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LOCAL UNIFORM LINEAR OPENNESS OF MULTIFUNCTIONS AND CALCULUS OF BOULIGAND –SEVERI AND CLARKE TANGENT SETS

Corneliu Ursescu

Abstract. The paper concerns calculus of certain tangent sets in the multifunction setting. To this end the multifunction linear openness is defined in linear topological spaces. The relationship with some metric notions is investigated and some metric openness results are derived.

1. Introduction

Let X and Y be generic spaces, let $F : X \to Y$ be a multifunction, and let $(a, b) \in X \times Y$. In this paper we are concerned with the estimation of certain tangent sets to $F^{-1}(b)$ at a. A generic tangency concept τ assigns a set $\tau_A(a) \subseteq X$ to a set $A \subseteq X$ and to a point $a \in X$, and $\tau_A(a)$ is called the τ-tangent set to A at a. To estimate $\tau_{F^{-1}(b)}(a)$, we follow the example set by Aubin [1] and define a multifunction $\tau_F(a, b) : X \to Y$ through the equality

$$\operatorname{graph}(\tau_F(a, b)) = \tau_{\operatorname{graph}(F)}(a, b).$$

Afterwards we compare the sets $\tau_{F^{-1}(b)}(a)$ and $(\tau_F(a, b))^{-1}(0)$. The inclusion

$$(\tau_F(a, b))^{-1}(0) \subseteq \tau_{F^{-1}(b)}(a)$$

is expected. Expected inclusions are proved (see Section 4) for some tangency concepts (see Section 2) by using linear openness of F (see Section 3).

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Multifunction linear openness is further investigated in linear topological spaces and, in normed spaces, it is characterized through a metric property, namely the multifunction ω-openness (see Section 5), which is powerful enough to characterize multifunction metric regularity (see Section 6). A counterexample (see Section 11) shows that it is possible for a closed graph multifunction $F : l^2(R) \to R$ to be ω-open at a point in case $\omega = 1$, but not metrically regular near any point.

Locally closed graph results concerning ω-openness are derived in general metric spaces (see Section 7) as well as in particular metric spaces, which resemble normed spaces (see Section 8). These results are refined in normed spaces (see Section 10) by using an elementary tangency concept and a classical tangency concept (see Section 9).

The counterexample in Section 11 shows also that some locally closed graph results concerning ω-openness do not hold in infinite dimensional normed spaces. In the final Section 12, there are discussed further properties of the tangency concepts described in Section 2.

2. Tangency Concepts

In this section we describe three specific tangency concepts of which definitions are based on a nonempty family $\mathcal{S}$ of nonempty sets $S \subseteq (0, +\infty)$ (cf. [28, pp. 106, 107]). The first tangency concept makes sense in linear spaces, whereas the other two tangency concepts make sense in linear topological spaces.

First, we denote by κ the tangency concept which assigns to a set $A \subseteq X$ and to a point $a \in X$ the set $\kappa_A(a)$ of all points $x \in X$ with the property that: there exists $S \in \mathcal{S}$ such that $a + sx \in A$ whenever $s \in S$.

Second, we denote by K the tangency concept which assigns to a set $A \subseteq X$ and to a point $a \in X$ the set $K_A(a)$ of all points $x \in X$ with the property that: for every neighborhood U of the origin in X there exists $S \in \mathcal{S}$ such that $\emptyset \neq (a + s(x + U)) \cap A$ whenever $s \in S$.

Third, we denote by $\mathcal{K}$ the tangency concept which assigns to a set $A \subseteq X$ and to a point $a \in X$ a set $\mathcal{K}_A(a) \subseteq X$ which depends on whether or not a belongs to the closure $\overline{A}$ of A. If $a \notin \overline{A}$, then $\mathcal{K}_A(a) = \emptyset$. If $a \in \overline{A}$, i.e. $(a+\Gamma) \cap A \neq \emptyset$ whenever Γ is a neighborhood of the origin in X, then $\mathcal{K}_A(a)$ is the set of all points $x \in X$ with the property that: for every neighborhood U of the origin in X there exist a neighborhood Γ of the origin in X and $S \in \mathcal{S}$ such that $\emptyset \neq (a+\gamma+s(x+U)) \cap A$ whenever $s \in S$, $\gamma \in \Gamma$ and $a + \gamma \in A$.

Obviously, $\kappa_A(a) \subseteq K_A(a)$, $\mathcal{K}_A(a) \subseteq K_A(a)$, $0 \in \kappa_A(a)$ if $a \in A$, and $0 \in \mathcal{K}_A(a)$ if $a \in \overline{A}$.

In the particular case that

$$\mathcal{S} = \{S \subseteq (0, +\infty); \forall r > 0, \emptyset \neq (0, r) \cap S\}, \tag{1}$$

the closed cone $K_A(a)$ originates from the half-lines which are tangent to A at a, a notion considered in the same journal volume by Bouligand [5, p. 32] and Severi [22, p. 99]. The seminal feature of the two papers is emphasized in [14, p. 133]. A rough sketch of the history of the matter is given in [27].

In the particular case that

$$\mathcal{S} = \{S \subseteq (0,+\infty); \exists r > 0, (0,r) \subseteq S\}, \tag{2}$$

the closed cone $K_A(a)$ has been considered in [24, p. 151], whereas the closed convex cone $\mathcal{K}_A(a)$, which originates from Clarke [6, Proposition 3.7], has been considered in [21, p. 335, eq. (2.6)].

In the general case, $K_A(a)$ and $\mathcal{K}_A(a)$ are closed, but $\kappa_A(a)$, $K_A(a)$, and $\mathcal{K}_A(a)$ may fail to be cones, whereas $\mathcal{K}_A(a)$ may fail to be convex. For example, in case of the family

$$\mathcal{S} = \{\{1\}\}, \tag{3}$$

it follows: $\kappa_A(a) = A - a$; $K_A(a) = \overline{A} - a$; $\mathcal{K}_A(a) = \overline{A} - a$ if $a \in \overline{A}$. Therefore $\kappa_A(a)$ is a cone if and only if $A - a$ is a cone, and so on. On the other hand, $\kappa_A(a)$, $K_A(a)$, and $\mathcal{K}_A(a)$ are always cones provided that $tS \in \mathcal{S}$ whenever $S \in \mathcal{S}$ and $t > 0$. Each of the families (1) and (2) enjoys this $\mathcal{S}$-property, but the family (3) does not. Neither does the family

$$\mathcal{S} = \{\{s\}; s \in (0,1]\}. \tag{4}$$

Further $\mathcal{S}$-properties and the corresponding κ-, K-, and $\mathcal{K}$-properties are discussed in Section 12.

Now, $y \in \kappa(a,b)(x)$ if and only if there exists $S \in \mathcal{S}$ such that $b + sy \in F(a + sx)$ whenever $s \in S$, therefore there always holds not only the expected inclusion

$$(\kappa_F(a,b))^{-1}(0) \subseteq \kappa_{F^{-1}(b)}(a),$$

but also the equality

$$\kappa_{F^{-1}(b)}(a) = (\kappa_F(a,b))^{-1}(0).$$

Further, $y \in K_F(a,b)(x)$ if and only if for every neighborhood U of the origin in X and for every neighborhood V of the origin in Y there exists $S \in \mathcal{S}$ such that

$$\emptyset \neq (b + s(y+V)) \cap F(a + s(x+U)) \tag{5}$$

whenever $s \in S$, therefore

$$K_{F^{-1}(b)}(a) \subseteq (K_F(a,b))^{-1}(0),$$

and the equality

$$K_{F^{-1}(b)}(a) = (K_F(a,b))^{-1}(0) \tag{6}$$

is strongly expected.

Finally, $y \in \mathcal{K}_F(a,b)(x)$ if and only if $(a,b) \in \overline{\mathrm{graph}(F)}$ and for every neighborhood U of the origin in X and for every neighborhood V of the origin in Y there exist a neighborhood Γ of the origin in X, a neighborhood Δ of the origin in Y, and $S \in \mathcal{S}$ such that

$$\emptyset \neq (b + \delta + s(y+V)) \cap F(a + \gamma + s(x+U)) \tag{7}$$

whenever $s \in S$, $\gamma \in \Gamma$, $\delta \in \Delta$, and $b + \delta \in F(a+\gamma)$. No elementary relation involves the sets $\mathcal{K}_{F^{-1}(b)}(a)$ and $(\mathcal{K}^{-1}(a,b))(0)$, but the inclusion

$$(\mathcal{K}_F(a,b))^{-1}(0) \subseteq \mathcal{K}_{F^{-1}(b)}(a) \tag{8}$$

is expected.

In case of the $\mathcal{S}$-family (2), instances of the equality (6) appear in [25, p. 564, eq. (C)], [31, pp. 148, 149, Theorem 2], whereas instances of the inclusion (8) appear in [3, p. 74, Corollary 2.2], [20, p. 153, eq. (5.2)], [21, p. 349, eq. (5.16)].

Relations (6) and (8) will be proved (see Section 4), in the particular case that $S \cap (0,r) \in \mathcal{S}$ for all $r > 0$ and for all $S \in \mathcal{S}$. Each of the $\mathcal{S}$-families (1) and (2) exemplifies the particular case above, but the $\mathcal{S}$-families (3) and (4) do not. In spite of this shortcoming, the tangency concept κ corresponding to the $\mathcal{S}$-family (4) is more valuable than the tangency concept K provided by the $\mathcal{S}$-family (1) in that the K-results of openness can be derived from better κ-results of openness (see Section 10).

3. Linear Openness

To begin with, we recall the notions of multifunction openness and multifunction near openness. Let X and Y be topological spaces, and let $F : X \to Y$ be a multifunction. The multifunction F is said to be *open at a point* $(x,y) \in \mathrm{graph}(F)$ if for every neighborhood U of x the set $F(U)$ is a neighborhood of y; the multifunction F is said to be open *on a set* $W \subseteq \mathrm{graph}(F)$ if F is open at every $(x,y) \in W$; the multifunction F is said to be *open* if F is open on $\mathrm{graph}(F)$.

A twin definition makes use of the closure of the set $F(U)$. The multifunction F is said to be *nearly open at a point* $(x,y) \in \mathrm{graph}(F)$ if for every neighborhood U of x the set $\overline{F(U)}$ is a neighborhood of y; the multifunction F is said to be *nearly open on a set* $W \subseteq \mathrm{graph}(F)$ if F is nearly open at every $(x,y) \in W$; the multifunction F is said to be *nearly open* if F is nearly open on $\mathrm{graph}(F)$.

Openness at a point implies near openness at that point, and the converse holds too under appropriate hypotheses ([23, p. 439, Lemma 3]).

Note that F is open if and only if F maps open subsets of X into open subsets of Y, whereas F is nearly open if and only if F maps open subsets of X into nearly open subsets of Y (cf. [13, p. 160]). Recall that a set S is said to be nearly open if $\overline{S}$ is a neighborhood of S. Near openness of sets appears in [17, p. 451]. Near openness of functions appears in [18, p. 47, Definition 47].

Now, let $(x, y) \in \operatorname{graph}(F)$, let $\mathcal{U}$ be a base for the neighborhood system of x, and let $\mathcal{V}$ be a base for the neighborhood system of y. Then F is open at (x, y) if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$V \subseteq F(U),$$

whereas F is nearly open at (x, y) if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$V \subseteq \overline{F(U)}.$$

Further, let X and Y be uniform spaces, let $\mathcal{U}$ be a base for the uniformity on X, and let $\mathcal{V}$ be a base for the uniformity on Y. In this case, the family of sets $\{U[x]; U \in \mathcal{U}\}$ is a base for the neighborhood system of x, whereas the family of sets $\{V[y]; V \in \mathcal{V}\}$ is a base for the neighborhood system of y. Therefore F is open at (x, y) if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$V[y] \subseteq F(U[x]), \tag{9}$$

whereas F is nearly open at (x, y) if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$V[y] \subseteq \overline{F(U[x])}. \tag{10}$$

Special openness and near openness of multifunctions can be defined by using the special inclusions (9) and (10).

The multifunction F is said to be *uniformly open on a set* $W \subseteq \operatorname{graph}(F)$ if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that the inclusion (9) holds whenever $(x, y) \in W$; the multifunction F is said to be *locally uniformly open* if for every $(a, b) \in \operatorname{graph}(F)$ there exists a neighborhood W of (a, b) such that F is uniformly open on $W \cap \operatorname{graph}(F)$; the multifunction F is said to be *uniformly open* if F is uniformly open on $\operatorname{graph}(F)$.

The multifunction F is said to be *uniformly nearly open on a set* $W \subseteq \operatorname{graph}(F)$ if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that the inclusion (10) holds whenever $(x, y) \in W$; the multifunction F is said to be *locally uniformly nearly open* if for every $(a, b) \in \operatorname{graph}(F)$ there exists a neighborhood W of (a, b) such that F is uniformly nearly open on $W \cap \mathrm{graph}(F)$; the multifunction F is said to be *uniformly nearly open* if F is uniformly nearly open on $\mathrm{graph}(F)$.

Uniform openness implies uniform near openness, and the converse holds too under appropriate hypotheses ([12, p. 202, **36** Lemma]). Local uniform openness implies local uniform near openness, and the converse holds too under appropriate hypotheses ([33, p. 145, Theorem 4]).

Sometimes the phrase "almost open" is used instead of the phrase "nearly open". The phrases "uniformly open" and "uniformly almost open" appear in [19, p. 505, **(2,1)** Theorem]. The properties labelled by these phrases appear earlier in [12, p. 202, **36** Lemma]. Function openness and function uniform openness, which appear in [12, pp. 90, 202], are prerequisites for introducing those yet unlabelled multifunction properties.

Finally, let X and Y be linear topological spaces, let $\mathcal{U}$ be a base for the neighborhood system of the origin in X, and let $\mathcal{V}$ be a base for the neighborhood system of the origin in Y. In this case, the family of sets $\{x+U; U \in \mathcal{U}\}$ is a base for the neighborhood system of x, whereas the family of sets $\{y+V; V \in \mathcal{V}\}$ is a base for the neighborhood system of y. Therefore F is open at (x, y) if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$y + V \subseteq F(x + U),$$

whereas F is nearly open at (x, y) if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$y + V \subseteq \overline{F(x + U)}.$$

The two inclusions above suggest for us to consider the special inclusions

$$y + sV \subseteq F(x + sU) \tag{11}$$

$$y + sV \subseteq \overline{F(x + sU)} \tag{12}$$

and the corresponding special definitions. A first definition introduces the notions of linear openness and near linear openness.

Definition 1. The multifunction F is said to be *linearly open at a point* $(x, y) \in \mathrm{graph}(F)$ if for every neighborhood U of the origin in X there exists a neighborhood V of the origin in Y such that the inclusion (11) holds whenever $s \in (0, 1]$; the multifunction F is said to be *linearly open on a set* $W \subseteq \mathrm{graph}(F)$ if F is linearly open at every $(x, y) \in W$; the multifunction F is said to be *linearly open* if F is linearly open on $\mathrm{graph}(F)$.

The multifunction F is said to be *nearly linearly open at a point* $(x, y) \in \mathrm{graph}(F)$ if for every neighborhood U of the origin in X there exists a neighborhood

V of the origin in Y such that the inclusion (12) holds whenever $s \in (0, 1]$; the multifunction F is said to be *nearly linearly open on a set* $W \subseteq \operatorname{graph}(F)$ if F is nearly linearly open at every $(x, y) \in W$; the multifunction F is said to be *nearly linearly open* if F is nearly linearly open on $\operatorname{graph}(F)$.

A second definition introduces the specialized notions of uniform linear openness and uniform near linear openness.

Definition 2. The multifunction F is said to be *uniformly linearly open on a set* $W \subseteq \operatorname{graph}(F)$ if for every neighborhood U of the origin in X there exists a neighborhood V of the origin in Y such that the inclusion (11) holds whenever $s \in (0, 1]$ and $(x, y) \in W$; the multifunction F is said to be *locally linearly open* if for every $(a, b) \in \operatorname{graph}(F)$ there exists a neighborhood W of (a, b) such that F is uniformly linearly open on $W \cap \operatorname{graph}(F)$; the multifunction F is said to be *uniformly linearly open* if F is uniformly linearly open on $\operatorname{graph}(F)$.

The multifunction F is said to be *uniformly nearly linearly open on a set* $W \subseteq \operatorname{graph}(F)$ if for every neighborhood U of the origin in X there exists a neighborhood V of the origin in Y such that the inclusion (12) holds whenever $s \in (0, 1]$ and $(x, y) \in W$; the multifunction F is said to be *locally nearly linearly open* if for every $(a, b) \in \operatorname{graph}(F)$ there exists a neighborhood W of (a, b) such that F is uniformly nearly linearly open on $W \cap \operatorname{graph}(F)$; the multifunction F is said to be *uniformly nearly linearly open* if F is uniformly nearly linearly open on $\operatorname{graph}(F)$.

Note uniform linear openness implies local uniform linear openness, but the converse may fail. The multifunction $F : R \to R$ given by $\operatorname{graph}(F) = \{(r, r); r \in R\}$ is uniformly linearly open, whereas the multifunction $F : R \to R$ given by $\operatorname{graph}(F) = \{(r, r); r \in R, |r| < 1\}$ is only locally uniformly linearly open.

4. Main Results

Now we are in a position to state the main result concerning relations (6) and (8).

Theorem 1. *Let X and Y be linear topological spaces, and let $(0, r) \cap S \in \mathcal{S}$ for all $S \in \mathcal{S}$ and for all $r > 0$. Let the multifunction F be locally uniformly linearly open. Then the relations* (6) *and* (8) *hold for all $(a, b) \in \operatorname{graph}(F)$.*

The global Theorem 1 above is a straightforward corollary of the point-wise Theorem 2 below.

Theorem 3. *Let X and Y be linear topological spaces, and let $(0, r) \cap S \in \mathcal{S}$ for all $S \in \mathcal{S}$ and for all $r > 0$. Let $(a, b) \in \operatorname{graph}(F)$ and let there exist a* *neighborhood W of $(a,b) \in \operatorname{graph}(F)$ such that F is uniformly linearly open on $W \cap \operatorname{graph}(F)$. Then the relations* (6) *and* (8) *hold.*

The proof of Theorem 2 will be given at the end of this section. This proof relies on two lemmas. A first lemma concerns the local character of the tangency concepts κ, K, and $\mathcal{K}$. We say that *a tangency concept τ has a local character* if $\tau_A(a) \subseteq \tau_{A\cap(a+P)}(a)$ whenever P is a neighborhood of the origin in X.

Lemma 1. *Let X be a linear topological space and let $(0,r)\cap S \in \mathcal{S}$ for all $S \in \mathcal{S}$ and for all $r > 0$. Then all of the tangency concepts κ, K, and $\mathcal{K}$ have a local character.*

Proof. Let P be a neighborhood of the origin in X. First we assert that, if $x \in \kappa_A(a)$, then $x \in \kappa_{A\cap(a+P)}(a)$. Choose a real number $r > 0$ such that $(0,r)x \subseteq P$. By the definition of κ tangency, there exists $S \in \mathcal{S}$ such that $a + sx \subseteq A$ whenever $s \in S$. Note $(0,r)\cap S \in \mathcal{S}$ and let $s \in (0,r)\cap S$. Since $s \in S$, it follows $a+sx \in A$. Since $s \in (0,r)$, it follows $sx \in P$. To conclude, $a + su \in A \cap (a + P)$, and our first assertion is justified.

Second we assert that, if $x \in K_A(a)$, then $x \in K_{A\cap(a+P)}(a)$. Let U be a neighborhood of the origin in X. We can suppose, taking a smaller U if necessary, that there exists a real number $r > 0$ such that $(0,r)(x+U) \subseteq P$. By the definition of K tangency, there exists $S \in \mathcal{S}$ such that $\emptyset \neq (a + s(x + U)) \cap A$ whenever $s \in S$. Note $(0,r) \cap S \in \mathcal{S}$ and let $s \in (0,r) \cap S$. Since $s \in S$, it follows there exists $u \in U$ such that $a + s(x + u) \in A$. Since $s \in (0,r)$ and $u \in U$, it follows $s(x+u) \in P$. To conclude, $a + s(x + u) \in A \cap (a + P)$, and our second assertion is justified.

Third we assert that, if $x \in \mathcal{K}_A(a)$, then $x \in \mathcal{K}_{A\cap(a+P)}(a)$. Let U be a neighborhood of the origin in X. We can suppose, taking a smaller U if necessary, that there exist a neighborhood Q of the origin in X and a real number $r > 0$ such that $Q + (0,r)(x+U) \subseteq P$. By the definition of $\mathcal{K}$ tangency, there exist a neighborhood Γ of the origin in X and $S \in \mathcal{S}$ such that $\emptyset \neq (a+\gamma+s(s+U))\cap A$ whenever $s \in S$, $\gamma \in \Gamma$, and $a + \gamma \in A$. Note $(0,r) \cap S \in \mathcal{S}$ and $\Gamma \cap Q$ is a neighborhood of the origin in X, let $s \in (0,r) \cap S$, and let $\gamma \in \Gamma \cap Q$ such that $a + \gamma \in A \cap (a + P)$. Since $s \in S$, $\gamma \in \Gamma$, and $a + \gamma \in A$, it follows there exists $u \in U$ such that $a + \gamma + s(x + u) \in A$. Since $\gamma \in Q$, $s \in (0,r)$, and $u \in U$, it follows $\gamma + s(x+u) \in P$. To conclude, $a+\gamma+s(x+u) \in A\cap(A+P)$, and our third assertion is justified. ∎

A second lemma concerns stronger versions of the relations (5) and (7), namely the inclusions

$$b + s(y + V) \subseteq F(a + s(x + U)), \tag{13}$$

$$b + \delta + s(y + V) \subseteq F(a + \gamma + s(x + U)). \tag{14}$$

Lemma 2. *Let X and Y be linear topological spaces, and let $S \cap (0, r) \in \mathcal{S}$ for all $r > 0$ and for all $S \in \mathcal{S}$. Let $W \subseteq X \times Y$ be an open set such that the multifunction F is uniformly linearly open on $W \cap \operatorname{graph}(F)$. Then for every neighborhood U of the origin in X there exists a neighborhood V of the origin in Y such that:*

- *for every $(a, b) \in W \cap \operatorname{graph}(F)$ and for every $(x, y) \in \operatorname{graph}(K_F(a, b))$ there exists $S \in \mathcal{S}$ such that the inclusion (13) holds whenever $s \in S$.*
- *for every $(a, b) \in W \cap \operatorname{graph}(F)$ and for every $(x, y) \in \operatorname{graph}(\mathcal{K}_F(a, b))$ there exist a neighborhood Γ of the origin in X, a neighborhood Δ of the origin in Y, and $S \in \mathcal{S}$ such that the inclusion (14) holds whenever $s \in S$, $\gamma \in \Gamma$, $\delta \in \Delta$, and $b + \delta \in F(a + \gamma)$.*

Proof. Let U be a neighborhood of the origin in X and consider a neighborhood U' of the origin in X such that $U' + U' \subseteq U$. Since F is uniformly linearly open on $W \cap \operatorname{graph}(F)$, it follows that there exists a neighborhood V' of the origin in Y such that $b' + sV' \subseteq F(a' + sU')$ whenever $s \in (0, 1]$ and $(a', b') \in W \cap \operatorname{graph}(F)$. Let V be a neighborhood of the origin in Y such that $V - V \subseteq V'$. Further, for every $(a', b') \in X \times Y$ and for every $(x, y) \in X \times Y$ denote by $\Sigma(a', b'; x, y)$ the set

$$\{s > 0; \emptyset \neq ((a', b') + s((x, y) + (U' \times V))) \cap W \cap \operatorname{graph}(F)\}.$$

We assert that, if $s \in \Sigma(a', b'; x, y) \cap (0, 1]$, then $b' + s(y + V) \subseteq F(a' + s(x + U))$. Indeed, if $(u', v') \in U' \times V$ and $(a', b') + s((x, y) + (u', v')) \in W \cap \operatorname{graph}(F)$, then $b' + s(y + V) \subseteq (b' + s(y + v')) + sV' \subseteq F(a' + s(x + u') + sU') \subseteq F(a' + s(x + U))$, and our assertion is justified.

Now, let $(a, b) \in W \cap \operatorname{graph}(F)$ and $(x, y) \in \operatorname{graph}(K_F(a, b))$. Since the tangency concept K has a local character and since W is a neighborhood of (a, b), we get $(x, y) \in K_{W \cap \operatorname{graph}(F)}(a, b)$, hence there exists $S \in \mathcal{S}$ such that $S \subseteq \Sigma(a, b; x, y)$. Since $S \cap (0, 1) \in \mathcal{S}$ and the inclusion (13) holds whenever $s \in S \cap (0, 1)$, it follows the first part of the conclusion.

Finally, let $(a, b) \in W \cap \operatorname{graph}(F)$ and $(x, y) \in \operatorname{graph}(\mathcal{K}_F(a, b))$. Since the tangency concept $\mathcal{K}$ has a local character and since W is a neighborhood of (a, b), we get $(x, y) \in \mathcal{K}_{W \cap \operatorname{graph}(F)}(a, b)$, hence there exist a neighborhood Γ of the origin in X, a neighborhood Δ of the origin in Y, and $S \in \mathcal{S}$ such that

$$S \subseteq \bigcap_{\substack{(\gamma, \delta) \in \Gamma \times \Delta \\ (a + \gamma, b + \delta) \in W \cap \operatorname{graph}(F)}} \Sigma(a + \gamma, b + \delta; x, y).$$

We can suppose, taking smaller Γ and Δ if necessary, that $(a+\Gamma)\times(b+\Delta)\subseteq W$. Since $S\cap(0,1)\in\mathcal{S}$ and the inclusion (14) holds whenever $s\in S\cap(0,1)$, $\gamma\in\Gamma$, $\delta\in\Delta$, and $b+\delta\in F(a+\gamma)$, it follows the second part of the conclusion. ■

Proof of Theorem 2. First, let $x\in(K_F(a,b))^{-1}(0)$ and let U be a neighborhood of the origin in X. Since $(x,0)\in\operatorname{graph}(K_F(a,b))$, it follows from Lemma 2 that there exists a neighborhood V of the origin in X and $S\in\mathcal{S}$ such that $b+s(0+V)\subseteq F(a+s(x+U))$ whenever $s\in S$. Since $0\in V$, we get $b\in F(a+s(x+U))$ whenever $s\in S$, hence $x\in K_{F^{-1}(b)}(a)$, and the equality (6) follows.

Finally, let $x\in(\mathcal{K}_F(a,b))^{-1}(0)$ and let U be a neighborhood of the origin in X. Since $(x,0)\in\operatorname{graph}(\mathcal{K}_F(a,b))$, it follows from Lemma 2 that there exist a neighborhood V of the origin in Y, a neighborhood Γ of the origin in X, a neighborhood Δ of the origin in Y, and $S\in\mathcal{S}$ such that $b+\delta+s(0+V)\subseteq F(a+\gamma+s(x+U))$ whenever $s\in S$, $\gamma\in\Gamma$, $\delta\in\Delta$ and $b+\delta\in F(a+\gamma)$. Since $0\in\Delta$ and $0\in V$, we get $b\in F(a+\gamma+s(x+U))$ whenever $s\in S$, $\gamma\in\Gamma$, and $b\in F(a+\gamma)$, hence $x\in\mathcal{K}_{F^{-1}(b)}(a)$, and the inclusion (8) follows. ■

5. Linear Openness in Normed Spaces

In this section we characterize uniform linear openness and uniform near linear openness in the general setting of linear topological spaces and in the particular setting of normed spaces.

Lemma 3. *Let X and Y be linear topological spaces.*

The multifunction F is uniformly linearly open on a set $W\subseteq\operatorname{graph}(F)$ if and only if for every neighborhood U of the origin in X there exist a neighborhood V of the origin in Y and a real number $r>0$ such that the inclusion (11) holds for all $s\in(0,r)$ and for all $(x,y)\in W$.

The multifunction F is uniformly nearly linearly open on a set $W\subseteq\operatorname{graph}(F)$ if and only if for every neighborhood U of the origin in X there exist a neighborhood V of the origin in Y and a real number $r>0$ such that the inclusion (12) holds for all $s\in(0,r)$ and for all $(x,y)\in W$.

Proof. To prove the former part of the result, let $W\subseteq\operatorname{graph}(F)$, denote by $q(U,V)$ the statement that the inclusion (11) holds for all $s\in(0,1]$ and for all $(x,y)\in W$, denote by $Q(U,V,r)$ the statement that the inclusion (11) holds for all $s\in(0,r)$ and for all $(x,y)\in W$, note $q(U,V)$ implies $Q(U,V,1)$, $Q(U,V,2)$ implies $q(U,V)$, $Q(U,V,r)$ is equivalent to $Q(tU,tV,r/t)$ whenever $t>0$, and $Q(U',V',r')$ implies $Q(U,V,r)$ whenever $r\le r'$, $V\subseteq V'$, and $U'\subseteq U$. Now, let F be uniformly linearly open on W and let U be a neighborhood of the origin in X. Then there exists a neighborhood V of the origin in X such that $q(U,V)$ is true and therefore there exists a real number $r > 0$, namely $r = 1$, such that $Q(U, V, r)$ is true. Conversely, let for every neighborhood U' of the origin in X there exist a neighborhood V' of the origin in X and a real number $r' > 0$ such that $Q(U', V', r')$ is true, and let U be a neighborhood of the origin in X. Choose a neighborhood U' of the origin in X such that $(0, 1]U' \subseteq U$, choose a neighborhood V' of the origin in Y and a real number $r' > 0$ such that $Q(U', V', r')$ is true, let $t = \min\{r'/2, 1\}$, and let $V = trV'$. Then $Q(U', V', r')$ is equivalent to $Q(tU', V, r'/t)$, which implies $Q(U, V, 2)$, hence $q(U, V)$ is true, and F is uniformly linearly open on W. The latter part of the result can be proved in a similar manner. ■

In case that X and Y are normed spaces, the result above can be further refined by using the inclusions

$$B(y, \omega\epsilon) \subseteq F(B(x, \epsilon)), \tag{15}$$

$$B(y, \omega\epsilon) \subseteq \overline{F(B(x, \epsilon))}, \tag{16}$$

where $\omega > 0$ is a real number. Throughout this paper, $B(c, r)$ stands for the open ball with center c and radius r.

Lemma 4. *Let X and Y be normed spaces.*

The multifunction F is uniformly linearly open on a set $W \subseteq \operatorname{graph}(F)$ if and only if there exist a real number $\omega > 0$ and a real number $\zeta > 0$ such that the inclusion (15) holds for all $\epsilon \in (0, \zeta)$ and for all $(x, y) \in W$.

The multifunction F is uniformly nearly linearly open on a set $W \subseteq \operatorname{graph}(F)$ if and only if there exist a real number $\omega > 0$ and a real number $\zeta > 0$ such that the inclusion (16) holds for all $\epsilon \in (0, \zeta)$ and for all $(x, y) \in W$.

Proof. Let $W \subseteq \operatorname{graph}(F)$, and recall the definition as well as the properties of the statement $Q(U, V, r)$ from within the proof of Lemma 3. If F is uniformly linearly open on W, then there exist a neighborhood V of the origin in Y and a real number $\zeta > 0$ such that $Q(B(0, 1), V, \zeta)$ is true. Choose a real number $\omega > 0$ such that $B(0, \omega \subseteq V$. Then $Q(B(0, 1), V, \zeta)$ implies $Q(B(0, 1), B(0, \omega), \zeta)$, i.e. inclusion (15) holds for all $\epsilon \in (0, \zeta)$ and for all $(x, y) \in W$. Conversely, let $Q(B(0, 1), B(0, \omega), \zeta)$ be true for some $\omega > 0$ and $\zeta > 0$, and let U be a neighborhood of the origin in X. Choose $t > 0$ such that $B(0, t) \subseteq U$, let $V = B(0, t\omega)$, and let $r = \zeta/t$. Then $Q(B(0, 1), B(0, \omega), \zeta)$ is equivalent to $Q(B(0, t), V, r)$, which implies $Q(U, V, r)$, hence F is uniformly linearly open on W. The second part of the result can be proved in a similar manner. ■

The result above can be further rephrased through the metric definitions based on the metric inclusions (15) and (16). The first definition introduces the notions of ω-openness and near ω-openness.

Definition 3. Let X and Y be metric spaces, and let $\omega > 0$ be a real number.

The multifunction F is said to be *ω-open at a point* $(x, y) \in \mathrm{graph}(F)$ if there exists a real number $\zeta > 0$ such that the inclusion (15) holds for all $\epsilon \in (0, \zeta)$; the multifunction F is said to be ω-open on a set $W \subseteq \mathrm{graph}(F)$ if F is ω-open at every $(x, y) \in W$; the multifunction F is said to be ω-open if F is ω-open on $\mathrm{graph}(F)$.

The multifunction F is said to be *nearly ω-open at a point* $(x, y) \in \mathrm{graph}(F)$ if there exists a real number $\zeta > 0$ such that the inclusion (16) holds for all $\epsilon \in (0, \zeta)$; the multifunction F is said to be nearly ω-open on a set $W \subseteq \mathrm{graph}(F)$ if F is nearly ω-open at every $(x, y) \in W$; the multifunction F is said to be nearly ω-open if F is nearly ω-open on $\mathrm{graph}(F)$.

The second definition introduces the specialized notions of uniform ω-openness and uniform near ω-openness.

Definition 4. Let X and Y be metric spaces, and let $\omega > 0$ be a real number.

The multifunction F is said to be *uniformly ω-open on a set* $W \subseteq \mathrm{graph}(F)$ if there exists a real number $\zeta > 0$ such that the inclusion (15) holds for all $\epsilon \in (0, \zeta)$ and for all $(x, y) \in W$; the multifunction F is said to be *locally ω-open* if for every $(a, b) \in \mathrm{graph}(F)$ there exists a neighborhood W of (a, b) such that F is uniformly ω-open on $W \cap \mathrm{graph}(F)$; the multifunction F is said to be *uniformly ω-open* if F is uniformly ω-open on $\mathrm{graph}(F)$.

The multifunction F is said to be *uniformly nearly ω-open on a set* $W \subseteq \mathrm{graph}(F)$ if there exists a real number $\zeta > 0$ such that the inclusion (16) holds for all $\epsilon \in (0, \zeta)$ and for all $(x, y) \in W$; the multifunction F is said to be *locally nearly ω-open* if for every $(a, b) \in \mathrm{graph}(F)$ there exists a neighborhood W of (a, b) such that F is uniformly nearly ω-open on $W \cap \mathrm{graph}(F)$; the multifunction F is said to be *uniformly nearly ω-open* if F is uniformly nearly ω-open on $\mathrm{graph}(F)$.

Theorem 3. *Let X and Y be normed spaces. The multifunction F is uniformly linearly open on a set $W \subseteq \mathrm{graph}(F)$ if and only if there exists a real number $\omega > 0$ such that F is uniformly ω-open on W. The multifunction F is uniformly nearly linearly open on a set $W \subseteq \mathrm{graph}(F)$ if and only if there exists a real number $\omega > 0$ such that F is uniformly nearly ω-open on W.*

In case that W is a singleton, we obtain a characterization of linear openness and near linear openness at a point. Let X and Y be normed spaces. The multifunction F is linearly open at a point $(x, y) \in \mathrm{graph}(F)$ if and only if there exists a real number $\omega > 0$ such that F is ω-open at (x, y). The multifunction F is nearly linearly open at a point $(x, y) \in \mathrm{graph}(F)$ if and only if there exists a real number $\omega > 0$ such that F is nearly ω-open at (x, y). In view of this characterization, linear openness at a point is the multifunction version of the function *fatness* defined in [34, p. 545].

In case that $W = \mathrm{graph}(F)$, we obtain characterizations of uniform linear openness and uniform near linear openness. Let X and Y be normed spaces. The multifunction F is uniformly linearly open if and only if there exists a real number $\omega > 0$ such that F is uniformly ω-open. The multifunction F is uniformly nearly linearly open if and only if there exists a real number $\omega > 0$ such that F is uniformly nearly ω-open.

Finally, we obtain characterizations of local uniform linear openness and local uniform near linear openness. Let X and Y be normed spaces. The multifunction F is locally uniformly linearly open if and only if for every point $(a, b) \in \mathrm{graph}(F)$ there exist a neighborhood W of (a, b) and a real number $\omega > 0$ such that F is uniformly ω-open on $W \cap \mathrm{graph}(F)$. The multifunction F is locally uniformly nearly linearly open if and only if for every point $(a, b) \in \mathrm{graph}(F)$ there exist a neighborhood W of (a, b) and a real number $\omega > 0$ such that F is uniformly nearly ω-open on $W \cap \mathrm{graph}(F)$.

A counterexample shows that local uniform linear openness does not necessarily imply local uniform ω-openness for any $\omega > 0$. Let $F : R \to R$ be a continuous, increasing function. Then $F(B(x, \epsilon)) = (F(x - \epsilon), F(x + \epsilon))$, so inclusion (15) means that both $F(x - \epsilon) \leq F(x) - \omega\epsilon$ and $F(x) + \omega\epsilon \leq F(x + \epsilon)$. Therefore, if F is differentiable at x and ω-open there for some $\omega > 0$, then $\omega \leq F'(x)$, hence $F'(x) > 0$. Conversely, if $F'(x) > 0$, then F is ω-open at x for every $\omega \in (0, F'(x))$. Now, let $F(x) = \arctan x$. Then F is locally uniformly linearly open, but there does not exists any $\omega > 0$ which renders F a locally uniformly ω-open function. In fact, there does not exists any $\omega > 0$ which renders F an ω-open function, that is, an ω-open function at every point x.

6. Metric Regularity and ω-Openness

In this section we use multifunction ω-openness to investigate the multifunction metric regularity, a theory based on the inequality

$$d(x, F^{-1}(y)) \leq d(y, F(x))/\omega \tag{17}$$

(cf. Definition 1 and Definition 1 (loc) in [11, p. 507]; see inequality (11) in [4, p. 235]. Here, $d(p, Q)$ stands for the distance from the point p to the set Q. For a detailed history of the matter we refer to [14, pp. 159-163, 462-463].

Let X and Y be metric spaces. The multifunction F is said to be *metrically regular on a set* $W \subseteq X \times Y$ if there exists a real number $\omega > 0$ such that the inequality (17) holds for all $(x, y) \in W$. The multifunction F is said to be *metrically regular near a point* $(a, b) \in \operatorname{graph}(F)$ if there exists a neighborhood W of (a, b) such that F is metrically regular on W.

Note the inequality (17) holds if and only if $y \in F(B(x, \epsilon))$ whenever $\epsilon > 0$ and $y \in B(F(x), \omega\epsilon)$ (cf. Definition 2 in [11, p. 508] and Proposition 2 in [11, p. 509]). Here, $B(C, r)$ stands for the union of the open balls $B(c, r)$ with $c \in C$.

Theorem 4. *Let X and Y be normed spaces. The multifunction F is metrically regular near a point $(a, b) \in \operatorname{graph}(F)$ if and only if there exists a neighborhood W of (a, b) such that F is uniformly linearly open on $W \cap \operatorname{graph}(F)$.*

Theorem 5 above is a straightforward corollary of Theorem 4 below.

Theorem 5. *Let X and Y be metric spaces. The multifunction F is metrically regular near a point $(a, b) \in \operatorname{graph}(F)$ if and only if there exist a neighborhood W of (a, b) and a real number $\omega > 0$ such that F is uniformly ω-open on $W \cap \operatorname{graph}(F)$.*

Theorem 5 above is a straightforward corollary of Lemma 5 below, a multifunction version of a remark in [7, pp. 11, 12]. This lemma connects the theory developed in this paper to the theory exposed in [14, pp. 56-70].

Lemma 5. *Let X and Y be metric spaces, let $(a, b) \in \operatorname{graph}(F)$, and let $\omega > 0$. The following two conditions are equivalent:*

- *there exists a neighborhood W of (a, b) such that for every $(x, y) \in W$ there holds the inequality* (17)*;*
- *there exists a neighborhood W of (a, b) such that F is uniformly ω-open on $W \cap \operatorname{graph}(F)$.*

Proof. First, let the former condition be satisfied and choose a neighborhood W of (a, b) such that the inequality (17) holds whenever $(x, y) \in W$. Further, choose a neighborhood U of a, a neighborhood V of b, and a real number $\zeta > 0$ such that $U \times B(V, \omega\zeta) \subseteq W$. Now, let $(x, y) \in (U \times V) \cap \operatorname{graph}(F)$ and let $\epsilon \in (0, \zeta)$. We assert that the inclusion (15) holds. Indeed, if $v \in B(y, \omega\epsilon)$, then $(x, v) \in W$ and $v \in B(F(x), \omega\epsilon)$, hence $v \in F(B(x, \epsilon))$.

Finally, let the latter condition be satisfied, choose a neighborhood W of (a, b) such that F is uniformly ω-open on $W \cap \operatorname{graph}(F)$, choose a real number $\zeta > 0$ such that the inclusion (15) holds whenever $\epsilon \in (0, \zeta)$ and $(x, y) \in W \cap \operatorname{graph}(F)$.

Since $B(b, \omega\epsilon) \subseteq F(B(a, \epsilon))$ whenever $\epsilon \in (0, \zeta)$,it follows $d(a, F^{-1}(y)) \leq d(y, b)/\omega$ whenever $y \in B(b, \omega\zeta)$, hence $d(x, F^{-1}(y)) \leq d(x, a) + d(y, b)/\omega$ whenever $x \in X$ and $y \in B(b, \omega\zeta)$.

Now, choose a real number $\alpha > 0$ such that $2\alpha \leq \zeta$ and $B(a, \alpha) \times B(b, 3\omega\alpha) \subseteq W$. We assert that the inequality (17) holds whenever $(x, y) \in B(a, \alpha) \times B(b, \omega\alpha)$.

Obviously, the inequality (17) holds in case $d(x,a)+d(y,b)/\omega \le d(y,F(x))/\omega$. It remains to discuss the case $d(y,F(x))/\omega < d(x,a)+d(b,y)/\omega$. Let $\epsilon > 0$ such that $d(y,F(x))/\omega < \epsilon < d(x,a)+d(y,b)/\omega$, and choose $z \in F(x)$ such that $d(y,z)/\omega < \epsilon$. Since $d(z,b) \le d(z,y)+d(y,b) < \omega\epsilon + d(y,b) < \omega d(x,a) + 2d(y,b) < 3\omega\alpha$, it follows $(x,z) \in W \cap \mathrm{graph}(F)$. Since $\epsilon < 2\alpha$, it follows $\epsilon \in (0,\zeta)$. Further, $y \in B(z,\omega\epsilon)$, hence $y \in F(B(x,\epsilon))$, $d(x,F^{-1}(y)) < \epsilon$, the inequality (17) holds, and the conclusion is established. ■

The counterexample in Section 11 shows that it is possible for a multifunction to be linearly open at a point even if that multifunction is not metrically regular at any point.

7. Locally Closed Graph Results; The General Metric Setting

In this section we investigate quantitative aspects of the qualitative property of local uniform ω-openness.

Let X and Y be metric spaces, let $F : X \to Y$ be a multifunction, and let $\omega > 0$ be a real number. In the following $\overline{R}$ stands for the set of real extended numbers, that is, $\overline{R} = R \cup \{\pm\infty\}$.

A necessary and sufficient condition that F be ω-open is that there exist a positive function $\eta : \mathrm{graph}(F) \to \overline{R}$ such that the inclusion (15) holds for all $(x,y) \in \mathrm{graph}(F)$ and for all $\epsilon \in (0,\eta(x,y))$. In such a case we say η materializes ω-openness of F.

A necessary and sufficient condition that F be locally uniformly ω-open is that there exist a positive function $\eta : \mathrm{graph}(F) \to \overline{R}$ which materializes the ω-openness of F and which enjoys the additional condition that for every $(a,b) \in \mathrm{graph}(F)$ there exists a neighborhood W of (a,b) such that

$$0 < \inf_{(x,y)\in W\cap \mathrm{graph}(F)} \eta(x,y).$$

This additional condition holds if, for example, the positive function η is lower semi continuous. In the following we focus our attention on the Lipschitz inequality

$$\eta(x',y') \le \eta(x,y) + \sup\{d(x,x'), d(y,y')/\omega\}.$$

which is always satisfied by the nonnegative function η_ω below, of which positivity characterizes local closeness of $\mathrm{graph}(F)$.

Denote by $\eta_\omega : \mathrm{graph}(F) \to \overline{R}$ the function which assigns to every $(x,y) \in \mathrm{graph}(F)$ the supremum of the real numbers $\epsilon > 0$ such that

$$(B(x,\epsilon) \times B(y,\omega\epsilon)) \cap \overline{\mathrm{graph}(F)} \subseteq \mathrm{graph}(F).$$

By convention, $\sup\emptyset = 0$. Note that $\mathrm{graph}(F)$ is closed if and only if $\eta_\omega(x,y) = +\infty$ for some $(x,y) \in \mathrm{graph}(F)$, in which case $\eta_\omega(x,y) = +\infty$ for all $(x,y) \in \mathrm{graph}(F)$. Note also that $\mathrm{graph}(F)$ is locally closed if and only if $\eta_\omega(x,y) > 0$ for all $(x,y) \in \mathrm{graph}(F)$.

To prove that η_ω satisfies the Lipschitz inequality, we use the abbreviations $\eta = \eta_\omega$, $z = (x,y)$, $z' = (x',y')$, and

$$d(z,z') = \sup\{d(x,x'), d(y,y')/\omega\},$$

and we note the equality

$$B(z,\epsilon) = B(x,\epsilon) \times B(y,\omega\epsilon).$$

We have to show that $\eta(z') \leq \eta(z) + d(z,z')$. The inequality is obvious if $\eta(z') \leq d(z,z')$. Now, let $d(z,z') < \eta(z')$ and $\epsilon \in (d(z,z'), \eta(z'))$. Since $\overline{B(z',\epsilon) \cap \mathrm{graph}(F)} \subseteq \mathrm{graph}(F)$ and $B(z, \epsilon - d(z,z')) \subseteq B(z',\epsilon)$, it follows $\overline{B(z,\epsilon - d(z,z')) \cap \mathrm{graph}(F)} \subseteq \mathrm{graph}(F)$, hence $\epsilon - d(z,z') \leq \eta(z)$. To conclude, $\eta(z') - d(z,z') \leq \eta(z)$, and the desired inequality is established.

Now, denote by H_ω the family of nonnegative functions $\eta : \mathrm{graph}(F) \to \overline{R}$ which satisfies the Lipschitz inequality as well as the inequality $\eta(x,y) \leq \eta_\omega(x,y)$ for all $(x,y) \in \mathrm{graph}(F)$. Clearly, $\eta_\omega \in H_\omega$ and H_ω contains at least a positive function, namely η_ω, if and only if F has a locally closed graph.

The next result characterizes the positive functions $\eta \in H_\omega$ which materialize ω-openness of F through the rather technical relation

$$\emptyset \neq B(v, d(v,y) - \theta\omega\epsilon) \cap B(y,\omega\epsilon) \cap F(B(x,\epsilon)). \tag{18}$$

Lemma 6. *Let the metric spaces X and Y be complete, and let the multifunction $F : X \to Y$ have a locally closed graph. Let $\eta \in H_\omega$ be a positive function. Then the following conditions are equivalent:*

- *for every $(x,y) \in \mathrm{graph}(F)$ and for every $\epsilon \in (0, \eta(x,y))$ there holds the inclusion (15);*
- *for every $(x,y) \in \mathrm{graph}(F)$, for every $v \in B(y, \omega \cdot \eta(x,y)) \setminus \{y\}$, and for every $\theta \in (0,1)$ there exists $\epsilon \in (0, d(v,y)/(\theta\omega))$ such that there holds relation (18).*

Proof. To show that the former condition implies the latter one, let $(x,y) \in \mathrm{graph}(F)$, let $v \in B(y, \omega \cdot \eta(x,y)) \setminus \{y\}$, let $\theta \in (0,1)$, set

$$I = (d(v,y)/\omega, d(v,y)/(\theta\omega)) \cap (0, \eta(x,y)),$$

note the interval I is nonempty, and choose any $\epsilon \in I$. Then v belongs to the right hand side of the relation (18), and the relation (18) holds.

Let now the latter condition be satisfied, let $(x, y) \in \mathrm{graph}(F)$, let $\epsilon \in (0, \eta(x, y))$, and let $v \in B(y, \omega\epsilon) \setminus \{y\}$. We must show that $v \in F(B(x, \epsilon))$. Let $\theta \in (d(v, y)/(\omega\epsilon), 1)$. Following the spirit of some ideas in [2, p. 195], [3, p. 76], [10, p. 572], and [15, p. 30] (cf. also [26, p. 222], [29, pp. 81, 82], [30, p. 404], [32, p. 208]), we endow the space $X \times Y$ with the metric

$$d_\omega((p, q), (p', q')) = \sup\{d(p, p'), d(q, q')/\omega\},$$

and we apply the variational principle of Ekeland [8, p. 324] to the function

$$(p, q) \in \overline{\mathrm{graph}(F)} \to d(v, q) \in R$$

in order to get a point $(a, b) \in \overline{\mathrm{graph}(F)}$ such that

$$d(v, b) + \theta\omega d_\omega((a, b), (x, y)) \leq d(v, y)$$

(see [2, p. 195] and [16, p. 815]) and such that

$$d(v, b) < d(v, q) + \theta\omega d_\omega((p, q), (a, b))$$

for all $(p, q) \in \overline{\mathrm{graph}(F)} \setminus \{(a, b)\}$. Since $d(v, y) < \theta\omega\epsilon$, it follows from the former inequality of the Ekeland principle that $d_\omega((a, b), (x, y)) < \epsilon$, hence $a \in B(x, \epsilon)$ and $b \in B(y, \omega\epsilon)$. Since $\epsilon < \eta_\omega(x, y)$, it follows $(a, b) \in \mathrm{graph}(F)$, hence $b \in F(a) \subseteq F(B(x, \epsilon))$. We have to show that $v = b$. Suppose, to the contrary, that $b \neq v$. Since $d(v, y) < \theta\omega\eta(x, y)$, it follows from the former inequality of the Ekeland principle that

$$d(v, b) + \theta\omega d_\omega((a, b), (x, y)) < \theta\omega\eta(x, y) \leq \theta\omega\eta(a, b) + \theta\omega d_\omega((a, b), (x, y)),$$

therefore $d(v, b) < \theta\omega\eta(a, b)$. Since $v \in B(b, \omega\eta(a, b)) \setminus \{b\}$, it follows from the second condition of the theorem that there exists $\alpha \in (0, d(v, b)/(\theta\omega))$ such that the set

$$S = B(v, d(v, b) - \theta\omega\alpha) \cap B(b, \omega\alpha) \cap F(B(a, \alpha))$$

is nonempty. Let $q \in S$. Since $q \in F(B(a, \alpha))$, it follows there exists $p \in B(a, \alpha)$ such that $q \in F(p)$. Since $d(q, b) < \omega\alpha$, it follows

$$d_\omega((p, q), (a, b)) < \alpha.$$

Since $d(q, v) < d(v, b) - \theta\omega\alpha$, it follows $q \neq b$, therefore $(p, q) \in \mathrm{graph}(F) \setminus \{(a, b)\}$. Further, it follows from the latter inequality of the Ekeland principle that $d(v, b) < [d(v, b) - \theta\omega\alpha] + [\theta\omega\alpha]$, a contradiction. To conclude, $v = b$. ■

We have not stated Lemma 6 only in case $\eta = \eta_\omega$ because an example shows it is possible for the two equivalent η-conditions to be false (hence useless) for $\eta = \eta_\omega$, but true (hence useful) for some $\eta \neq \eta_\omega$.

Let the metric space X consists of the vertices a, b, and c of an isosceles triangle of which base ab has the length 1, and of which legs ac and bc have the length 2. Further, let $\mathrm{graph}(F) = \{(a,a),(b,b)\}$ and let $\omega = 1$. Then $\eta_\omega(x,y) = +\infty$ for all $(x,y) \in \mathrm{graph}(F)$, but the inclusion (15) does not hold for any $(x,y) \in \mathrm{graph}(F)$ and for any $\epsilon \in (2, \eta_\omega(x,y))$ because $B(y,\epsilon) = X \not\subseteq F(X) = F(B(x,\epsilon))$. Finally, let $\eta(x,y) = 1$ for all $(x,y) \in \mathrm{graph}(F)$. Then the inclusion (15) does hold for all $(x,y) \in \mathrm{graph}(F)$ and for all $\epsilon \in (0, \eta(x,y))$ because $B(y,\epsilon) = \{y\} = F(x) = F(B(x,\epsilon))$.

In the next section we show that the anomaly above does not hold in a particular metric setting.

8. Locally Closed Graph Results: A Particular Metric Setting

Recall that the metric space Y is said to resemble normed spaces if

$$B(B(y,\delta),\delta') = B(y,\delta+\delta')$$

for all $y \in Y$, $\delta > 0$, and $\delta' > 0$ (see Definition 2.2 in [32, p. 204] and the definition of γ-convexity in [9, p. 271]). Equivalently, Y resembles normed spaces if and only if $B(y,\delta) \cap B(y',\delta') \neq \emptyset$ whenever $y \in Y$, $y' \in Y$, $\delta > 0$, $\delta' > 0$, and $\delta + \delta' > d(y,y')$.

Lemma 7. *Let the metric spaces X and Y be complete, let the metric space Y resemble normed spaces, and let the multifunction $F : X \to Y$ have a locally closed graph. Then the following conditions are equivalent:*

- *for every $(x,y) \in \mathrm{graph}(F)$ and for every $\epsilon \in (0, \eta_\omega(x,y))$ there holds the inclusion* (15)*;*
- *for every $(x,y) \in \mathrm{graph}(F)$, for every $v \in Y \setminus \{y\}$, and for every $\theta \in (0,1)$ there exists $\epsilon \in (0, d(v,y)/(\theta\omega))$ such that there holds the relation* (18).

Proof. To show that the former condition implies the latter one, let $(x,y) \in \mathrm{graph}(F)$, let $v \in B(y, \omega \cdot \eta(x,y)) \setminus \{y\}$, let $\theta \in (0,1)$, set

$$I = (d(v,y)/\omega, d(v,y)/(\theta\omega)) \cap (0, \eta(x,y)),$$

note the interval I is nonempty, and choose any $\epsilon \in I$. Then $B(v, d(v,y) - \theta\omega\epsilon) \cap B(y, \omega\epsilon)$ is a nonempty subset of $F(B(x,\epsilon))$, and the relation (18) holds. The fact that the latter condition implies the former one follows from Lemma 6. ∎

Theorem 6. *Let X and Y be complete metric spaces, let the metric space Y resemble normed spaces, and let the set* $\mathrm{graph}(F)$ *be locally closed. Then the following conditions are equivalent:*

- *for every* $(x, y) \in \mathrm{graph}(F)$ *and for every* $\epsilon \in (0, \eta_\omega(x, y))$ *there holds the inclusion* (15)*;*
- *the multifunction F is locally uniformly ω-open;*
- *the multifunction F is ω-open;*
- *the multifunction F is nearly ω-open;*
- *for every* $(x, y) \in \mathrm{graph}(F)$ *and for every* $\zeta > 0$ *there exists* $\epsilon \in (0, \zeta)$ *such that there holds the inclusion* (16).

Proof. It suffices to show that the last condition of the result implies the latter condition of Lemma 7. Let $(x, y) \in \mathrm{graph}(F)$, let $v \in Y \setminus \{y\}$, and let $\theta > 0$. According to the last condition of the result, there exists $\epsilon \in (0, d(v, y)/(\theta\omega))$ such that there holds the inclusion (16). Since Y resembles normed spaces it follows the intersection of the open balls $B(y, \omega\epsilon)$ and $B(v, d(v, y) - \theta\omega\epsilon)$ is nonempty. Since this intersection is a nonempty open subset of $\overline{F(B(x, \epsilon))}$, the relation (18) holds, the latter condition of Lemma 7 is satisfied, and the proof is accomplished. ■

Theorem 6 extends Theorem 2.4 in [32, p. 205], where $\mathrm{graph}(F)$ is closed. It extends also part of Theorem 7 in [30, p. 408], where X and Y are Banach spaces. To close this section, we recall a result which derives linear openness of

$$F = G - H_1^{-1} - \cdots - H_n^{-1} \tag{19}$$

from linear openness of G and from linear openness of each H_i. Here the multifunction $G : X \to Y$ and the multifunctions $H_i : Y \to X$ provide the multifunction $F : X \to Y$ through the equality

$$F(x) = G(x) - H_1^{-1}(x) - \cdots - H_n^{-1}(x),$$

which implies the equality

$$\mathrm{domain}(F) = \mathrm{domain}(G) \cap \mathrm{range}(H_1) \cap \cdots \cap \mathrm{range}(H_n).$$

The result in [30, p. 405, 406] derives ω-openness of F from ψ-openness of G and from χ_i-openness of each H_i in case that

$$\omega = \psi - \chi_1^{-1} - \cdots - \chi_n^{-1} \tag{20}$$

is a positive real number.

Theorem 7. *Let X and Y be Banach spaces, and let F be given by* (19). *Let* $\mathrm{domain}(F)$ *be nonempty. Let* $\mathrm{graph}(G)$ *and each* $\mathrm{graph}(H_i)$ *be locally closed. If G is ψ-open, if each H_i is χ_i-open, and if ω given by* (20) *is positive, then F is ω-open.*

If ω given by (20) is null, then F given by (19) may lack linear openness. A counterexample is provided in case $\mathrm{graph}(G) = \{(x, \psi x); x \in R\}$ and each $\mathrm{graph}(H_i) = \{(x, \chi_i x); x \in R\}$, in which case $\mathrm{graph}(F) = \{(x, \omega x); x \in R\}$.

9. Elementary vs. Classical Tangency

In this section we discuss the relationship between the tangency concept K provided by the $\mathcal{S}$-family (1) and the tangency concept κ provided by the $\mathcal{S}$-family (4). We refer to these tangency concepts as the classical K and the elementary κ.

In the beginning of this section, we consider the linear topological setting. In case of the elementary tangency concept κ, the equalities

$$\kappa_A(a) = \bigcup_{s \in (0,1]} (1/s)(A - a) = \bigcup_{t \geq 1} t(A - a)$$

imply the inclusion $A - a \subseteq \kappa_A(a)$, hence $F(u) - y \subseteq \kappa_F(x, y)(u - x)$ for every $u \in X$. Therefore no matter which openness property of F at (x, y) implies the corresponding openness property of $\kappa_F(x, y)$ at $(0, 0)$. For example, linear openness at (x, y) of F implies linear openness at $(0, 0)$ of $\kappa_F(x, y)$, but the converse may fail. A counterexample is provided by

$$\mathrm{graph}(F) = \{(r, r); r \in R, r = \pm 1/n, n \in N\},$$

in which case F is not linearly open at $(0, 0)$, although $\kappa_F(0, 0)$ is linearly open at $(0, 0)$, for

$$\mathrm{graph}(\kappa_F(0, 0)) = \{(r, r); r \in R\}$$

However, near linear openness at $(0, 0)$ of all of the multifunctions $\kappa_F(x, y)$ with $(x, y) \in \mathrm{graph}(F)$ may be equivalent to linear openness of F (see Section 10).

In case of the classical tangency concept K, the equalities

$$K_A(a) = \bigcap_{r > 0} \overline{\bigcup_{s \in (0,r)} (1/s)(A - a)} = \bigcap_{r > 0} \overline{\bigcup_{t > r} t(A - a)}$$

imply the inclusion $K_A(a) \subseteq \overline{\kappa_A(a)}$, hence

$$\mathrm{graph}(K_F(x, y)) \subseteq \overline{\mathrm{graph}(\kappa_F(x, y))}.$$

The topological Lemma 8 below shows that no matter which near openness property of $K_F(x, y)$ implies the corresponding property of $\kappa_F(x, y)$.

Lemma 8. *Let X and Y be topological spaces, and let $F : X \to Y$ and $G : X \to Y$ be multifunctions. Then* $\operatorname{graph}(F) \subseteq \overline{\operatorname{graph}(G)}$ *if and only if* $F(U) \subseteq \overline{G(U)}$ *for every open set* $U \subseteq X$.

Proof. First, let $\operatorname{graph}(F) \subseteq \overline{\operatorname{graph}(G)}$, let $U \subseteq X$ be an open set, let $y \in F(U)$, and let V be a neighborhood of y. We have to show that there exists $v \in V$ such that $v \in G(U)$. Since $y \in F(x)$ for some $x \in U$ and since $U \times V$ is a neighborhood of $(x, y) \in \overline{\operatorname{graph}(G)}$, it follows there exists $u \in U$ and $v \in V$ such that $(u, v) \in \operatorname{graph}(G)$. To conclude, $v \in G(U)$.

Conversely, let $F(U) \subseteq \overline{G(U)}$ for every open set $U \subseteq X$, let $(x, y) \in \operatorname{graph}(F)$, and let W be a neighborhood of (x, y). We have to show that there exists $(u, v) \in W$ such that $(u, v) \in \operatorname{graph}(G)$. Choose an open neighborhood U of x and a neighborhood V of y such that $U \times V \subseteq W$. Since $y \in \overline{G(U)}$, it follows there exists $v \in V$ such that $v \in G(U)$. Choose $u \in U$ such that $v \in G(u)$. To conclude, $(u, v) \in W$ and $(u, v) \in \operatorname{graph}(G)$. ■

Since $t\kappa_A(a) \subseteq \kappa_A(a)$ whenever $t \geq 1$, it follows

$$t\kappa_F(x, y)(u) \subseteq \kappa_F(x, y)(tu) \tag{21}$$

whenever $u \in X$ and $t \geq 1$, and we obtain some useful characterizations of linear openness as well as near linear openness of $\kappa_F(x, y)$ at $(0, 0)$.

Recall that linear openness of $\kappa_F(x, y)$ at $(0, 0)$ means that for every neighborhood U of the origin in X there exists a neighborhood v of the origin in Y such that for every $s \in (0, 1]$ there holds the inclusion

$$sV \subseteq \kappa_F(x, y)(sU). \tag{22}$$

Because the inclusion (21) holds for all $t \geq 1$, we get the equivalence of the following three conditions:

- for every neighborhood U of the origin in X there exists a neighborhood v of the origin in Y such that for every $s > 0$ there holds the inclusion (22);
- $\kappa_F(x, y)$ is linearly open at $(0, 0)$;
- for every neighborhood U of the origin in X there exists a neighborhood v of the origin in Y such that for every $r > 0$ there exists $s \in (0, r)$ such that there holds the inclusion (22).

Recall also that near linear openness of $\kappa_F(x, y)$ at $(0, 0)$ means that for every neighborhood U of the origin in X there exists a neighborhood U of the origin in Y such that for every $s \in (0, 1]$ there holds the inclusion

$$sV \subseteq \overline{\kappa_F(x, y)(sU)}. \tag{23}$$

Because the inclusion (21) holds for all $t \geq 1$, we get the equivalence of the following three conditions:

- for every neighborhood U of the origin in X there exists a neighborhood V of the origin in Y such that for every $s > 0$ there holds the inclusion (23);
- $\kappa_F(x, y)$ is nearly linearly open at $(0, 0)$;
- for every neighborhood U of the origin in X there exists a neighborhood V of the origin in Y such that for every $r > 0$ there exists $s \in (0, r)$ such that there holds the inclusion (23).

Since $tK_A(a) \subseteq K_A(a)$ whenever $t > 0$, it follows $tK_A(a) = K_A(a)$ whenever $t > 0$, hence

$$tK_F(x, y)(u) = K_F(x, y)(tu)$$

whenever $u \in X$ and $t > 0$, and we obtain some useful characterizations of linear openness as well as near linear openness of $K_F(x, y)$ at $(0, 0)$.

The multifunction $K_F(x, y)$ is linearly open at $(0, 0)$ if and only if for every neighborhood U of the origin in X there exists a neighborhood V of the origin in Y such that

$$V \subseteq K_F(x, y)(U). \tag{24}$$

The multifunction $K_F(x, y)$ is nearly linearly open at $(0, 0)$ if and only if for every neighborhood U of the origin in X there exists a neighborhood V of the origin in Y such that

$$V \subseteq \overline{K_F(x, y)(U)}. \tag{25}$$

Linear openness and near linear openness of $K_F(x, y)$ at $(0, 0)$ are equivalent if the linear topological space X is finite dimensional. Indeed, if X is finite dimensional, then we can rephrase the two openness properties by using only compact neighborhoods U of the origin in X, and the two inclusions (24) and (25) coincide because the closed graph multifunction $K_F(x, y)$ maps compact sets into closed sets (see [12, pp. 203, 204]).

In view of Lemma 8, $\kappa_F(x, y)$ is nearly linearly open at $(0, 0)$ if so is $K_F(x, y)$. If X is finite dimensional, then the converse implication holds too, i.e. $\kappa_F(x, y)$ is nearly linearly open at $(0, 0)$ if and only if so is $K_F(x, y)$. The counterexample in Section 11 shows that the equivalence above may fail if the space X is not finite dimensional. There $\mathrm{graph}(K_F(x, y)) = \{(0, 0)\}$, although $\mathrm{graph}(F)$ is closed and the inclusion (15) holds for all $\epsilon > 0$. The equivalence above follows from Lemma 9 below, which concerns the inclusion

$$\bigcap_{\gamma>0} \bigcup_{s\in(0,r)} (1/s)\overline{\kappa_F(x, y)(sU)} \subseteq K_F(x, y)(U). \tag{26}$$

Lemma 9. *Let X and Y be linear topological spaces. Let κ be the elementary tangency concept and let K be the classical tangency concept. Then the inclusion* (26) *holds for every point $(x, y) \in X \times Y$ and for every compact set $U \subseteq X$.*

Proof. Let $(x, y) \in X \times Y$, let $U \subseteq X$ be a compact set, let v be a point of the left hand side of the inclusion (26), and suppose by contradiction that $v \notin K_F(x, y)(U)$, i.e.

$$(U \times \{v\}) \cap K_{\mathrm{graph}(F)}(x, y) = \emptyset.$$

Equivalently, for every $u \in U$ there exist a neighborhood W of the origin in $X \times Y$ and a real number $r > 0$ such that

$$((x, y) + (0, r)((u, v) + W)) \cap \mathrm{graph}(F) = \emptyset.$$

Since $U \times \{v\}$ is compact, it follows there exist a neighborhood W of the origin in $X \times Y$ and a real number $r > 0$ such that

$$((x, y) + (0, r)((U \times \{v\}) + W)) \cap \mathrm{graph}(F) = \emptyset$$

(see [28, p. 108, Lemma 3 (a)]). Since $sv \in \overline{\kappa_F(x, y)(sU)}$ for some $s \in (0, r)$ and since $V' = \{v'; (0, v') \in W\}$ is a neighborhood of the origin in Y, it follows there exists $v' \in V'$ such that $s(v + v') \in \kappa_F(x, u)(sU)$. Further there exists $u \in U$ such that $s(v + v') \in \kappa_F(x, y)(su)$. Further there exists $\sigma \in (0, 1]$ such that $y+\sigma s(v+v') \in F(x+\sigma su)$. To conclude, $(x, y)+(\sigma s)((u, v)+(0, v')) \in \mathrm{graph}(F)$ and $\sigma s \in (0, r)$, a contradiction. ∎

10. Locally Closed Results: The Normed Space Setting

Let X and Y be normed spaces, let $F : X \to Y$ be a multifunction and let $\omega > 0$ be a real number.

The first result of this section equates ω-openness of F at all the points $(x, y) \in \mathrm{graph}(F)$ to either ω-openness or near ω-openness of all the multifunctions $\kappa_F(x, y)$ at $(0, 0)$. Here κ is the elementary tangency concept, i.e. the tangency concept corresponding to the $\mathcal{S}$-family (4).

Recall ω-openness of $\kappa_F(x, y)$ at $(0, 0)$ means that there exists a real number $\zeta > 0$ such that

$$B(0, \omega\epsilon) \subseteq \kappa_F(x, y)(B(0, \epsilon)) \tag{27}$$

for all $\epsilon \in (0, \zeta)$. Because the inclusion (21) holds for all $t \geq 1$, we get the equivalence of the following three conditions:

- for every $\epsilon > 0$ there holds the inclusion (21);
- the multifunction $\kappa_F(x, y)$ is ω-open at $(0, 0)$;
- for every $\zeta > 0$ there exists $\epsilon \in (0, \zeta)$ such that there holds the inclusion (27).

Recall also near ω-openness of $\kappa_F(x, y)$ at $(0, 0)$ means that there exists a real number $\zeta > 0$ such that

$$B(0, \omega\epsilon) \subseteq \overline{\kappa_F(x, y)(B(0, \epsilon))} \tag{28}$$

for all $\epsilon \in (0, \zeta)$. Because the inclusion (21) holds for all $t \geq 1$, we get the equivalence of the following conditions:

- for every $\epsilon > 0$ there holds the inclusion (28);
- the multifunction $\kappa_F(x, y)$ is nearly ω-open at $(0, 0)$;
- for every $\zeta > 0$ there exists $\epsilon \in (0, \zeta)$ such that there holds the inclusion (28).

The last two conditions above should be compared to the last two conditions in Theorem 6.

Theorem 8. *Let X and Y be Banach spaces, and let the set* $\operatorname{graph}(F)$ *be locally closed. Let κ be the elementary tangency concept. The following conditions are equivalent:*

- *the multifunction F is ω-open;*
- *for every $(x, y) \in \operatorname{graph}(F)$ the multifunction $\kappa_F(x, y)$ is ω-open at $(0, 0)$;*
- *for every $(x, y) \in \operatorname{graph}(F)$ the multifunction $\kappa_F(x, y)$ is nearly ω-open at $(0, 0)$.*

Proof. It suffices to show that the last condition of the theorem implies the latter condition of Lemma 7. Let $(x, y) \in \operatorname{graph}(F)$, let $v \in Y \setminus \{y\}$, and let $\theta \in (0, 1)$. According to the last condition of the theorem, there exists $\epsilon \in (0, d(v, y)/(\theta\omega))$ such that there holds the inclusion (28). Since the intersection of the open balls $B(0, \omega\epsilon)$ and $B(v - y, d(v, y) - \theta\omega\epsilon)$ is a nonempty open subset of $\overline{\kappa_F(x, y)(B(0, \epsilon))}$, it follows the set

$$Q = B(0, \omega\epsilon) \cap B(v - y, d(v, y) - \theta\omega\epsilon) \cap \kappa_F(x, y)(B(0, \epsilon))$$

is nonempty. Let $q \in Q$. Since $q \in \kappa_F(x, y)(B(0, \epsilon))$, it follows there exists $p \in B(0, \epsilon)$ such that $q \in \kappa_F(x, y)(p)$, hence there exists $s \in (0, 1]$ such that $y + sq \in F(x + sp) \subseteq F(B(x, s\epsilon))$. Note $s\epsilon \in (0, d(v, y)/(\theta\omega))$. Since $d(y + sq, v) = \|(1 - s)(y - v) + s(y - v + q)\| \leq (1 - s)d(v, y) + sd(q, v - y)$ and since

$q \in B(v-y, d(v,y)-\theta\omega\epsilon)$, it follows $d(y+sq, v) \leq d(v,y)-\theta\omega s\epsilon$. To conclude, $y+sq$ belongs to the set

$$B(v, d(v,y)-\theta\omega s\epsilon) \cap B(y, \omega s\epsilon) \cap F(B(x, s\epsilon)),$$

the latter condition of Lemma 7 is satisfied, and the proof is accomplished. ∎

The second result of this section derives ω-openness of F at all the points $(x,y) \in \operatorname{graph}(F)$ from near ω-openness of all the multifunctions $K_F(x,y)$ at $(0,0)$. Here K is the classical tangency concept, i.e. the K tangency concept corresponding to the family (1).

Note $K_F(x,y)$ is ω-open at $(0,0)$ if and only if

$$B(0,\omega) \subseteq K_F(x,y)(B(0,1)), \tag{29}$$

whereas $K_F(x,y)$ is nearly ω-open at $(0,0)$ if and only if

$$B(0,\omega) \subseteq \overline{K_F(x,y)(B(0,1))}. \tag{30}$$

Theorem 9. *Let X and Y be Banach spaces, and let the set* $\operatorname{graph}(F)$ *be locally closed. Let K be the classical tangency concept. If for every $(x,y) \in$* $\operatorname{graph}(F)$ *the multifunction $K_F(x,y)$ is nearly ω-open at $(0,0)$, then the multifunction F is ω-open.*

Proof. Let $(x,y) \in \operatorname{graph}(F)$. Since $K_F(x,y)$ is nearly ω-open at $(0,0)$, it follows from Lemma 8 that also $\kappa_F(x,y)$ is nearly ω-open at $(0,0)$. According to Theorem 8, F is ω-open. ∎

Theorem 10 below improves on Theorem 9 above in the particular setting of finite dimensional spaces X. The result equates ω-openness of F at all the points $(x,y) \in \operatorname{graph}(F)$ to either ω-openness or near ω-openness of all the multifunctions $K_F(x,y)$ at $(0,0)$.

Theorem 10. *Let X and Y be Banach spaces, let the space X be finite dimensional, and let the set* $\operatorname{graph}(F)$ *be locally closed. Let K be the classical tangency concept. Then the following conditions are equivalent:*

- *the multifunction F is ω-open;*
- *for every $(x,y) \in$* $\operatorname{graph}(F)$ *the multifunction $K_F(x,y)$ is ω-open at* $(0,0)$*;*
- *for every $(x,y) \in$* $\operatorname{graph}(F)$ *the multifunction $K_F(x,y)$ is nearly ω-open at* $(0,0)$.

Proof. In view of Theorem 9, we have to show that the first condition implies the second one. Let F be ω-open and let $(x,y) \in \operatorname{graph}(F)$. Since F is also nearly ω-open at (x,y), it follows $\kappa_F(x,y)$ is nearly ω-open at $(0,0)$. According to Lemma 9, $B(0,\omega) \subseteq K_F(x,y)\left(\overline{B(0,1)}\right)$. Further, $B(0,\sigma\omega) \subseteq K_F(x,y)(\overline{B(0,\sigma)}) \subseteq K_F(x,y)(B(0,1))$ for all $\sigma \in (0,1)$, hence the inclusion (29) holds, and $K_F(x,y)$ is ω-open at $(0,0)$. ∎

At the end of this section we note the condition of ω-openness of F connects Theorem 6 and each of the Theorems 8, 9, and 10.

11. A Counterexample

Consider the infinite dimensional Hilbert space $l^2(R)$, which means that, if $x \in l^2(R)$, then $\|x\| = \sqrt{\sum_{r\in R}|x(r)|^2}$. For every $s \in R$ define the Kronecker function $\delta_s : R \to R$ by $\delta_s(r) = 1$ if $r = s$ as well as by $\delta_s(r) = 0$ if $r \neq s$, and note $\delta_s \in l^2(R)$ as well as $\|\delta_s\| = 1$. Define the multifunction $F : l^2(R) \to R$ by $\operatorname{domain}(F) = \{s\delta_s; s \in R\}$ and $F(s\delta_s) = \{s\}$. Obviously $0 \in F(0)$ and F is 1-open at $(0,0)$, namely

$$F(B(0,\epsilon)) = B(0,\epsilon)$$

for all $\epsilon > 0$. Nevertheless, in case of the classical tangency concept K, the multifunction $K_F(0,0)$ is not open at $(0,0)$ because

$$\operatorname{graph}(K_F(0,0)) = \{(0,0)\}.$$

To prove this equality, let $q \in K_F(0,0)(p)$, which means there exist sequences $\epsilon_n \in (0,1/n)$, $p_n \in B(p,1/n)$, and $q_n \in B(q,1/n)$ such that $\epsilon_n q_n \in F(\epsilon_n p_n)$. Further there exits a sequence $s_n \in R$ such that $\epsilon_n p_n = s_n\delta_{s_n}$ and $\epsilon_n q_n = s_n$. Clearly, s_n/ϵ_n converges to q and $|s_n|/\epsilon_n$ converges to $\|p\|$, hence $\|p\| = |q|$. We assert that $(p,q) = (0,0)$. Suppose, by contradiction, that $q \neq 0$. We can suppose, taking a subsequence in necessary, that $|q_n| \neq 0$, hence $s_n \neq 0$. Since s_n converges to 0, we can suppose, taking a further subsequence if necessary, that $s_n \neq s_{n'}$ whenever $n \neq n'$. Then $\delta_{s_n} = (\epsilon_n/s_n)p_n$ converges to $(1/q)p$, which contradicts the fact that $\|\delta_{s_n} - \delta_{s_{n'}}\| = \sqrt{2}$ whenever $s_n \neq s_{n'}$.

Since $\|s\delta_s - s'\delta_{s'}\| = \sqrt{|s|^2 + |s'|^2}$ whenever $s \neq s'$, it follows

$$\operatorname{domain}(F) \cap B(s\delta_s, |s|) = \emptyset$$

whenever $s \neq 0$, hence F has a closed graph, and moreover, F is not open at any point $(x,y) \in \operatorname{graph}(F) \setminus \{(0,0)\}$. According to Theorem 5, the multifunction F

is not metrically regular near $(0, 0)$. A direct proof of this fact follows. If $x = s\delta_s$, $y = s'$, and $s \neq s'$, then $d(x, F^{-1}(y))/d(y, F(x)) = \sqrt{|s|^2 + |s'|^2}/|s - s'|$, so

$$\sup_{(x,y)\in W} \frac{d(x, F^{-1}(y))}{d(y, F(x))} = +\infty$$

for every neighborhood W of $(0, 0)$.

12. Miscellanea

Let $\mathcal{S}$ be a nonempty family of nonempty sets $S \subseteq R$, and consider the tangency concepts κ, K, and $\mathcal{K}$ described in Section 2.

Resuming our general discussion, we note that $\kappa_A(a) \subseteq K_A(a)$, $K_A(a)$ and $\mathcal{K}_A(a)$ are closed sets, $\mathcal{K}_A(a) \subseteq K_A(a)$, and moreover, $0 \in \mathcal{K}_A(a) \subseteq K_A(a)$ whenever $a \in \overline{A}$.

If $t > 0$ and if $(1/t)S \in \mathcal{S}$ whenever $S \in \mathcal{S}$, then $t\kappa_A(a) \subseteq \kappa_A(a)$, $tK_A(a) \subseteq K_A(a)$, and $t\mathcal{K}_A(a) \subseteq \mathcal{K}_A(a)$. A proof of the $\mathcal{K}$-inclusion is given next. Let $x \in \mathcal{K}_A(a)$. We have to show that $tx \in \mathcal{K}_A(a)$. Let U be a neighborhood of the origin in X. Since $(1/t)U$ is a neighborhood of the origin in X, it follows there exists a neighborhood Γ of the origin in X and $S \in \mathcal{S}$ such that $\emptyset \neq (a + \gamma + s(x + (1/t)U)) \cap A$ whenever $s \in S$, $\gamma \in \Gamma$, and $a + \gamma \in A$. Note $(1/t)S \in \mathcal{S}$, let $s \in (1/t)S$, and let $\gamma \in \Gamma$ such that $a + \gamma \in A$. Then $\emptyset \neq (a + \gamma + s(tx + U)) \cap A$ because $(a + \gamma + s(tx + U) = (a + \gamma + (st)(x + (1/t)U)$ and $st \in S$, hence $tx \in \mathcal{K}_A(a)$.

A corollary of the statement above is that, if $tS \in \mathcal{S}$ whenever $t > 0$ and $S \in \mathcal{S}$, then the sets $\kappa_A(a)$, $K_A(a)$, and $\mathcal{K}_A(a)$ are cones.

If $S' \cap S'' \in \mathcal{S}$ whenever $S' \in \mathcal{S}$ and $S'' \in \mathcal{S}$, and if $(0, r) \cap S \in \mathcal{S}$ whenever $S \in \mathcal{S}$ and $r > 0$, then $\mathcal{K}_A(a) + \mathcal{K}_A(a) \subseteq \mathcal{K}_A(a)$. A proof of this inclusion is given next. Let $x' \in \mathcal{K}_A(a)$ and let $x'' \in \mathcal{K}_A(a)$. We have to show that $x' + x'' \in \mathcal{K}_A(a)$. Let U be a neighborhood of the origin in X. Choose a neighborhood U^* of the origin in X such that $U^* + U^* \subseteq U$. By the definition of $\mathcal{K}$ tangent sets, there exist $S' \in \mathcal{S}$ and a neighborhood Γ' of the origin in X such that $\emptyset \neq (a + \gamma + s(x' + U^*)) \cap A$ whenever $s \in S'$, $\gamma \in \Gamma'$, and $a + \gamma \in A$. Since $\mathcal{K}$ tangency has a local character, it follows $x'' \in \mathcal{K}_{A\cap(a+\Gamma')}(a)$, hence there exist $S'' \in \mathcal{S}$ and a neighborhood Γ'' of the origin in X such that $\emptyset \neq (a + \gamma + s(x'' + U^*)) \cap A \cap (a + \Gamma')$ whenever $s \in S''$, $\gamma \in \Gamma''$ and $a + \gamma \in A \cap (a + \Gamma')$. Note $S' \cap S'' \in \mathcal{S}$ and $\Gamma' \cap \Gamma''$ is a neighborhood of the origin in X, let $s \in S$, and let $\gamma \in \Gamma' \cap \Gamma''$ such that $a + \gamma \in A$. Since $s \in S''$, $\gamma \in \Gamma''$, and $a + \gamma \in A \cap (a + \Gamma')$, it follows there exists $u'' \in U^*$ such that $a + \gamma + s(x'' + u'') \in A \cap (a + \Gamma')$. Since $s \in S'$, $\gamma + s(x'' + u'') \in \Gamma'$, and $a + \gamma + s(x'' + u'') \in A$, it follows there exists $u' \in U^*$

such that $a+\gamma+s(x''+u'')+s(x'+u')\in A$. Since $u'+u''\in U$ it follows $\emptyset\neq(a+\gamma+s(x'+x''+U))\cap A$, hence $x'+x''\in\mathcal{K}_A(a)$. ■

A convexity proof is given in [20, p. 146, 147, Theorem 1].

Let $\{X_i\}_{i\in I}$ be a family of linear topological spaces and let the cartesian product $X=\prod_{i\in I}X_i$ be endowed with the product topology. Further, for every $i\in I$ let $A_i\subseteq X_i$ and $a_i\in X_i$, and let $a=(a_i)_{i\in I}$ and $A=\prod_{i\in I}A_i$. Then there hold the inclusions

$$\begin{aligned}\kappa_A(a) &\subseteq \prod_{i\in I}\kappa_{A_i}(a_i),\\ K_A(a) &\subseteq \prod_{i\in I}K_{A_i}(a_i),\\ \mathcal{K}_A(a) &\subseteq \prod_{i\in I}\mathcal{K}_{A_i}(a_i).\end{aligned}$$

A proof of the $\mathcal{K}$-inclusion is given next. The inclusion is obvious if $a\notin\overline{A}$, because $\mathcal{K}_A(a)=\emptyset$. Assume that $a\in\overline{A}$, i.e. $a_i\in\overline{A_i}$ whenever $i\in I$, let $x\in\mathcal{K}_A(a)$, let $x=(x_i)_{i\in I}$, let $i\in I$, and let U_i be a neighborhood of the origin in X_i. Further, let $U_j=X_j$ for every $j\in I\setminus\{i\}$ and note $U=\prod_{j\in I}U_j$ is a neighborhood of the origin in X. By the definition of $\mathcal{K}$ tangent sets, there exist a neighborhood Γ of the origin in X and $S\in\mathcal{S}$ such that the specific relation

$$\emptyset\neq(a+\gamma+s(x+U))\cap A$$

holds whenever $s\in S$, $\gamma\in\Gamma$ and $a+\gamma\in A$. Further, there exists a family $\{\Gamma_j\}_{j\in J}$ such that $\prod_{i\in I}\Gamma_i\subseteq\Gamma$ and Γ_j is a neighborhood of the origin in X_j whenever $j\in I$. Finally, let $s\in S$ and $\gamma_i\in\Gamma_i$ such that $a_i+\gamma_i\in A_i$. By hypothesis, for every $j\in I\setminus\{i\}$ there exists $\gamma_j\in\Gamma_j$ such that $a_j+\gamma_j\in A_j$. Now let $\gamma=(\gamma_j)_{j\in I}$. Since $\gamma\in\Gamma$, it follows the specific relation above holds, hence $\emptyset\neq(a_i+\gamma_i+s(x_i+U_i))\cap A_i$, $x_i\in\mathcal{K}_{A_i}(a_i)$, and $x\in\prod_{i\in I}(\mathcal{K}_{A_i}(a_i))$.

If $S'\cap S''\in\mathcal{S}$ whenever $S'\in\mathcal{S}$ and $S''\in\mathcal{S}$, then the three inclusions above can be improved to the corresponding equalities

$$\begin{aligned}\kappa_A(a) &= \prod_{i\in I}\kappa_{A_i}(a_i),\\ K_A(a) &= \prod_{i\in I}K_{A_i}(a_i),\\ \mathcal{K}_A(a) &= \prod_{i\in I}(\mathcal{K}_{A_i}(a_i)).\end{aligned}$$

A proof of the $\mathcal{K}$-equality is given next. We have to prove the $\mathcal{K}$-inclusion $\prod_{i\in I}(\mathcal{K}_{A_i}(a_i))\subseteq\mathcal{K}_A(a)$, which is obvious if $a_i\notin\overline{A_i}$ for some $i\in I$, because $\mathcal{K}_{A_i}(a_i)=$

$\emptyset$ for that $i \in I$. Assume that $a_i \in \overline{A_i}$ whenever $i \in I$, let $x_i \in \mathcal{K}_{A_i}(a_i)$ for every $i \in I$, let $x = (x_i)_{i \in I}$, and let U be a neighborhood of the origin in X. Then, there exist a family $\{U_i\}_{i \in I}$ and a finite family $J \subseteq I$ such that $\prod_{i \in I} U_i \subseteq U$, U_i is a neighborhood of the origin in X_i whenever $i \in I$, and $U_i = X_i$ whenever $i \notin J$. By the definition of $\mathcal{K}$ tangent sets, for every $i \in J$ there exist a neighborhood Γ_i of the origin in X_i and $S_i \in \mathcal{S}$ such that the i-specific relation

$$\emptyset \neq (a_i + \gamma_i + s(x_i + U_i)) \cap A_i$$

holds whenever $s \in S_i$, $\gamma_i \in \Gamma_i$ and $a_i + \gamma_i \in A_i$. Let $S = \cap_{i \in J} S_i$ and note $S \in \mathcal{S}$. Further, let $\Gamma_i = X_i$ for all $i \notin I$, let $\Gamma = \prod_{i \in I} \Gamma_i$, and note Γ is a neighborhood of the origin in X. Finally, let $s \in S$ and $\gamma \in \Gamma$ such that $a + \gamma \in A$. Let $\gamma = (\gamma_i)_{i \in I}$. If $i \in J$, then the i-specific relation above holds because $s \in S_i$, $\gamma_i \in \Gamma_i$, and $a_i + \gamma_i \in A_i$. If $i \notin J$, then the i-specific relation above holds because $U_i = X_i$ and A_i is nonempty. To conclude, $\emptyset \neq (a + \gamma + s(x + U)) \cap A$, and $x \in \mathcal{K}_A(a)$.

The equality $K_{A \times B}(a, b) = K_A(a) \times K_B(b)$ has been proved in [25, p. 569].

The elementary tangency concept κ does not have a local character, i.e. the inclusion $\kappa_A(a) \subseteq \kappa_{A \cap U}(a)$ may fail if U is a neighborhood of a. Therefore, if W is a neighborhood of (x, y) and $F_W : X \to Y$ is given by the equality $\operatorname{graph}(F_W) = W \cap \operatorname{graph}(F)$, then the inclusion $\operatorname{graph}(\kappa_F(x, y)) \subseteq \operatorname{graph}(\kappa_{(F_W)}(x, y))$ may fail, and moreover, if $(x, y) \in \operatorname{graph}(F)$, then neither ω-openness nor near ω-openness of $\kappa_F(x, y)$ at $(0, 0)$ would imply the corresponding openness of $\kappa_{(F_W)}(x, y)$. Nevertheless the two implications above do hold.

To justify the latter implication above we have to show that, if $\kappa_F(x, y)$ is nearly ω-open at $(0, 0)$, then for every $\zeta > 0$ there exists $\epsilon \in (0, \zeta)$ such that the W-inclusion

$$B(0, \omega\epsilon) \subseteq \overline{\kappa_{(F_W)}(x, y)(B(0, \epsilon))}$$

holds. Let $\zeta > 0$. We can suppose, taking a smaller ζ if necessary, that $B(x, \zeta) \times B(y, \zeta\omega) \subseteq W$. Now, choose $\epsilon \in (0, \zeta)$ such that the inclusion (28) holds. We assert that the W-inclusion holds too. Let $v \in B(0, \omega\epsilon)$ and let V be a neighborhood of v. We can suppose, taking a smaller V if necessary, that $V \subseteq B(0, \omega\epsilon)$. Then there exist $v' \in V$ and $u' \in B(0, \epsilon)$ such that $v' \in \kappa_F(x, y)(u')$. Further, there exists $s \in (0, 1]$ such that $y + sv' \in F(x + su')$. Since $x + su' \in B(x, \zeta)$ and $y + sv' \in B(y, \omega\zeta)$, it follows $y + sv' \in F_W(x, y)(x + su')$, $v' \in \kappa_{F_W}(x, y)(u')$, and v belongs to the right hand side of the W-inclusion.

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Corneliu Ursescu
"Octav Mayer"
Institute of Mathematics,
Romanian Academy,
Iaşi Branch,
8 Copou Blvd.
700505 Iaşi,
Romania
E-mail: corneliuursescu@yahoo.com