{V} = \bigcup_{z \in F(x)} z + t_z c - \operatorname{int} C(x)$ is a neighborhood of $F(x)$ and that $\mathcal{V} \cap W(x) = \emptyset$. Since F is $(-C(x))$-u.s.c. at x, there exists a neighborhood U_1 of x such that

$$F(u) \subset \mathcal{V} - \operatorname{int} C(x) = \mathcal{V}, \text{ for all } u \in U_1.$$

Since W is u.s.c. at x, by statement (ii) of Proposition 2.1, there exists a neighborhood U_2 of x such that

$$W(x) \supset W(u), \text{ for all } u \in U_2.$$

Hence $U_2 \cap U_2 \subset \mathfrak{S}$. Since $x \in \mathfrak{S}$ is arbitrary, $\mathfrak{S}$ is open. ∎

Proposition 2.3. *Let E be a nonempty subset of a topological space and $\mathbb{Z}$ a real topological vector space. Let $C : E \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $W : E \to 2^{\mathbb{Z}}$ defined by $W(x) = \mathbb{Z} \setminus \big(-\operatorname{int} C(x)\big)$. Suppose $F : E \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. If W is u.s.c. on E and for each $x \in E$, F is $(-C(x))$-l.s.c. at x, then the set $\mathfrak{T} = \{x \in E : F(x) \cap (-\operatorname{int} C(x)) \neq \emptyset\}$ is open.*

Proof. Let $x \in \mathfrak{T}$. Since F is $(-C(x))$-l.s.c. at x, there exists a neighborhood U_1 of x such that

$$F(u) \cap (-\operatorname{int} C(x)) \neq \emptyset, \text{ for all } u \in U_1.$$

Since W is u.s.c. at x, by statement (ii) of Proposition 2.1, there exists a neighborhood U_2 of x such that

$$W(x) \supset W(u), \text{ for all } u \in U_2.$$

Hence $U_2 \cap U_2 \subset \mathfrak{T}$. Since $x \in \mathfrak{T}$ is arbitrary, $\mathfrak{T}$ is open. ∎

Definition 2.3. Let X and $\mathbb{Z}$ be two real vector spaces. Suppose that K is a nonempty convex set of X and that $T : X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$.

(i) T is said to be convex on K if for each $x_1,\ x_2 \in K$ and $\mu \in [0,1]$

$$T(\mu x_1 + (1-\mu)x_2) \supset \mu T(x_1) + (1-\mu)T(x_2);$$

(ii) T is said to be concave on K if for each $x_1,\ x_2 \in K$ and $\mu \in [0,1]$

$$T(\mu x_1 + (1-\mu)x_2) \subset \mu T(x_1) + (1-\mu)T(x_2);$$

(iii) T is said to be affine on K if T is convex and concave on K.

Definition 2.4. (C-compactness [8]). Let C be a nonempty convex cone in a Hausdorff topological space Z. We say $E \subset Z$ is C-*compact* if any cover of E of the form

$$\{\mathcal{U}_\alpha + C : \alpha \in I,\ \mathcal{U}_\alpha \text{ are open}\}$$

admits a finite subcover.

Definition 2.5. (C-semicontinuity, See also (8]). Let X and Z be a topological space and a real topological vector space, respectively. Let $T : X \to 2^Z \setminus \{\emptyset\}$ and $C : X \to 2^Z$, which has proper convex cone values. Let $x' \in X$.

(i) T is said to be $C(x)$-*lower semicontinuous* ($C(x)$-l.s.c.) at x if for each $\mathcal{V}$, an open set of Z with $T(x) \cap \mathcal{V} \neq \emptyset$, there exists a neighborhood $\mathcal{U}$ of x such that

$$T(u) \cap \big(\mathcal{V} + \operatorname{int} C(x)\big) \neq \emptyset, \text{ for all } u \in \mathcal{U}.$$

T is said to be C-*lower semicontinuous* (C-l.s.c.) on X if T is $C(x)$-l.s.c. at x for every $x \in Z$.

(ii) T is said to be $C(x)$-*upper semicontinuous* ($C(x)$-u.s.c.) at x, if for each neighborhood $\mathcal{V}_{T(x)}$ of $T(x)$ there exists a neighborhood $\mathcal{U}_x$ of x such that

$$T(u) \subset \mathcal{V}_{T(x)} + \operatorname{int} C(x), \text{ for all } u \in \mathcal{U}_x.$$

T is said to be C-*upper semicontinuous* (C-u.s.c.) on X if T is $C(x)$-u.s.c. at x for every $x \in Z$.

Definition 2.6. (Generalized C-quasiconvexity). Let X be a vector space, and Z also a vector space with a proper solid convex cone C. Suppose that K is a convex subset of X and that $T : K \to 2^Z \setminus \{\emptyset\}$. Then T is said to be *generalized* C-*quasiconvex on* K if for each $z \in Z$,

$$A(z) := \{x \in K : T(x) \subset z - C\}$$

is convex or empty.

Definition 2.7. (Extended C-quasiconvexity). Let X be a vector space, and Z also a vector space with a proper solid convex cone C. Suppose that K is a convex subset of X and that $T : K \to 2^Z \setminus \{\emptyset\}$. Then T is said to be *extended C-quasiconvex on K* if for each $z \in Z$,

$$A(z) := \{x \in K : T(x) \cap (z - C) \neq \emptyset\}$$

is convex or empty.

Definition 2.8. (C-quasiconcavity, [13]). Let X be a nonempty convex subset of a real topological vector space and Z a real topological vector space. Let $T : X \to 2^Z \setminus \{\emptyset\}$. Suppose that $C : X \to 2^Z$ has proper solid convex cone values. We say that T is *C-quasiconcave* on X if for each $x_1, x_2 \in X$ and $z \in Z$, $T(x_1) \not\subset z - \operatorname{int} C(x_1)$ and $T(x_2) \not\subset z - \operatorname{int} C(x_2)$ imply

$$T(x_\mu) \not\subset z - \operatorname{int} C(x_\mu), \text{ for all } x_\mu \in (x_1, x_2).$$

We also say that T is *strictly C-quasiconcave* on X if for each $x_1, x_2 \in X$ and $z \in Z$, $T(x_1) \not\subset z - \operatorname{int} C(x_1)$ and $T(x_2) \not\subset z - \operatorname{int} C(x_2)$ imply

$$T(x_\mu) \not\subset z - \operatorname{cl} C(x_\mu), \text{ for all } x_\mu \in (x_1, x_2).$$

Definition 2.9. (C-proper quasiconcavity, [13]). Let X be a nonempty convex subset of a real topological vector space and Z a real topological vector space. Let $T : X \to 2^Z \setminus \{\emptyset\}$. Suppose that $C : X \to 2^Z$ has proper solid convex cone values. We say that T is *C-proper quasiconcave* on X if for each $x_1, x_2 \in X$ and $z \in Z$, $T(x_1) \cap \big(z - \operatorname{int} C(x_1)\big) = \emptyset$ and $T(x_2) \cap \big(z - \operatorname{int} C(x_2)\big) = \emptyset$ imply

$$T(x_\mu) \cap \big(z - \operatorname{int} C(x_\mu)\big) = \emptyset, \text{ for all } x_\mu \in (x_1, x_2).$$

We also say that T is *strictly C-properly quasiconcave* on X if for each $x_1, x_2 \in X$ and $z \in Z$, $T(x_1) \cap \big(z - \operatorname{int} C(x_1)\big) = \emptyset$ and $T(x_2) \cap \big(z - \operatorname{int} C(x_2)\big) = \emptyset$ imply

$$T(x_\mu) \cap \big(z - \operatorname{cl} C(x_\mu)\big) = \emptyset, \text{ for all } x_\mu \in (x_1, x_2).$$

Remark 1. If T is single-valued and C has constant values, Definitions 2.8 and 2.9 reduce to the definition of $(-C)$-proper quasiconvexity.

Definition 2.10. (C-weak quasiconcavity). Let X be a nonempty convex subset of a real topological vector space, $\mathbb{Z}$ a real topological vector space and $C : X \to 2^{\mathbb{Z}}$ with a proper solid convex cone values. Suppose $T : X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. We say that T is *C-weakly quasiconcave* on X if for each $x_1, x_2 \in X$, $T(x_1) \not\subset -\operatorname{int} C(x_1)$ and $T(x_2) \not\subset -\operatorname{int} C(x_2)$ imply

$$T(x_\mu) \not\subset -\operatorname{int} C(x_\mu), \text{ for all } x_\mu \in (x_1, x_2)$$

and also $T(x_1) \not\subset -\operatorname{int} C(x_1)$ and $T(x_2) \not\subset -\operatorname{cl} C(x_2)$ imply

$$T(x_\mu) \not\subset -\operatorname{cl} C(x_\mu), \text{ for all } x_\mu \in (x_1, x_2).$$

We say that T is *strictly C-weakly quasiconcave* on X if for each $x_1, x_2 \in X$, $T(x_1) \not\subset -\operatorname{int} C(x_1)$ and $T(x_2) \not\subset -\operatorname{int} C(x_2)$ imply

$$T(x_\mu) \not\subset -\operatorname{cl} C(x_\mu), \text{ for all } x_\mu \in (x_1, x_2).$$

Example 1. Let $X = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ and $\mathbb{Z} = \mathbb{R}^2$. Let

$$C(x, y) = \{(u, v) \in \mathbb{Z} : u \cos x + v \sin x \geq 0\}.$$

Suppose

$$T(x, y) = (x - \frac{\pi}{4})(\sin x, -\cos x) + \{(y - x - t)(\cos x, \sin x) : t \in [0, 1]\}.$$

Then T is C-weakly quasiconcave on X.

Definition 2.11. (C-weak proper quasiconcavity). Let X be a nonempty convex subset of a real topological vector space, $\mathbb{Z}$ a real topological vector space and $C : X \to 2^{\mathbb{Z}}$ with a proper solid convex cone values. Suppose $T : X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. We say that T is *C-weakly properly quasiconcave* on X if for each $x_1, x_2 \in X$, $T(x_1) \cap (z - \operatorname{int} C(x_1)) \neq \emptyset$ and $T(x_2) \cap (z - \operatorname{int} C(x_2)) \neq \emptyset$ imply

$$T(x_\mu) \cap (z - \operatorname{int} C(x_\mu)) \neq \emptyset, \text{ for all } x_\mu \in (x_1, x_2)$$

and also $T(x_1) \cap (z - \operatorname{int} C(x_1)) \neq \emptyset$ and $T(x_2) \cap (z - \operatorname{cl} C(x_2)) \neq \emptyset$ imply

$$T(x_\mu) \cap (z - \operatorname{cl} C(x_\mu)) \neq \emptyset, \text{ for all } x_\mu \in (x_1, x_2).$$

We say that T is *strictly C-weakly properly quasiconcave* on X if for each $x_1, x_2 \in X$, $T(x_1) \cap (z - \operatorname{int} C(x_1)) \neq \emptyset$ and $T(x_2) \cap (z - \operatorname{int} C(x_2)) \neq \emptyset$ imply

$$T(x_\mu) \cap (z - \operatorname{cl} C(x_\mu)) \neq \emptyset, \text{ for all } x_\mu \in (x_1, x_2).$$

Example 1. Let $X = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ and $\mathbb{Z} = \mathbb{R}^2$. Let

$$C(x, y) = \{(u, v) \in \mathbb{Z} : u \cos x + v \sin x \geq 0\}.$$

Suppose

$$T(x, y) = (x - \frac{\pi}{4})(\sin x, -\cos x) + \operatorname{co}\{(y - x)(\cos x, \sin x), (0, 0)\}.$$

Then T is C-weakly properly quasiconcave on X.

Definition 2.12. (C-diagonally quasiconcavity; see also, [5]). Let X be a nonempty convex subset of a real vector space, $\mathbb{Z}$ a real vector space and $C : X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Suppose $T : X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$.

(i) T is said to be *Type I C-diagonally quasiconcave in its second argument*, if for any finite subset A of X and any $x \in \operatorname{co} A$, there exists $y \in A$ such that $T(x, y) \not\subset -\operatorname{int} C(x)$.

(ii) T is said to be *Type II C-diagonally quasiconcave in its second argument*, if for any finite subset A of X and any $x \in \operatorname{co} A$, there exists $y \in A$ such that $T(x, y) \cap (-\operatorname{int} C(x)) = \emptyset$.

Proposition 2.4. *Let X be a nonempty convex subset of a real vector space, $\mathbb{Z}$ a real vector space and $C : X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Suppose that $T : X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. We also assume the following two conditions:*

(i) *for each $x \in X$ $T(x, x) \not\subset -\operatorname{int} C(x)$;*

(ii) *for each $x \in X$ $T(x, \cdot)$ is generalized $C(x)$-quasiconvex on X.*

Then T is Type I C-diagonally quasiconcave in its second argument.

Proof. Suppose to the contrary that T is not C-diagonally quasiconcave in its second argument. Then there exist $x, x_1, \ldots, x_n \in X$ such that $x \in \operatorname{co}\{x_1, \ldots, x_n\}$ and $T(x, x_i) \subset -\operatorname{int} C(x)$, for each $i = 1, \ldots, n$. Therefore by condition (ii), we have $T(x, x) \subset -\operatorname{int} C(x)$. However this contradicts to condition (i). Accordingly T is Type I C-diagonally quasiconcave in its second argument. ■

Proposition 2.5. *Let X be a nonempty convex subset of a real vector space, $\mathbb{Z}$ a real vector space and $C : X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Suppose that $T : X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. We also assume the following two conditions:*

(i) *for each $x \in X$ $T(x, x) \cap (-\operatorname{int} C(x)) = \emptyset$;*

(ii) *for each $x \in X$ $T(x, \cdot)$ is extended $C(x)$-quasiconvex on X.*

Then T is Type II C-diagonally quasiconcave in its second argument.

Proof. Suppose to the contrary that T is not Type II C-diagonally quasiconcave in its second argument. Then there exist $x, x_1, \ldots, x_n \in X$ such that $x \in \operatorname{co}\{x_1, \ldots, x_n\}$ and $T(x, x_i) \cap (-\operatorname{int} C(x)) \neq \emptyset$, for each $i = 1, \ldots, n$. Therefore by condition (ii), we have $T(x, x) \cap (-\operatorname{int} C(x)) = \emptyset$. However this contradicts to condition (i). Accordingly T is Type II C-diagonally quasiconcave in its second argument. ■

Definition 2.13. (Intersectional mapping, [13]. Let X be a topological space and $\mathbb{Z}$ a nonempty set. Let $T, G : X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$, respectively. We say G is an *intersectional mapping* of T, if for each $x \in X$ there exist a neighborhood $\mathcal{U}_x$ of x such that

$$G(x) \subset \bigcap_{u \in \mathcal{U}_x} T(u).$$

Proposition 2.6. [13, Proposition 2.2] *Let X be a nonempty subset of a topological space and $\mathbb{Z}$ a real topological vector spaces, respectively. Let $C : X \to 2^{\mathbb{Z}}$, which has proper solid convex cone values. Supose that $W : X \to 2^{\mathbb{Z}}$ defined by*

$$\mathcal{W}(x) = \mathbb{Z} \setminus \operatorname{int} C(x)$$

has closed graph. Then C has at least one intersectional mapping, which has solid convex cone values.

Proposition 2.7. *Let E be a nonempty subset of a topological space and $\mathbb{Z}$ a real topological vector space. Let $C : E \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $D : E \to 2^{\mathbb{Z}}$ an intersectional mapping of C with solid convex cone values. Suppose $F : E \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ and $W : E \to 2^{\mathbb{Z}}$, defined by $W(x) = \mathbb{Z} \setminus \big(-\operatorname{int} C(x)\big)$. We also assume the following conditions:*

- (i) *W has closed graph;*
- (ii) *F is $(-D)$-u.s.c on X*
- (iii) *$F(x)$ is $(-D(x))$-compact for each $x \in X$.*

Then the set $\mathfrak{S} = \{x \in E : F(x) \subset -\operatorname{int} C(x)\}$ is open.

Proof. Let $x \in \mathfrak{S}$ and $d \in \operatorname{int} D(x)$. Then for each $z \in F(x)$ there exists a positive number $t_z > 0$ such that

$$z + t_z d \in -\operatorname{int} C(x).$$

Note that $\mathcal{V} = \bigcup_{z \in F(x)} z + t_z d - \operatorname{int} C(x)$ is a neighborhood of $F(x)$. Hence by condition (iii), there exist $z_1, \ldots, z_n \in F(x)$ such that

$$\bigcup_{i=1}^{n} \big(z_i + t_{z_i} d - \operatorname{int} D(x)\big) \supset F(x).$$

By condition (ii), there exists a neighborhood U_1 of x such that

$$\bigcup_{i=1}^{n} \big(z_i + t_{z_i} d \in -\operatorname{int} D(x)\big) \supset F(u), \text{ for all } u \in U_1.$$

Futhermore by condition (i), there exists a neighborhood U_2 of x such that

$$\{z_1 + t_{z_1} d, \ldots, z_n + t_{z_n} d\} \subset -\operatorname{int} C(u), \text{ for all } u \in U_2.$$

Since D is an intersectional mapping of C, there exists a neighborhood U_3 of x such that $D(x) \subset C(u)$ for all $u \in U_3$. Hence for each $u \in U_2 \cap U_3$ we have

$$\bigcup_{i=1}^{n} \big(z_i + t_{z_i} d - \operatorname{int} D(x)\big) \subset \operatorname{int} C(u).$$

Therefore we have

$$F(u) \subset \operatorname{int} C(u), \text{ for all } u \in \bigcap_{i=1}^{3} U_i.$$

Hence $\bigcap_{i=1}^{3} U_i \subset \mathfrak{S}$. Since $x \in \mathfrak{S}$ is arbitrary, $\mathfrak{S}$ is open. ■

Proposition 2.8. *Let E be a nonempty subset of a topological space and $\mathbb{Z}$ a real topological vector space. Let $C : E \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $D : E \to 2^{\mathbb{Z}}$ an intersectional mapping of C with solid convex cone values. Suppose $F : E \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ and $W : E \to 2^{\mathbb{Z}}$, defined by $W(x) = \mathbb{Z} \setminus \big(-\operatorname{int} C(x)\big)$. We also assume the following conditions:*

(i) *W has closed graph;*

(ii) *F is $(-D)$-l.s.c on X.*

Then the set $\mathfrak{T} = \{x \in E : F(x) \cap (-\operatorname{int} C(x)) \neq \emptyset\}$ is open.

Proof. Let $x \in \mathfrak{T}$. Then there exists $z \in -\operatorname{int} C(x)$ such that

$$T(x) \cap (z - \operatorname{int} D(x)) \neq \emptyset.$$

By condition (ii), there exists a neighborhood U_1 of x such that

$$T(u) \cap (z - \operatorname{int} D(x)) \neq \emptyset, \text{ for all } u \in U_1.$$

Futhermore by condition (i), there exists a neighborhood U_2 of x such that

$$z \subset -\operatorname{int} C(u), \text{ for all } u \in U_2.$$

Since D is an intersectional mapping of C, there exists a neighborhood U_3 of x such that $D(x) \subset C(u)$ for all $u \in U_3$. Hence for each $u \in U_2 \cap U_3$ we have

$$z - \operatorname{int} D(u) \subset \operatorname{int} C(u).$$

Therefore we have

$$F(u) \cup (-\operatorname{int} C(u)) \neq \emptyset, \text{ for all } u \in \bigcap_{i=1}^{3} U_i.$$

Hence $\bigcap_{i=1}^{3} U_i \subset \mathfrak{T}$. Since $x \in \mathfrak{T}$ is arbitrary, $\mathfrak{T}$ is open. ■

Definition 2.14. Let X be a topological space and Y an nonemptyset. A set-valued map $F : X \to 2^Y$ is said to *have open lower sections*, if the set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for every $y \in Y$.

Lemma 2.1. ([11]). *Let X be a topological space and Y a convex set of a real topological vector space. Let $F, G : X \to 2^Y$ be two set-valued maps with open lower sections. Then:*

(i) *the set-valued map* $H : X \to 2^Y$*, defined by* $H(x) = \operatorname{co}\big(F(x)\big)$ *for all* $x \in X$*, has open lower sections;*

(ii) *the set-valued map* $J : X \to 2^Y$*, defined by* $J(x) = F(x) \cap G(x)$ *for all* $x \in X$*, has open lower sections.*

Lemma 2.2. [10, Fan-Browder fixed-point theorem]. *Let* X *be a nonempty compact convex subset of a real Hausdorff topological vector space. Suppose that* $F : X \to 2^X$ *is a set-valued map with nonempty convex values and open lower sections. Then* F *has a fixed point.*

3. Existence of Solution for PGQVEP

In this section we drive some existence results for PGQVEP.

Theorem 3.1. *Let* X *be a nonempty subset of a real topological vector space and* $\mathbb{Z}$ *a real topological vector space,* $\mathbb{P}$ *an index set and* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values. Let* $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$*. Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$*. Also we assume that the following conditions:*

(i) *for each* $p \in \mathbb{P}$ $F(p, \cdot, \cdot)$ *is Type I* C*-diagonally quasiconcave in the third argument;*

(ii) X *is compact and convex;*

(iii) K *has convex values and has open lower sections;*

(iv) *for each fixed* $p \in \mathbb{P}$ *and* $x \in X$*, the set*

$$\{y \in X : F(p, x, y) \subset -\operatorname{int} C(p, x)\}$$

is open.

(v) *for each fixed* $p \in \mathbb{P}$ *the set* $\{x \in X : K(p, x) \cap \{y \in X : F(p, x, y) \subset -\operatorname{int} C(p, x)\} \neq \emptyset\}$

Then Ω *is nonempty for each* $p \in \mathbb{P}$*. Moreover,* Φ *is nonempty for each* $p \in \mathbb{P}$ *if conditions* (i)*,* (iv)*, and* (v) *are replaced by the following* (vi)*,* (vii)*, and (viii):*

(vi) *for each* $p \in \mathbb{P}$ $F(p, \cdot, \cdot)$ *is Type II* C*-diagonally quasiconcave in the third argument;*

(vii) *the set* $\{y \in X : F(p, x, y) \cap (-\operatorname{int} C(p, x)) \neq \emptyset\}$ *is open, for each* $p \in \mathbb{P}$ *and* $x \in X$*.*

($viii$) *for each fixed* $p \in \mathbb{P}$ *the set* $\{x \in X : K(p, x) \cap \{y \in X : F(p, x, y) \cap -\operatorname{int} C(p, x)) \neq \emptyset\} \neq \emptyset$ *is cloded.*

Proof. Let $p \in \mathbb{P}$ and for $x \in X$,

$$T(p,x) := \{y \in X : F(p,x,y) \subset -\operatorname{int} C(p,x)\}.$$

Then by condition (i), we have

$$x \notin \operatorname{co}\big(T(p,x)\big), \text{ for each } x \in X. \tag{1}$$

Let $G(p,x) = K(p,x) \cap \operatorname{co}\big(T(p,x)\big)$. By condition (iii), K has convex values and K has open lower sections. Hence there exists $x' \in X$ such that $x' \in K(p,x')$ by Lemma 2.2. If for every $x \in X$, $G(p,x) = \emptyset$, then $x' \in \Omega(p)$. Thus we may suppose $G(p,x) \neq \emptyset$ for some $x \in X$. By condition (iv), T has open lower sections. Therefore by Lemma 2.1, G has open lower sections. Clearly G has convex values. Let $H : X \to 2^X$ be defined by

$$H(p,x) = \begin{cases} G(p,x), & \text{if } G(p,x) \neq \emptyset, \\ K(p,x), & \text{otherwise.} \end{cases}$$

Hence H has convex values and, by condition (v), open lower sections. Accordingly by Lemma 2.2, there exists $\hat{x} \in X$ such that $\hat{x} \in H(\hat{x})$. Because of (1), $\hat{x} \in K(p,\hat{x})$ and $G(p,\hat{x}) = \emptyset$. Hence $\hat{x} \in S(p)$. Therefore $S(p) \neq \emptyset$. Since $p \in \mathbb{P}$ is arbitrary, we have $S(p) \neq \emptyset$ for each $p \in \mathbb{P}$.

In above proof, let Ω, (i) and (iv) be replaced by Φ, (v) and (vi), respectively and let

$$T(p,x) = \{y \in X : F(p,x,y) \cap (-\operatorname{int} C(p,x)) \neq \emptyset\}$$

Then we obtain upper semicontinuity of Φ on $\mathbb{P} \times X$. ∎

Next we concider sufficient condition for assumptions (iv) and (vi) of Theorem 3.1.

Proposition 3.1. *Let E be a nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector space with a proper solid convex cone C Suppose $F : E \to 2^{\mathbb{Z}} \backslash \{\emptyset\}$. If F is $(-C)$-u.s.c. on E, then the set $\{x \in E : F(x) \subset -\operatorname{int} C\}$ is open. If F is $(-C)$-l.s.c. on E, then the set $\{x \in E : F(x) \cap (-\operatorname{int} C) \neq \emptyset\}$ is open.*

Proof. In Propositions 2.2 and 2.3 assuming C has constant value, respectively, we obtain the results. ∎

The following result is a consequence of Theorem 3.1 and Propositions 2.4, 2.5 and 3.1.

Corollary 3.1. *In Theorem 3.1,*

(i) *condition* (i) *can be replaced by the following conditions:*

(a) *for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, x) \not\subset -\operatorname{int} C(p, x)$;

(b) *for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, \cdot)$ *is generalized* $C(p, x)$*-quasiconvex on* X;

(ii) *condition* (vi) *can be replaced by the following conditions:*

(a) *for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, x) \cap (-\operatorname{int} C(p, x)) = \emptyset$;

(b) *for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, \cdot)$ *is extended* $C(p, x)$*-quasiconvex on* X;

(iii) *condition* (iv) *can be replaced by the following condition:*

(a) *for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, \cdot)$ *is* $-C(p, x)$*-u.s.c. on* X;

(iv) *condition* (vii) *can be replaced by the following condition:*

(a) *for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, \cdot)$ *is* $-C(p, x)$*-l.s.c. on* X.

To investigate upper and lower semicontinuities of Ω and Φ, we need to require closedness of $K(p, x)$ for each $p \in \mathbb{P}$ and $x \in X$. The following theorem is useful.

Theorem 3.2. *Let* X *be a nonempty subset of a real topological vector space and* $\mathbb{Z}$ *a real topological vector space,* $\mathbb{P}$ *an index set and* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values. Let* $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. *Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. *Also we assume that the following conditions:*

(i) $\Omega(p)$ *is nonempty for each* $p \in \mathbb{P}$;

(ii) *the set* $\{y \in X : F(p, x, y) \subset -\operatorname{int} C(p, x)\}$ *is open for each fixed* $p \in \mathbb{P}$ *and* $x \in X$.

Then

$$\overline{\Omega}(p) = \{x \in \operatorname{cl} K(p, x) : F(p, x, y) \not\subset -\operatorname{int} C(p, x),\ \textit{for all } y \in \operatorname{cl} K(p, x)\}$$

is nonempty for each $p \in \mathbb{P}$.

Proof. Let $p \in \mathbb{P}$ and $x \in \Omega(p)$. Suppose to the contrary that there exists $y \in \operatorname{cl} K(p, x)$ such that $F(p, x, y) \subset -\operatorname{int} C(p, x)$. Then by condition (ii), there exists a neighborhood $\mathcal{U}$ of y such that

$$F(p, x, y') \subset -\operatorname{int} C(p, x), \text{ for all } v' \in \mathcal{U}.$$

Clearly $\mathcal{U} \cap K(p, x) \neq \emptyset$. This contradicts to the fact that $x \in \Omega(p)$. Hence $\overline{\Omega}(p)$ is nonempty. Since $p \in \mathbb{P}$ is arbitrary, $\overline{\Omega}(p)$ is nonempty for each $p \in \mathbb{P}$. ∎

Theorem 3.3. *Let X be a nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector space, $\mathbb{P}$ an index set and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

(i) $\Psi(p)$ *is nonempty for each* $p \in \mathbb{P}$*;*

(ii) *the set* $\{y \in X : F(p, x, y) \cap (-\operatorname{int} C(p, x)) \neq \emptyset\}$ *is open for each fixed* $p \in \mathbb{P}$ *and* $x \in X$.

Then

$$\overline{\Psi}(p) = \{x \in \operatorname{cl} K(p, x) : F(p, x, y) \cap (-\operatorname{int} C(p, x)) = \emptyset,\ \forall y \in \operatorname{cl} K(p, x)\}$$

is nonempty for each $p \in \mathbb{P}$.

Proof. Let $p \in \mathbb{P}$ and $x \in \Psi(p)$. Suppose to the contrary that there exists $y \in \operatorname{cl} K(p, x)$ such that $F(p, x, y) \cap (-\operatorname{int} C(p, x)) \neq \emptyset$. Hence by condition (ii), there exists a neighborhood $\mathcal{U}$ of y such that

$$F(p, x, y') \cap (-\operatorname{int} C(p, x)) \neq \emptyset, \text{ for all } v' \in \mathcal{U}.$$

Clearly $\mathcal{U} \cap K(p, x) \neq \emptyset$. This contradicts to the fact that $x \in \Psi(p)$. Hence $\overline{\Psi}(p)$ is nonempty. Since $p \in \mathbb{P}$ is arbitrary, $\overline{\Psi}(p)$ is nonempty for each $p \in \mathbb{P}$. ■

The following result is a consequence of Theorems 3.1, 3.2 and 3.3.

Theorem 3.4. *Let X be a nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector space, $\mathbb{P}$ an index set and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K, K' : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

(i) *for each* $p \in \mathbb{P}$ $F(p, \cdot, \cdot)$ *is Type I* C*-diagonally quasiconcave in the third argument;*

(ii) X *is compact and convex;*

(iii) K' *has convex values and open lower sections;*

(iv) $K(p, x) = \operatorname{cl} K'(p, x)$ *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(v) *the set* $\{y \in X : F(p, x, y) \subset -\operatorname{int} C(p, x)\}$ *is open, for each fixed* $p \in \mathbb{P}$ *and* $x \in X$.

(vi) *for each fixed* $p \in \mathbb{P}$ *the set* $\{x \in X : K(p, x) \cap \{y \in X : F(p, x, y) \subset -\operatorname{int} C(p, x)\} \neq \emptyset\}$ *is closed.*

Then Ω is nonempty for each $p \in \mathbb{P}$. Moreover, Φ is nonempty for each $p \in \mathbb{P}$ if conditions (i), (v), *and* (vi) *are replaced by the following* (vii), ($viii$), *and* (ix)*:*

(vii) *for each* $p \in \mathbb{P}$ $F(p, \cdot, \cdot)$ *is Type II* C*-diagonally quasiconcave in the third argument;*

($viii$) *the set* $\{(p, x, y) \in \mathbb{P} \times X : F(p, x, y) \cap (-\operatorname{int} C(p, x)) \neq \emptyset\}$ *is open.*

(ix) *for each fixed* $p \in \mathbb{P}$ *the set* $\{x \in X : K(p, x) \cap \{y \in X : F(p, x, y) \cap -\operatorname{int} C(p, x)) \neq \emptyset\} \neq \emptyset$ *is cloded.*

Remark 2. Conditions (iii) and (iv) and the condition that K has closed convex values and open lower sections are quite different. For example let $\mathbb{P} = \{1\}$, $X = [0, 1]$, $A = (0, \frac{1}{2})$ and $B = (\frac{1}{2}, 1)$. Suppese that $K : \mathbb{P} \times X \to 2^X$ is defined by

$$K(p, x) = \operatorname{cl}\big(xA + (1 - x)B\big).$$

Then K satisfies conditions (iii) and (iv) but K does not have open lower sections.

The following result is a consequence of Theorem 3.4 and Propositions 2.4, 2.5 and 3.1.

Corollary 3.2. *In Theorem 3.4,*

(i) *condition* (i) *can be replaced by the following conditions:*
  - *(a) for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, x) \not\subset -\operatorname{int} C(p, x)$*;*
  - *(b) for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, \cdot)$ *is generalized* $C(p, x)$*-quasiconvex on* X*;*

(ii) *condition* (vii) *can be replaced by the following conditions:*
  - *(a) for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, x) \cap (-\operatorname{int} C(p, x)) = \emptyset$*;*
  - *(b) for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, \cdot)$ *is extended* $C(p, x)$*-quasiconvex on* X*;*

(iii) *condition* ($viii$) *can be replaced by the following condition:*
  - *(a) for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, \cdot)$ *is* $-C(p, x)$*-u.s.c. on* X*;*

(iv) *condition* (iv) *can be replaced by the following condition:*
  - *(a) for each* $p \in \mathbb{P}$ *and* $x \in X$, $F(p, x, \cdot)$ *is* $-C(p, x)$*-l.s.c. on* X.

Let

$$\Omega'(p) := \{x \in K(p, x) : F(p, x, y) \not\subset -\operatorname{cl} C(p, x),\ \forall y \in K(p, x)\}$$

and

$$\Phi'(p) := \{x \in K(p, x) : F(p, x, y) \cap (-\operatorname{cl} C(p, x)) = \emptyset,\ \forall y \in K(p, x)\}.$$

Theorem 3.5. *Let* X *be a nonempty subset of a real topological vector space and* $\mathbb{Z}$ *a real topological vector space,* $\mathbb{P}$ *an index set and* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values. Let* $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$ *and* $c : \mathbb{P} \times X \to \mathbb{Z}$ *with* $c(p, x) \in \operatorname{int} C(p, x)$ *for each* $p \in \mathbb{P}$ *and* $x \in X$. *Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. *Also we assume that the following conditions:*

(i) *for each* $p \in \mathbb{P}$, $F(p, \cdot, \cdot) - c(p, \cdot)$ *is Type I* C*-diagonally quasiconcave in the third argument;*

(ii) X *is compact and convex;*

(iii) K *has convex values and open lower sections;*

(iv) $\{y \in X : F(p, x, y) - c(p, x) \subset -\operatorname{int} C(p, x)\}$ *is open, for each fixed* $p \in \mathbb{P}$ *and* $x \in X$.

(v) *for each fixed* $p \in \mathbb{P}$ *the following set is closed,* $\{x \in X : K(p, x) \cap \{y \in X : F(p, x, y) - c(p, x) \subset -\operatorname{int} C(p, x)\} \neq \emptyset\}$.

Then $\Omega'(p)$ *is nonempty for each* $p \in \mathbb{P}$. *Moreover,* $\Phi'(p)$ *is nonempty for each* $p \in \mathbb{P}$ *if conditions* (i) *and* (iv) *are replaced by the following* (vi) *and* (vii)*:*

(vi) *for each* $p \in \mathbb{P}$ $F(p, \cdot, \cdot) - c(p, x)$ *is Type II* C*-diagonally quasiconcave in the third argument;*

(vii) *the set* $\{y \in X : (F(p, x, y) - c(p, x)) \cap (-\operatorname{int} C(p, x)) \neq \emptyset\}$ *is open, for each* $p \in \mathbb{P}$ *and* $x \in X$.

($viii$) *for each fixed* $p \in \mathbb{P}$ *the following set is closed,* $\{x \in X : K(p, x) \cap \{y \in X : F(p, x, y) - c(p, x) \cap (-\operatorname{int} C(p, x))\} \neq \emptyset\}$.

Proof. First, we show nonemptyness of $\Omega'(p)$ for each $p \in \Gamma$. Let $F'(p, x, y) = F(p, x, y) - c(p, x)$. Then by Theorem 3.1, for each $p \in \mathbb{P}$ there exists $x \in X$ such that $x \in K(p, x)$ and

$$F'(p, x, y) \not\subset -\operatorname{int} C(p, x), \text{ for all } y \in K(p, x),$$

i.e.,

$$F(p, x, y) - c(p, x) \not\subset -\operatorname{int} C(p, x), \text{ for all } y \in K(p, x). \tag{2}$$

Since $c(p, x) \in \operatorname{int} C(p, x)$ for each $p \in \mathbb{P}$ and $x \in X$, (2) implies

$$F(p, x, y) \not\subset -\operatorname{cl} C(p, x), \text{ for all } y \in K(p, x).$$

Hence $\Omega'(p) \neq \emptyset$ for each $p \in \mathbb{P}$.

Next, we show nonemptyness of $\Phi'(p)$ for each $p \in \Gamma$. Let $F'(p, x, y) = F(p, x, y) - c(p, x)$. Then by Theorem 3.1, for each $p \in \mathbb{P}$ there exists $x \in X$ such that $x \in K(p, x)$ and

$$F'(p, x, y) \cap (-\operatorname{int} C(p, x)) = \emptyset, \text{ for all } y \in K(p, x),$$

i.e.,

$$F(p, x, y) - c(p, x) \cap (-\operatorname{int} C(p, x)) = \emptyset, \text{ for all } y \in K(p, x). \tag{3}$$

Since $c(p,x) \in \operatorname{int} C(p,x)$ for each $p \in \mathbb{P}$ and $x \in X$, (3) implies

$$F(p,x,y) \cap (-\operatorname{int} C(p,x)) = \emptyset, \text{ for all } y \in K(p,x).$$

Hence $\Phi'(p) \neq \emptyset$ for each $p \in \mathbb{P}$. ∎

Next corollary follows from Theorem 3.5, Propositions 2.4, 2.5 and 3.1.

Corollary 3.3. *In Theorem 3.5,*

(i) *condition* (i) *can be replaced by the following conditions:*

(a) $F(p,x,x) - c(p,x) \not\subset -\operatorname{int} C(p,x)$ *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(b) $F(p,x,\cdot) - c(p,\cdot)$ *is generalized* $C(p,x)$*-quasiconvex on* X *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(ii) *condition* (vi) *can be replaced by the following conditions:*

(a) $(F(p,x,x) - c(p,x)) \cap (-\operatorname{int} C(p,x)) = \emptyset$ *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(b) $(F(p,x,\cdot) - c(p,\cdot))$ *is extended* $C(p,x)$*-quasiconvex on* X *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iii) *condition* (iv) *can be replaced by the following condition:*

(a) $F(p,x,\cdot) - c(p,\cdot)$ *is* $-C(p,x)$*-u.s.c. on* X *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iv) *condition* (vii) *can be replaced by the following condition:*

(a) $F(p,x,\cdot) - c(p,\cdot)$ *is* $-C(p,x)$*-l.s.c. on* X *for each* $p \in \mathbb{P}$ *and* $x \in X$.

Theorem 3.6. *Let* X *be a nonempty subset of a real topological vector space and* $\mathbb{Z}$ *a real topological vector space,* $\mathbb{P}$ *an index set and* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values. Let* $K, K' : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$ *and* $c : \mathbb{P} \times X \to \mathbb{Z}$ *with* $c(p,x) \in \operatorname{int} C(p,x)$ *for each* $p \in \mathbb{P}$ *and* $x \in X$. *Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. *Also we assume that the following conditions:*

(i) *for each* $p \in \mathbb{P}$ $F(p,\cdot,\cdot) - c(p,\cdot)$ *is Type I* C*-diagonally quasiconcave in the third argument;*

(ii) X *is compact and convex;*

(iii) K' *has convex values and open lower sections;*

(iv) $K(p,x) = \operatorname{cl} K'(p,x)$ *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(v) *the set* $\{y \in X : F(p,x,y) - c(p,x) \subset -\operatorname{int} C(p,x)\}$ *is open for each fixed* $p \in \mathbb{P}$ *and* $x \in X$.

(vi) *for each fixed* $p \in \mathbb{P}$ *the following set is closed,* $\{x \in X : K(p,x) \cap \{y \in X : F(p,x,y) - c(p,x) \subset -\operatorname{int} C(p,x)\} \neq \emptyset\}$.

Then Ω' is nonempty for each $p \in \mathbb{P}$. Moreover, Φ' is nonempty for each $p \in \mathbb{P}$ if conditions (i) and (iv) are replaced by the following (v) and (vi):

(vii) *for each $p \in \mathbb{P}$ $F(p, \cdot, \cdot) - c(p, x)$ is Type II C-diagonally quasiconcave in the third argument;*

($viii$) *the set $\{y \in X : (F(p, x, y) - c(p, x)) \cap (-\operatorname{int} C(p, x)) \neq \emptyset\}$ is open, for each $p \in \mathbb{P}$ and $x \in X$.*

(ix) *for each fixed $p \in \mathbb{P}$ the following set is closed, $\{x \in X : K(p, x) \cap \{y \in X : F(p, x, y) - c(p, x) \cap (-\operatorname{int} C(p, x))\} \neq \emptyset\}$.*

Proof. First, we show nonemptyness of $\Omega'(p)$ for each $p \in \Gamma$. Let $F'(p, x, y) = F(p, x, y) - c(p, x)$. Then by Theorem 3.1, for each $p \in \mathbb{P}$ there exists $x \in X$ such that $x \in K'(p, x)$ and

$$F'(p, x, y) \not\subset -\operatorname{int} C(p, x), \text{ for all } y \in K'(p, x),$$

i.e.,

$$F(p, x, y) - c(p, x) \not\subset -\operatorname{int} C(p, x), \text{ for all } y \in K'(p, x).$$

By condition (vii), we have

$$F(p, x, y) - c(p, x) \not\subset -\operatorname{int} C(p, x), \text{ for all } y \in K(p, x). \tag{4}$$

Since $c(p, x) \in \operatorname{int} C(p, x)$, (4) implies

$$F(p, x, y) \not\subset -\operatorname{cl} C(p, x), \text{ for all } y \in K(p, x).$$

Hence $\Omega'(p)$ is nonempty for each $p \in \mathbb{P}$.

Next, we show nonemptyness of $\Phi'(p)$ for each $p \in \Gamma$. Let $F'(p, x, y) = F(p, x, y) - c(p, x)$. Then by Theorem 3.1, for each $p \in \mathbb{P}$ there exists $x \in X$ such that $x \in K'(p, x)$ and

$$F'(p, x, y) \cap (-\operatorname{int} C(p, x)) = \emptyset, \text{ for all } y \in K'(p, x),$$

i.e.,

$$F(p, x, y) - c(p, x) \cap (-\operatorname{int} C(p, x)) = \emptyset, \text{ for all } y \in K'(p, x).$$

By condition (vii), we have

$$F(p, x, y) - c(p, x) \cap (-\operatorname{int} C(p, x)) = \emptyset, \text{ for all } y \in K(p, x). \tag{5}$$

Since $c(p, x) \in \operatorname{int} C(p, x)$, (5) implies

$$F(p, x, y) \cap (-\operatorname{int} C(p, x)) = \emptyset, \text{ for all } y \in K(p, x).$$

Hence $\Phi'(p)$ is nonempty for each $p \in \mathbb{P}$. ■

The following result follows Theorem 3.6, Propositions 2.4, 2.5 and 3.1.

Corollary 3.4. *In Theorem 3.6,*

(i) *condition* (i) *can be replaced by the following conditions:*

(a) $F(p,x,x) - c(p,x) \not\subset -\operatorname{int} C(p,x)$ *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(b) $F(p,x,\cdot) - c(p,\cdot)$ *is generalized* $C(p,x)$*-quasiconvex on* X *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(ii) *condition* (vii) *can be replaced by the following conditions:*

(a) $(F(p,x,x) - c(p,x)) \cap (-\operatorname{int} C(p,x)) = \emptyset$ *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(b) $(F(p,x,\cdot) - c(p,\cdot))$ *is extended* $C(p,x)$*-quasiconvex on* X *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iii) *condition (v) can be replaced by the following condition:*

(a) $F(p,x,\cdot) - c(p,\cdot)$ *is* $-C(p,x)$*-u.s.c. on* X *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iv) *condition* ($viii$) *can be replaced by the following condition:*

(a) $F(p,x,\cdot) - c(p,\cdot)$ *is* $-C(p,x)$*-l.s.c. on* X *for each* $p \in \mathbb{P}$ *and* $x \in X$.

4. Upper Semicontinuity of the Solution Mapping

In this section we show that the solution mappings Ω of PGQVEP and Φ of PEQVEP are upper semicontinuous on $\mathbb{P}$, respectively, under suitable assumptions.

Theorem 4.1. *Let* X *be a nonempty subset of a real topological vector space and* $\mathbb{Z}$ *a real topological vector space,* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values and* $\mathbb{P}$ *a topological space. Let* $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$*. Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$*. Also we assume that the following conditions:*

(i) $\Omega(p)$ *is nonempty for each* $p \in \mathbb{P}$*;*

(ii) X *is compact;*

(iii) $K(p,x)$ *is compact for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iv) K *is u.s.c. on* $\mathbb{P} \times X$*;*

(v) *the set* $\{(p,x,y) \in \mathbb{P} \times X : F(p,x,y) \subset -\operatorname{int} C(p,x)\}$ *is open.*

Then Ω *is u.s.c. on* $\mathbb{P}$*. Moreover,* Φ *is u.s.c. on* $\mathbb{P}$ *if condition (v) is replaced by the following one:*

(vi) *the set* $\{(p,x,y) \in \mathbb{P} \times X : F(p,x,y) \cap (-\operatorname{int} C(p,x)) \neq \emptyset\}$ *is open.*

Proof. Since X is compact, it suffices to show that Ω has closed graph. Let $p \in \mathbb{P}$. Suppose $\{p_\mu\} \subset \mathbb{P}$ is a net with $p_\mu \to p$ and $\{x_\mu\} \subset X$ is a net with $x_\mu \in \Omega(p_\mu)$ for all $\mu \in \mathcal{M}$. Then $x_\mu \in K(p_\mu, x_\mu)$. By condition (ii), there exists a subnet $\{x_\nu\} \subset \{x_\mu\}$ such that $x_\nu \to x \in X$. Hence we may assume, without loss of generality, $p_\mu \to p$ and $x_\mu \to x$ with $x_\mu \in \Omega(p_\mu)$. Because of conditions (ii),

(iii) and (iv), K has closed graph. Note that $x_\mu \in \Omega(p_\mu)$ implies $x_\mu \in K(p_\mu, x_\mu)$. Therefore we have $x \in K(p, x)$. Suppose to the contrary that $x \notin \Omega(p)$. Then there exists $y \in K(p, x)$ such that

$$F(p, x, y) \subset -\mathrm{int}\, C(p, x). \tag{6}$$

Then by condition (v), there exists a neighborhood $\mathcal{U}$ of (p, x, y) such that

$$f(p', x', y') \in -\mathrm{int}\, C(p', x'), \text{ for all } (p', x', y') \in \mathcal{U}. \tag{7}$$

This contradicts to the fact that $p_\mu \to p,\ x_\mu \to x$ and $x_\mu \in \Omega(p_\mu)$ for all $\mu \in \mathcal{M}$. Hence $x \in \Omega(p)$. Therefore Ω is u.s.c. on $\mathbb{P}$.

Let Ω and (v) be replaced by Φ and (vi), respectively. Let (6) and (7) be replaced by the following (8) and (9), respectively in above proof:

$$F(p, x, y) \cap (-\mathrm{int}\, C(p, x)) \neq \emptyset \tag{8}$$

and

$$f(p', x', y') \cap (-\mathrm{int}\, C(p', x')) \neq \emptyset, \text{ for all } (p', x', y') \in \mathcal{U}. \tag{9}$$

Consequently, Φ is u.s.c. on $\mathbb{P}$. ∎

Theorem 4.2. *Let X be a nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector space, $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $\mathbb{P}$ a topological space. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

- (i) *$\Omega(p)$ is nonempty for each $p \in \mathbb{P}$;*
- (ii) *X is compact;*
- (iii) *$K(p, x)$ is compact for each $p \in \mathbb{P}$ and $x \in X$;*
- (iv) *K is u.s.c. on $\mathbb{P} \times X$;*
- (v) *$W : \mathbb{P} \times X \to 2^Z$ defined by $W(p, x) = \mathbb{Z} \setminus \big(-\mathrm{int}\, C(p, x)\big)$ is u.s.c. on $\mathbb{P} \times X$;*
- (vi) *F is $\big(-C\big)$-u.s.c. on $\mathbb{P} \times X \times X$.*

Then Ω is u.s.c. on $\mathbb{P}$. Moreover, Φ is u.s.c. on $\mathbb{P}$ if condition (vi) is replaced by the following one:

- (vii) *F is $\big(-C\big)$-l.s.c. on $\mathbb{P} \times X \times X$.*

Proof. Upper semicontinuity of Ω follows from Theorem 4.1 and Proposition 2.2. Upper semicontinuity of Φ follows from Theorem 4.1 and Proposition 2.3. ■

Example 3. Let $\mathbb{P} = [0,1]$, $\mathbb{X} = \mathbb{R}$, $X = [0, \frac{\pi}{2}]$, $Z = \mathbb{R}^2$, $A = (0, \frac{\pi}{4})$ and $B = (\frac{\pi}{4}, \frac{\pi}{2})$. Let

$$K(p,x) = \mathrm{cl}\left(p\frac{2x}{\pi}A + \frac{2(1-x)}{\pi}B\right),$$

$$C(p,x) = \begin{cases} \{(u,v) \in Z : u \geq 0\}, & x \in [0, \frac{\pi}{6}) \text{ and } p \in [0, \frac{1}{2}), \\ \{(u,v) \in Z : u \geq 0 \text{ and } v \geq 0\}, & x \in [\frac{\pi}{6}, \frac{\pi}{3}] \text{ or } p \in [\frac{1}{2}, 1], \\ \{(u,v) \in Z : v \geq 0\}, & x \in (\frac{\pi}{3}, 1]) \text{ and } p \in [0, \frac{1}{2}). \end{cases}$$

Suppose that

$$F(p,x,y) = \begin{cases} \mathrm{co}\left\{\begin{pmatrix} -p \\ -p \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}, & \text{if } x \leq y, x \in [0, \frac{\pi}{6}) \text{ and } p \in [0, \frac{1}{2}), \\ \mathrm{co}\left\{\begin{pmatrix} -p \\ -p \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}, & \text{if } x \leq y, x \in [\frac{\pi}{6}, \frac{\pi}{3}] \text{ or } p \in [\frac{1}{2}, 1], \\ \mathrm{co}\left\{\begin{pmatrix} -p \\ -p \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right\}, & \text{if } x \leq y, x \in (\frac{\pi}{3}, 1] \text{ and } p \in [0, \frac{1}{2}), \\ \left\{\begin{pmatrix} -p \\ -p \end{pmatrix}\right\}, & \text{otherwise.} \end{cases}$$

Then by Theorem 3.4 and Corollary 3.2, for each $p \in \mathbb{P}$, $\Omega(p) \neq \emptyset$. We also observe that every condition of Theorem 4.2 is satisfied. Accordingly Ω is u.s.c. on $\mathbb{P}$. Indeed,

$$\Omega(p) = \begin{cases} \mathrm{co}\left\{\frac{\pi}{6}, \frac{\pi}{4}\right\}, & p = 0, \\ \left\{\frac{\pi}{6}\right\}, & p \in (0,1), \\ \mathrm{co}\left\{\frac{\pi}{6}, \frac{\pi}{3}\right\}, & p = 1. \end{cases}$$

The following result is a consequence of Theorem 4.1 and Proposition 2.7.

Theorem 4.3. *Let X be a nonempty subset of a real topological vector space $\mathbb{X}$ and $\mathbb{Z}$ a real topological vector space, $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $\mathbb{P}$ a topological space. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ is a vector-valued function and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is an intersectional mapping of C with solid convex cone values. Also we assume that the following conditions:*

(i) $\Omega(p)$ *is nonempty for each* $p \in \mathbb{P}$*;*

(ii) X *is compact;*

(iii) $K(p,x)$ *is compact for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iv) K *is u.s.c. on* $\mathbb{P} \times X$*;*

(v) $W : \mathbb{P} \times X \to 2^{Z}$ *defined by* $W(p,x) = \mathbb{Z} \setminus \big(-\operatorname{int} C(p,x)\big)$ *has closed graph;*

(vi) F *is* $\big(-D\big)$*-u.s.c on* $\mathbb{P} \times X \times X$*;*

(vii) $F(p,x,y)$ *is* $\big(-D(p,x)\big)$*-compact for each* $p \in \mathbb{P}$*,* $x \in X$ *and* $y \in X$.

Then Ω *is u.s.c. on* $\mathbb{P}$.

The following result is a consequence of Theorem 4.1 and Proposition 2.8.

Theorem 4.4. *Let* X *be a nonempty subset of a real topological vector space* $\mathbb{X}$ *and* $\mathbb{Z}$ *a real topological vector space,* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values and* $\mathbb{P}$ *a topological space. Let* $K : \mathbb{P} \times X \to 2^{X} \setminus \{\emptyset\}$*. Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ *is a vector-valued function and that* $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *is an intersectional mapping of* C *with solid convex cone values. Also we assume that the following conditions:*

(i) $\Phi(p)$ *is nonempty for each* $p \in \mathbb{P}$*;*

(ii) X *is compact;*

(iii) $K(p,x)$ *is compact for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iv) K *is u.s.c. on* $\mathbb{P} \times X$*;*

(v) $W : \mathbb{P} \times X \to 2^{Z}$ *defined by* $W(p,x) = \mathbb{Z} \setminus \big(-\operatorname{int} C(p,x)\big)$ *has closed graph;*

(vi) F *is* $\big(-D\big)$*-l.s.c on* $\mathbb{P} \times X \times X$.

Then Φ *is u.s.c. on* $\mathbb{P}$.

Example 4. Let $\mathbb{P}$, $\mathbb{X}$, X, $\mathbb{Z}$ and K be the same as those in Exmaple 3. Let

$$C(p,x) = \left\{ \begin{array}{rl} & \left\{(u,v) \in Z : \left\langle \binom{\cos(x+p)}{\sin(x+p)}, \binom{u}{v} \right\rangle \geq 0 \right\}, \\ D(p,x) = & \left\{(u,v) \in Z : \begin{pmatrix} \cos x' & \sin x' \\ \cos x'' & \sin x'' \end{pmatrix} \binom{u}{v} \in \mathbb{R}^2_+ \right\}, \end{array} \right.$$

where $x' = x + p - \dfrac{\pi}{32}$, $x'' = x + p + \dfrac{\pi}{32}$. Suppose

$$F(p,x,y) = \begin{cases} \operatorname{co}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, (1 + (y - x)) \begin{pmatrix} \cos(x+p) \\ \sin(x+p) \end{pmatrix} \right\}, & \text{if } x \leq y, \\ \left\{ p \begin{pmatrix} -\cos(x+p) \\ -\sin(x+p) \end{pmatrix} \right\}, & \text{otherwise.} \end{cases}$$

Then by Theorem 3.4 and Corollary 3.2, for each $p \in \mathbb{P}$, $\Phi(p) \neq \emptyset$. We also observe that every condition of Theorem 4.4 is satisfied. Accordingly S is u.s.c. on $\mathbb{P}$. Indeed

$$\Phi(p) = \begin{cases} \left\{\dfrac{\pi}{6}\right\} & p \in (0,1], \\ \left[\dfrac{\pi}{6}, \dfrac{\pi}{4}\right] & p = 0. \end{cases}$$

5. Lower Semicontinuity of the Solution Mapping

We next establish that the solution mappings Ω of PGQVEP and Φ of PEQVEP are lower semicontinuous on $\mathbb{P}$ under suitable assumptions.

Theorem 5.1. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

- (i) *$\Omega'(p)$ is nonempty for each $p \in \mathbb{P}$;*
- (ii) *$K(p, x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*
- (iii) *for each $p \in \mathbb{P}$ $K(p, \cdot)$ is affine on X;*
- (iv) *K is u.s.c. on $\mathbb{P} \times X$;*
- (v) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$, is l.s.c. on $\mathbb{P}$;*
- (vi) *the set $\{(p, x, y) \in \mathbb{P} \times X \times X : F(p, x, y) \not\subset -\mathrm{cl}\, C(p, x) = \emptyset\}$ is open;*
- (vii) *F is C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then Ω is l.s.c. on $\mathbb{P}$. Moreover, Φ is l.s.c. on $\mathbb{P}$ if conditions (vi) and (vii) are replaced by the following (viii) and (ix)

- $(viii)$ *the set $\{(p, x, y) \in \mathbb{P} \times X : F(p, x, y) \cap (-\mathrm{cl}\, C(p, x)) \neq \emptyset\}$ is open;*
- (ix) *F is C-weakly properly quasiconcave on $\mathbb{P} \times X \times X$.*

Proof. Suppose $p \in \mathbb{P}$. Let $\mathcal{V}$ be an open set of X with $\mathcal{V} \cap \Omega(p) \neq \emptyset$ and $x_1 \in \mathcal{V} \cap \Omega(p)$. If $x_1 \notin \Omega'(p)$, we can choose $x_2 \in \Omega'(p)$. Let $x_\mu = \mu x_1 + (1-\mu)x_2$, $\mu \in (0, 1)$. Then by condition (vii) for each $\mu \in (0, 1)$, we have

$$F(p, x_\mu, y) \not\subset -\mathrm{cl}\, C(p, x_\mu), \text{ for all } y \in \mu K(p, x_1) + (1-\mu)K(p, x_2).$$

Since $K(p, \cdot)$ is convex for each $p \in \mathbb{P}$, $x_\mu \in K(p, x_\mu)$. Also since $K(p, \cdot)$ is concave for each $p \in \mathbb{P}$,

$$K(p, x_\mu) \subset \mu K(p, x_1) + (1-\mu)K(p, x_2).$$

Hence we have $x_\mu \in \Omega'(p)$ for all $\mu \in (0,1)$. Thus there exists $x \in \mathcal{V} \cap \Omega'(p) \cap \operatorname{co}\{x_1, x_2\}$. If $x_1 \in \Omega'(p)$, then let $x = x_1$.

By condition (vi) for each $y \in K(p,x)$ there exist corredponding neighborhoods P_y of p, U_y of u, and V_y of y such that

$$F(q,u,v) \not\subset -\operatorname{cl} C(q,u), \text{ for all } q \in P_y, \; u \in U_y \text{ and } v \in V_y. \tag{10}$$

Since $K(p,x)$ is compact, there exist $\{y_1,\dots,y_n\} \subset K(p,x)$ such that $\bigcup_{i=1}^n V_{y_i} \supset K(p,x)$. Therefore we have

$$F(q,u,v) \not\subset -\operatorname{cl} C(q,u), \text{ for all } q \in \bigcap_{i=1}^n P_{y_i}, \; u \in \bigcap_{i=1}^n U_{y_i} \text{ and } v \in \bigcup_{i=1}^n V_{y_i}.$$

By condition (iv) there exist neighborhoods P of p and U of x such that

$$K(q,u) \subset \bigcup_{i=1}^n V_{y_i}, \text{ for all } q \in P \text{ and } u \in U.$$

Because of condition (v) there exists a neighborhood P' of p such that for each $q \in P'$ we have

$$u' \in K(q,u'), \text{ for some } u' \in \bigcap_{i=1}^n U_{y_i} \cap \mathcal{V}.$$

Let $\mathcal{P} = \bigcap_{i=1}^n P_{y_i} \cap P \cap P'$. For each $p' \in \mathcal{P}$, there exists corresponding $x' \in \bigcap_{i=1}^n U_{y_i} \cap \mathcal{V}$ such that $x' \in K(p',x')$ and

$$F(p',x',y') \notin -\operatorname{cl} C(p',x'), \text{ for all } y' \in K(p',x'). \tag{11}$$

Therefore for each $p' \in \mathcal{P}$, $\Omega(p') \cap \mathcal{V} \neq \emptyset$. Thus Ω is l.s.c. at p. Since $p \in \mathbb{P}$ is arbitrary, Ω is l.s.c. on $\mathbb{P}$.

Let Ω, (vi) and (vii) be replaced by Φ, (viii) and (ix), respectively. Let (10) and (11) be replaced by the following (12) and (13), respectively in the above proof:

$$F(p,x,y) \cap (-\operatorname{cl} C(p,x)) \neq \emptyset \tag{12}$$

and

$$F(p',x',y') \cap (-\operatorname{cl} C(p',x')) \neq \emptyset, \text{ for all } (p',x',y') \in \mathcal{U}. \tag{13}$$

Then we can obtain lower semicontinuity of Φ on $\mathbb{P}$. ∎

Next we investigate condition (vi) and (viii).

Proposition 5.1. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

(*i*) *F is $C(p,x)$-l.s.c. at (p,x,y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(*ii*) *C is u.s.c. on $\mathbb{P} \times X$.*

Then the set $\mathfrak{U} = \{(p,x,y) \in \mathbb{P} \times X \times X : F(p,x,y) \not\subset -\operatorname{cl} C(p,x)\}$ is open.

Proof. Let $(p,x,y) \in \mathfrak{U}$, i.e., $F(p,x,y) \not\subset -\operatorname{cl} C(p,x)$. Then there exists $z \in F(p,x,y)$ such that $z \notin -\operatorname{cl} C(p,x)$ and $F(p,x,y) \cap (z + \operatorname{int} C(p,x)) \neq \emptyset$. Therefore by condition (i), there exists a neighborhood $\mathcal{U}$ of (p,x,y) such that

$$F(q,u,v) \cap (z + \operatorname{int} C(p,x)) \neq \emptyset, \text{ for all } (q,u,v) \in \mathcal{U}.$$

By condition (ii) and Proposition 2.1, there exists a neighborhood $\mathcal{V}$ of (p,x) such that

$$\big(z + \operatorname{cl} C(p,x)\big) \cap \big(-\operatorname{cl} C(q,u)\big) = \emptyset, \text{ for all } (q,u) \in \mathcal{V}.$$

Accordingly for each $(q,u,v) \in \mathcal{U} \cap \big(\mathcal{V} \times X\big)$ we have

$$F(q,u,v) \not\subset -\operatorname{cl} C(q,u).$$

Hence $\mathcal{U} \cap \big(\mathcal{V} \times X\big) \subset \mathfrak{U}$. Since $(p,x,y) \in \mathfrak{U}$ is arbitrary, $\mathfrak{U}$ is open. ∎

Proposition 5.2. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

(*i*) *F is $C(p,x)$-u.s.c. at (p,x,y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(*ii*) *C is u.s.c. on $\mathbb{P} \times X$.*

Then the set $\mathfrak{U}' = \{(p,x,y) \in \mathbb{P} \times X \times X : F(p,x,y) \cap (-\operatorname{cl} C(p,x)) = \emptyset\}$ is open.

Proof. Let $(p,x,y) \in \mathfrak{U}'$, i.e., $F(p,x,y) \cap (-\operatorname{cl} C(p,x)) = \emptyset$. Therefore by condition (i), there exists a neighborhood $\mathcal{U}$ of (p,x,y) such that

$$F(q,u,v) \cap (-\operatorname{cl} C(p,x)) = \emptyset, \text{ for all } (q,u,v) \in \mathcal{U}.$$

By condition (ii) and Proposition 2.1, there exists a neighborhood $\mathcal{V}$ of (p,x) such that

$$\big(z + \operatorname{cl} C(p,x)\big) \cap \big(-\operatorname{cl} C(q,u)\big) = \emptyset, \text{ for all } (q,u) \in \mathcal{V}.$$

Accordingly for each $(q,u,v) \in \mathcal{U} \cap \big(\mathcal{V} \times X\big)$ we have

$$F(q,u,v) \cap (-\operatorname{cl} C(q,u)) = \emptyset.$$

Hence $\mathcal{U} \cap \big(\mathcal{V} \times X\big) \subset \mathfrak{U}'$. Since $(p,x,y) \in \mathfrak{U}'$ is arbitrary, $\mathfrak{U}'$ is open. ■

Proposition 5.3. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is a intersectional mapping of C with solid convex cone values. Also we assume that the following conditions:*

(i) *F is $D(p,x)$-l.s.c. at (p,x,y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(ii) *C has closed graph.*

Then the set $\mathfrak{U} = \{(p,x,y) \in \mathbb{P} \times X \times X : F(p,x,y) \not\subset -\mathrm{cl}\, C(p,x)\}$ is open.

Proof. Let $(p,x,y) \in \mathfrak{U}$. Then there exists $z \in F(p,x,y)$ such that

$$z \notin -\mathrm{cl}\, C(p,x).$$

Let $d \in \big(-\mathrm{cl}\, C(p,x)\big)^{\mathrm{c}} \cap \big(z - \mathrm{int}\, D(p,x)\big)$. By condition (i), there exists a neighborhood $\mathcal{U}$ of (p,x,y) such that

$$F(q,u,v) \cap (d + \mathrm{int}\, D(p,x)) \neq \emptyset, \text{ for all } (q,u,v) \in \mathcal{U}.$$

Since C has closed graph and D is an intersectional mapping of C, there exists a neighborhood $\mathcal{V}$ of (p,x) such that

$$\big(d + \mathrm{cl}\, D(p,x)\big) \cap \big(-\mathrm{cl}\, C(q,u)\big) = \emptyset, \text{ for all } (q,u) \in \mathcal{V}.$$

Accordingly for each $(q,u,v) \in \mathcal{U} \cap \big(\mathcal{V} \times X\big)$ we have

$$F(q,u,v) \notin -\mathrm{cl}\, C(q,u), \text{ for all } (q,u,v) \in \mathcal{U} \cap \big(\mathcal{V} \times X\big).$$

Hence $\mathcal{U} \cap \big(\mathcal{V} \times X\big) \subset \mathfrak{U}$. Since $(p,x,y) \in \mathfrak{U}$ is arbitrary, $\mathfrak{U}$ is open. ■

Proposition 5.4. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is a intersectional mapping of C with solid convex cone values. Also we assume that the following conditions:*

(i) *F is $D(p,x)$-u.s.c. at (p,x,y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(ii) *$F(p,x,y)$ is $D(p,x)$-comapct for each $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(iii) *C has closed graph.*

Then the set $\mathfrak{U}' = \{(p,x,y) \in \mathbb{P} \times X \times X : F(p,x,y) \cap (-\mathrm{cl}\, C(p,x)) = \emptyset\}$ *is open.*

Proof. Let $(p,x,y) \in \mathfrak{U}$, i.e., $F(p,x,y) \cap (-\mathrm{cl}\, C(p,x)) = \emptyset$ and let $d \in \mathrm{int}\, D(p,x)$. Then for each $z \in F(p,x,y)$ there exists corresponding positive number $t_z > 0$ such that

$$z - t_z d \notin -\mathrm{cl}\, C(p,x).$$

Note that $\bigcup_{z \in F(p,x,y)} z - t_z d + \mathrm{int}\, D(p,x) \supset F(p,x,y)$. By condition (ii), there exists $z_1, \ldots, z_n \in F(p,x,y)$ such that

$$\bigcup_{i=1}^{n} z_i - t_{z_i} d + \mathrm{int}\, D(p,x) \supset F(p,x,y).$$

By condition (i), there exists a neighborhood $\mathcal{U}$ of (p,x,y) such that

$$F(q,u,v) \subset \bigcup_{i=1}^{n} z_i - t_{z_i} d + \mathrm{int}\, D(p,x), \text{ for all } (q,u,v) \in \mathcal{U}.$$

Since D is an intersectional mapping of C and C has closed graph, there exists a neighborhood $\mathcal{V}$ of (p,x) such that

$$\left(\bigcup_{i=1}^{n} z_i - t_{z_i} d + \mathrm{int}\, D(p,x)\right) \cap (-\mathrm{cl}\, C(q,u)) = \emptyset, \text{ for all } (q,u) \in \mathcal{V}.$$

Accordingly for each $(q,u,v) \in \mathcal{U} \cap (\mathcal{V} \times X)$ we have

$$F(q,u,v) \cap (-\mathrm{cl}\, C(q,u)) = \emptyset, \text{ for all } (q,u,v) \in \mathcal{U} \cap (\mathcal{V} \times X).$$

Hence $\mathcal{U} \cap (\mathcal{V} \times X) \subset \mathfrak{U}'$. Since $(p,x,y) \in \mathfrak{U}'$ is arbitrary, $\mathfrak{U}'$ is open. ■

Theorem 5.2. *Let* X *and* $\mathbb{P}$ *be two nonempty subsets of two real topological vector spaces, respectively,* $\mathbb{Z}$ *a real topological vector space and* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values. Let* $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$*. Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$*. Also we assume that the following conditions:*

- (i) $\Omega'(p)$ *is nonempty for each* $p \in \mathbb{P}$*;*
- (ii) $K(p,x)$ *is convex and compact for each* $p \in \mathbb{P}$ *and* $x \in X$*;*
- (iii) *for each* $p \in \mathbb{P}$ $K(p,\cdot)$ *is affine on* X*;*
- (iv) K *is u.s.c. on* $\mathbb{P} \times X$*;*
- (v) $\mathcal{F} : \mathbb{P} \to 2^X$*, defined by* $\mathcal{F}(p) = \{x \in X : x \in K(p,x)\}$*, is l.s.c. on* $\mathbb{P}$*;*
- (vi) F *is* $C(p,x)$*-l.s.c. at* (p,x,y) *for every* $p \in \mathbb{P}$*,* $x \in X$ *and* $y \in X$*;*

(vii) *C is u.s.c. on $\mathbb{P} \times X$;*

($viii$) *F is C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then Ω is l.s.c. on $\mathbb{P}$.

Proof. The result follows from Theorem 5.1 and Proposition 5.1. ∎

Theorem 5.3. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

(i) *$\Omega'(p)$ is nonempty for each $p \in \mathbb{P}$;*

(ii) *$K(p, x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*

(iii) *for each $p \in \mathbb{P}$ $K(p, \cdot)$ is affine on X;*

(iv) *K is u.s.c. on $\mathbb{P} \times X$;*

(v) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$, is l.s.c. on $\mathbb{P}$;*

(vi) *F is $C(p, x)$-u.s.c. at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(vii) *C is u.s.c. on $\mathbb{P} \times X$;*

($viii$) *F is C-weakly properly quasiconcave on $\mathbb{P} \times X \times X$.*

Then Φ is l.s.c. on $\mathbb{P}$.

Proof. The result follows from Theorem 5.1 and Proposition 5.2. ∎

Theorem 5.4. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is a intersectional mapping of C with solid convex cone values. Also we assume that the following conditions:*

(i) *$\Omega'(p)$ is nonempty for each $p \in \mathbb{P}$;*

(ii) *$K(p, x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*

(iii) *for each $p \in \mathbb{P}$ $K(p, \cdot)$ is affine on X;*

(iv) *K is u.s.c. on $\mathbb{P} \times X$;*

(v) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$, is l.s.c. on $\mathbb{P}$;*

(vi) *F is $D(p, x)$-l.s.c. at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(vii) *C has closed graph;*

($viii$) *F is C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then Ω is l.s.c. on $\mathbb{P}$.

Proof. The result follows from Theorem 5.1 and Proposition 5.3. ■

Example 5. Let $\mathbb{P},\ \mathbb{X},\ X,\ K,\ \mathbb{Z},\ C$ and D be the same as those in Exmaple 4. Suppose

$$F'(p,x,y) = (x-\frac{\pi}{4})\begin{pmatrix} \sin(x+p) \\ -\cos(x+p) \end{pmatrix} + (y-x)\begin{pmatrix} \cos(x+p) \\ \sin(x+p) \end{pmatrix}$$

and

$$F(p,x,y) = \begin{cases} \text{co}\left\{M(p), F'(p,x,y) - p(p-2)\begin{pmatrix} \cos(x+p) \\ \sin(x+p) \end{pmatrix}\right\} & \text{if } p<1, \\ \{F'(p,x,y)\} & \text{otherwise,} \end{cases}$$

where

$$M(p) = \left\{p\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \alpha \in [0,2\pi)\right\}.$$

Then by Corollary 3.3, we have

$$\Omega'(p) = \{x \in K(p,x) : F(p,x,y) \not\subset -\text{cl}\, C(p,x),\ \text{for all } y \in K(p,x)\} \neq \emptyset$$

for each $p \in \mathbb{P}$. Hence by Theorem , Ω is l.s.c. on $\mathbb{P}$. Indeed,

$$\Omega(p) = \begin{cases} \left\{x \in X : \frac{\pi}{6} \leq x \leq \frac{\pi}{6} + \frac{2}{3}p(2-p)\right\}, & \text{if } p \in [0,1), \\ \left\{x \in X : x = \frac{\pi}{6}\right\}, & \text{otherwise.} \end{cases}$$

Theorem 5.5. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is a intersectional mapping of C with solid convex cone values. Also we assume that the following conditions:*

(*i*) *$\Phi'(p)$ is nonempty for each $p \in \mathbb{P}$;*

(*ii*) *$K(p,x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*

(*iii*) *for each $p \in \mathbb{P}$ $K(p,\cdot)$ is affine on X;*

(*iv*) *K is u.s.c. on $\mathbb{P} \times X$;*

(*v*) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p,x)\}$, is l.s.c. on $\mathbb{P}$;*

(*vi*) *F is $D(p,x)$-u.s.c. at (p,x,y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(*vii*) *$F(p,x,y)$ is $D(p,x)$-compact for each $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

($viii$) *C has closed graph;*

(ix) *F is C-weakly properly quasiconcave on $\mathbb{P} \times X \times X$.*

Then Φ is l.s.c. on $\mathbb{P}$.

Proof. The result follows from Theorem 5.1 and Proposition 5.4. ∎

Example 6. Let $\mathbb{P}$, $\mathbb{X}$, X, K, $\mathbb{Z}$, C, D and F' be the same as those in Exmaple 5. Suppose

$$M(p) = \left\{ \begin{pmatrix} p \\ p \end{pmatrix} \right\}.$$

Then by Corollary 3.3 and Theorem 5.4, Φ is l.s.c. on $\mathbb{P}$. Indeed,

$$\Phi(p) = \begin{cases} \left\{x \in X : \dfrac{\pi}{6} \le x \le \dfrac{\pi}{6} + \dfrac{2}{3} p(2-p)\right\}, & \text{if } p \in [0,1), \\ \left\{x \in X : x = \dfrac{\pi}{6}\right\}, & \text{otherwise.} \end{cases}$$

Theorem 5.6. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

(i) *$\Omega(p)$ has at least two elements for each $p \in \mathbb{P}$;*

(ii) *$K(p, x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*

(iii) *for each $p \in \mathbb{P}$, $K(p, \cdot)$ is affine on X;*

(iv) *K is u.s.c. on $\mathbb{P} \times X$;*

(v) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$, is l.s.c. on $\mathbb{P}$;*

(vi) *the set $\{(p, x, y) \in \mathbb{P} \times X \times X : F(p, x, y) \not\subset -\operatorname{cl} C(p, x)\}$ is open;*

(vii) *F is strictly C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then Ω is l.s.c. on $\mathbb{P}$.

Proof. Suppose $p \in \mathbb{P}$. Let $\mathcal{V}$ be an open set of X with $\mathcal{V} \cap \Omega(p) \neq \emptyset$ and $x_1 \in \mathcal{V} \cap \Omega(p)$. Let $x_2 \in \Omega(p) \setminus \{x_1\}$. By condition (vii), we can choose $x \in \mathcal{V} \cap \Omega'(p)$. Therefore by Theorem 5.1, Then Ω is l.s.c. on $\mathbb{P}$. ∎

Theorem 5.7. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

(i) $\Phi(p)$ *has at least two elements for each* $p \in \mathbb{P}$*;*

(ii) $K(p, x)$ *is convex and compact for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iii) *for each* $p \in \mathbb{P}$*,* $K(p, \cdot)$ *is affine on* X*;*

(iv) K *is u.s.c. on* $\mathbb{P} \times X$*;*

(v) $\mathcal{F} : \mathbb{P} \to 2^X$*, defined by* $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$*, is l.s.c. on* $\mathbb{P}$*;*

(vi) *the set* $\{(p, x, y) \in \mathbb{P} \times X \times X : F(p, x, y) \cap (-\mathrm{cl}\, C(p, x)) = \emptyset\}$ *is open;*

(vii) F *is strictly* C*-weakly properly quasiconcave on* $\mathbb{P} \times X \times X$.

Then Φ *is l.s.c. on* $\mathbb{P}$.

Proof. Suppose $p \in \mathbb{P}$. Let $\mathcal{V}$ be an open set of X with $\mathcal{V} \cap \Phi(p) \neq \emptyset$ and $x_1 \in \mathcal{V} \cap \Phi(p)$. Let $x_2 \in \Phi(p) \setminus \{x_1\}$. By condition (vii), we can choose $x \in \mathcal{V} \cap \Phi'(p)$. Therefore by Theorem 5.1, Then Φ is l.s.c. on $\mathbb{P}$. ■

Theorem 5.8. *Let* X *and* $\mathbb{P}$ *be two nonempty subsets of two real topological vector spaces, respectively,* $\mathbb{Z}$ *a real topological vector space and* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values. Let* $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$*. Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$*. Also we assume that the following conditions:*

(i) $\Omega(p)$ *has at least two elements for each* $p \in \mathbb{P}$*;*

(ii) $K(p, x)$ *is convex and compact for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iii) *for each* $p \in \mathbb{P}$*,* $K(p, \cdot)$ *is affine on* X*;*

(iv) K *is u.s.c. on* $\mathbb{P} \times X$*;*

(v) $\mathcal{F} : \mathbb{P} \to 2^X$*, defined by* $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$*, is l.s.c. on* $\mathbb{P}$*;*

(vi) F *is* $C(p, x)$*-l.s.c. at* (p, x, y) *for every* $p \in \mathbb{P}$*,* $x \in X$ *and* $y \in X$*;*

(vii) C *is u.s.c. on* $\mathbb{P} \times X$.

($viii$) F *is strictly* C*-weakly quasiconcave on* $\mathbb{P} \times X \times X$.

Then Ω *is l.s.c. on* $\mathbb{P}$.

Proof. The result follows from Theorem 5.6 and Proposition 5.1. ■

Theorem 5.9. *Let* X *and* $\mathbb{P}$ *be two nonempty subsets of two real topological vector spaces, respectively,* $\mathbb{Z}$ *a real topological vector space and* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values. Let* $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$*. Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$*. Also we assume that the following conditions:*

(i) $\Phi(p)$ *has at least two elements for each* $p \in \mathbb{P}$*;*

(ii) $K(p, x)$ *is convex and compact for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iii) *for each* $p \in \mathbb{P}$*,* $K(p, \cdot)$ *is affine on* X*;*

(iv) *K is u.s.c. on $\mathbb{P} \times X$;*

(v) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$, is l.s.c. on $\mathbb{P}$;*

(vi) *F is $C(p, x)$-u.s.c. at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(vii) *$F(p, x, y)$ is $D(p, x)$-compact for each $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

$(viii)$ *C is u.s.c. on $\mathbb{P} \times X$.*

(ix) *F is strictly C-weakly properly quasiconcave on $\mathbb{P} \times X \times X$.*

Then Φ is l.s.c. on $\mathbb{P}$.

Proof. The result follows from Theorem 5.7 and Proposition 5.2. ■

Theorem 5.10. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is a intersectional mapping of C with solid convex cone values. Also we assume that the following conditions:*

(i) *$\Omega(p)$ has at least two elements for each $p \in \mathbb{P}$;*

(ii) *$K(p, x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*

(iii) *for each $p \in \mathbb{P}$ $K(p, \cdot)$ is affine on X;*

(iv) *K is u.s.c. on $\mathbb{P} \times X$;*

(v) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$, is l.s.c. on $\mathbb{P}$;*

(vi) *F is $D(p, x)$-l.s.c. at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*

(vii) *C has closed graph.*

$(viii)$ *F is C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then Ω is l.s.c. on $\mathbb{P}$.

Proof. The result follows from Theorem 5.6 and Proposition 5.3. ■

Theorem 5.11. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$ and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is a intersectional mapping of C with solid convex cone values. Also we assume that the following conditions:*

(i) *$\Phi(p)$ has at least two elements for each $p \in \mathbb{P}$;*

(ii) *$K(p, x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*

(iii) *for each $p \in \mathbb{P}$ $K(p, \cdot)$ is affine on X;*

(iv) *K is u.s.c. on $\mathbb{P} \times X$;*

(v) $\mathcal{F} : \mathbb{P} \to 2^X$, *defined by* $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$, *is l.s.c. on* $\mathbb{P}$*;*

(vi) F *is* $D(p, x)$*-u.s.c. at* (p, x, y) *for every* $p \in \mathbb{P}$, $x \in X$ *and* $y \in X$*;*

(vii) $F(p, x, y)$ *is* $D(p, x)$*-compact for each* $p \in \mathbb{P}$, $x \in X$ *and* $y \in X$*;*

($viii$) C *has closed graph.*

(ix) F *is* C*-weakly properly quasiconcave on* $\mathbb{P} \times X \times X$.

Then Ω *is l.s.c. on* $\mathbb{P}$.

Proof. The result follows from Theorem 5.7 and Proposition 5.4. ∎

6. Continuity of the Solution Mapping

By combining results established in Sections 4 and 5, we have the following results concering continuity of the solution mappings Ω and Φ, respectively.

Theorem 6.1. *Let* X *and* $\mathbb{P}$ *be two nonempty subsets of two real topological vector spaces, respectively,* $\mathbb{Z}$ *a real topological vector space and* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values. Let* $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. *Suppose that* $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. *Also we assume that the following conditions:*

(i) $\Omega'(p)$ *is nonempty for each* $p \in \mathbb{P}$*;*

(ii) X *is compact;*

(iii) $K(p, x)$ *is convex and compact for each* $p \in \mathbb{P}$ *and* $x \in X$*;*

(iv) *for each* $p \in \mathbb{P}$ $K(p, \cdot)$ *is affine on* X*;*

(v) K *is u.s.c. on* $\mathbb{P} \times X$*;*

(vi) $\mathcal{F} : \mathbb{P} \to 2^X$, *defined by* $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$, *is l.s.c. on* $\mathbb{P}$*;*

(vii) F *is* C*-weakly quasiconcave on* $\mathbb{P} \times X \times X$*;*

($viii$) *the set* $\{(p, x, y) \in \mathbb{P} \times X \times X : F(p, x, y) \not\subset -\mathrm{cl}\, C(p, x) = \emptyset\}$ *is open;*

(ix) *the set* $\{(p, x, y) \in \mathbb{P} \times X : F(p, x, y) \subset -\mathrm{int}\, C(p, x)\}$ *is open.*

Then Ω *is continuous on* $\mathbb{P}$. *Moreover,* Φ *is continuous on* $\mathbb{P}$ *if conditions (vii), (viii) and (ix) are replaced by the following (x), (xi) and (xii)*

(x) F *is* C*-weakly properly quasiconcave on* $\mathbb{P} \times X \times X$*;*

(xi) *the set* $\{(p, x, y) \in \mathbb{P} \times X : F(p, x, y) \cap (-\mathrm{cl}\, C(p, x)) \neq \emptyset\}$ *is open;*

(xii) *the set* $\{(p, x, y) \in \mathbb{P} \times X : F(p, x, y) \cap (-\mathrm{int}\, C(p, x)) \neq \emptyset\}$ *is open.*

Proof. The result follows from Theorems 4.1 and 5.1. ■

Theorem 6.2. *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $F : \mathbb{P} \times X \times X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$. Also we assume that the following conditions:*

(i) *$\Omega(p)$ is nonempty for each $p \in \mathbb{P}$;*

(ii) *X is compact;*

(iii) *$K(p, x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*

(iv) *for each $p \in \mathbb{P}$ $K(p, \cdot)$ is affine on X;*

(v) *K is u.s.c. on $\mathbb{P} \times X$;*

(vi) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p, x)\}$, is l.s.c. on $\mathbb{P}$;*

(vii) *F is strictly C-weakly quasiconcave on $\mathbb{P} \times X \times X$;*

$(viii)$ *the set $\{(p, x, y) \in \mathbb{P} \times X \times X : F(p, x, y) \not\subset -\mathrm{cl}\, C(p, x) = \emptyset\}$ is open;*

(ix) *the set $\{(p, x, y) \in \mathbb{P} \times X : F(p, x, y) \subset -\mathrm{int}\, C(p, x)\}$ is open.*

Then Ω is continuous on $\mathbb{P}$. Moreover, Φ is continuous on $\mathbb{P}$ if conditions (vii), $(viii)$ and (ix) are replaced by the following (x), (xi) and (xii)

(x) *F is strictly C-weakly properly quasiconcave on $\mathbb{P} \times X \times X$;*

(xi) *the set $\{(p, x, y) \in \mathbb{P} \times X : F(p, x, y) \cap (-\mathrm{cl}\, C(p, x)) \neq \emptyset\}$ is open;*

(xii) *the set $\{(p, x, y) \in \mathbb{P} \times X : F(p, x, y) \cap (-\mathrm{int}\, C(p, x)) \neq \emptyset\}$ is open.*

Proof. The result follows from Theorems 4.1, 5.6 and 5.7. ■

The following result is a consequence of Theorems 6.1, 6.2 and Propositions 3.1, 5.3 and 5.4.

Corollary 6.1. *In Theorems 6.1 and 6.2,*

(i) *condition (viii) can be replaced by the following conditions:*

(a) F is $D(p, x)$-l.s.c. at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;

(b) C has closed graph;

(ii) *condition (xi) can be replaced by the following conditions:*

(a) F is $D(p, x)$-u.s.c. at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;

(b) $F(p, x, y)$ is $D(p, x)$-comapct for each $p \in \mathbb{P}$, $x \in X$ and $y \in X$;

(c) C has closed graph;

(iii) *condition(ix) can be replaced by the following conditions:*

(a) $W : \mathbb{P} \times X \to 2^Z$ defined by $W(p, x) = \mathbb{Z} \setminus \big(-\mathrm{int}\, C(p, x)\big)$ has closed graph;

(b) *F is $\big(-D\big)$-u.s.c on $\mathbb{P}\times X\times X$;*

(c) *$F(p,x,y)$ is $\big(-D(p,x)\big)$-compact for each $p\in\mathbb{P}$, $x\in X$ and $y\in X$;*

(iv) *condition (xii) can be replaced by the following conditions:*

(a) *$W:\mathbb{P}\times X\to 2^Z$ defined by $W(p,x)=\mathbb{Z}\setminus\big(-\mathrm{int}\,C(p,x)\big)$ has closed graph;*

(b) *F is $\big(-D\big)$-l.s.c on $\mathbb{P}\times X\times X$.*

Example 7. Let $\mathbb{P}$, $\mathbb{X}$, X, K, $\mathbb{Z}$, C, D and F' be the same as those in Exmaple 5. Suppose

$$G(p,x,y)=\left\{\begin{pmatrix}(x+p)\cos(x+p)\\(y+p)\sin(y+p)\end{pmatrix}\right\}.$$

and

$$F(p,x,y)=\mathrm{co}\,\{F'(p,x,y),G(p,x,y)\}.$$

Then by Corollaries 3.2 and 6.1, Ω is continuous on $\mathbb{P}$. Indeed,

$$\Omega(p)=\big\{x\in X:\frac{\pi}{6}\le x\le\frac{\pi}{6}+\frac{2}{3}p(2-p)\big\}.$$

Example 8. Let $\mathbb{P}$, $\mathbb{X}$, X, K, $\mathbb{Z}$, C, D and G be the same as those in Exmaple 7. Suppose

$$F(p,x,y)=\mathrm{co}\,\left\{F'(p,x,y),-(p^2-4)\begin{pmatrix}\cos(x+p)\\\sin(x+p)\end{pmatrix}\right\}.$$

Then by Corollaries 3.2 and 6.1, Ω is continuous on $\mathbb{P}$. Indeed,

$$\Phi(p)=\big\{x\in X:\frac{\pi}{6}\le x\le\frac{\pi}{6}+\frac{2}{3}p(2-p)\big\}.$$

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