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CONTROLLABILITY FOR A CLASS OF DEGENERATE FUNCTIONAL DIFFERENTIAL INCLUSIONS IN A BANACH SPACE

Y. C. Liou, V. Obukhovskii and J. C. Yao*

Abstract. We study the controllability problem for a system governed by a degenerate semilinear functional differential inclusion in a Banach space with infinite delay. Notice that we are not assuming that the generalized semigroup generated by the linear part of inclusion is compact. Instead we suppose that the multivalued nonlinearity satisfies the regularity condition expressed in terms of the Hausdorff measure of noncompactness. It allows to obtain the general controllability principle in the terms of the topological degree theory for condensing multivalued operators. Two realizations of this principle are considered.

1. Introduction

The investigation of controllability problems for systems governed by differential and functional differential inclusions in Banach spaces attracts the attention of many researchers (see, e.g., [3, 6, 7, 12, 18, 19] and references therein). Notice that recently some controllability results were obtained also for inclusions with infinite delay (see, e.g., [10, 16]).

Let us mention, however, that some of these works (see, e.g., [16]) contain the assumption of compactness of the semigroup generated by the linear part of inclusion, as well as the supposition of the controllability of corresponding linear

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*Corresponding author

system, i.e., the invertibility of the linear controllability operator W. But it is known (see [20, 21]) that in infinite-dimensional case these hypotheses are in contradiction to each other.

In the present paper, extending some results of [18], we consider systems governed by degenerate (Sobolev type) functional differential inclusions with infinite delay assuming that the linear part of inclusion generates an arbitrary generalized C_0-semigroup. At the same time we suppose that the multivalued nonlinearity satisfies a regularity condition expressed in terms of the Hausdorff measure of noncompactness. Let us mention that the solvability of some boundary value problems for functional differential inclusions of that type was studied in the paper [2].

The paper is organized as follows. In Section 2 we give the necessary preliminaries from the fields of multivalued maps, measures of noncompactness, condensing operators and the corresponding topological degree theory. We present also necessary information concerning multivalued linear operators and generalized semigroups which they generate. At last, we present the axiomatic descrition of phase space given by Hale and Kato [13]. In Section 3 we describe the problem and introduce main assumptions. We define the multivalued operator Γ whose fixed points are generating solutions of the problem. We study the properties of Γ, in particular, we prove that it is condensing w.r.t. an appropriate vector-valued measure of noncompactness (Proposition 6). This approach allows to apply the technique of topological degree theory for condensing multivalued operators (see, e.g., [17, 8, 15]) and to obtain a general controllability result (Theorem 4). Two examples demonstrating the realization of this principle are presented.

2. Preliminaries

2.1. Multimaps and Measures of Noncompactness

Let X be a metric space, Y a normed space, $P(Y)$ denote the collection of all nonempty subsets of Y. We denote:

$$K(Y) = \{D \in P(Y) : D \text{ is compact}\};$$
$$Kv(Y) = \{D \in K(Y) : D \text{ is convex}\}.$$

We recall some notions (see e.g. [9, 15] for further details).

Definition 1. A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is upper semicontinuous (u.s.c.) if $\mathcal{F}^{-1}(\mathcal{V}) = \{x \in X : \mathcal{F}(x) \subset \mathcal{V}\}$ is an open subset of X for every open set $\mathcal{V} \subset Y$.

Sometimes we will denote a multimap by the symbol $\mathcal{F} : X \multimap Y$.

Definition 2. Let $\mathcal{E}$ be a Banach space and $(\mathcal{A}, \geq)$ a partially ordered set. A function $\beta : P(\mathcal{E}) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in $\mathcal{E}$ if

$$\beta(\overline{co}\,\Omega) = \beta(\Omega) \quad \text{for every} \quad \Omega \in P(\mathcal{E}).$$

A MNC β is called:

(i) monotone, if $\Omega_0, \Omega_1 \in P(\mathcal{E})$, $\Omega_0 \subseteq \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(ii) nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in \mathcal{E}$, $\Omega \in P(\mathcal{E})$;

(iii) invariant with respect to reflection through the origin, if $\beta(-\Omega) = \beta(\Omega)$ for every $\Omega \in P(\mathcal{E})$;

(iv) semiadditive, if $\beta(\Omega_0 \cup \Omega_1) = \max\{\beta(\Omega_0), \beta(\Omega_1)\}$ for every $\Omega_0, \Omega_1 \in P(\mathcal{E})$;

If $\mathcal{A}$ is a cone in a normed space, we say that the MNC β is

(v) algebraically semiadditive, if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in P(\mathcal{E})$;

(vi) regular, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

(vii) real, if $\mathcal{A}$ is $[0, +\infty]$ with the natural order.

As an example of MNC satisfying all above properties we can consider *the Hausdorff MNC*

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

Another examples can be presented by the following real measures of noncompactness defined on the space of bounded continuous functions $C([a, b]; E)$ on finite or infinite interval $[a, b]$ with the values in a Banach space E:

(i) *the modulus of fiber noncompactness*

$$\varphi(\Omega) = \sup_{t \in [a,b]} \chi_E(\Omega(t)),$$

where χ_E is the Hausdorff MNC in E and $\Omega(t) = \{y(t) : y \in \Omega\}$;

(ii) *the modulus of equicontinuity* defined as

$$\mathrm{mod}_C(\Omega) = \lim_{\delta \to 0} \sup_{y \in \Omega} \max_{|t_1 - t_2| \leq \delta} \|y(t_1) - y(t_2)\|.$$

It should be mentioned that these MNCs satisfy all above-mentioned properties except regularity.

To formulate the next assertion, recall the following notions. A multifunction $\Phi : [0,T] \to P(E)$ is said to be: (i) *integrable* if it admits a selection ϕ,

$$\phi(t) \in \Phi(t) \text{ for } a.e.\ t \in [0,T],$$

belonging to the space $L^1([0,T];E)$ of Bochner integrable functions; (ii) *integrably bounded* if there exists a function $\mu \in L^1_+([0,T])$ such that

$$\|\Phi(t)\| := \sup_{\psi \in \Phi(t)} \|\psi\| \leq \mu(t) \text{ for } a.e.\ t \in [0,T].$$

For an integrable multifunction $\Phi : [0,T] \to P(E)$, let us denote by $\mathcal{S}_\Phi$ the set of all its integrable selections. Then for any $t \in [0,T]$ *the multivalued integral* of Φ on $[0,t]$ is defined in the following way:

$$\int_0^t \Phi(s)ds = \{\int_0^t \phi(s)ds : \phi \in \mathcal{S}_\Phi\}.$$

Proposition 1. (See Theorem 4.2.3 of [15]). *Let E be a separable Banach space and $\Phi : [0,T] \to P(E)$ an integrable, integrably bounded multifunction such that*

$$\chi_E\left(\Phi(t)\right) \leq q(t) \text{ for } a.e.\ t \in [0,T],$$

where $q(\cdot) \in L^1_+([0,T])$. Then

$$\chi_E\left(\int_0^t \Phi(s)ds\right) \leq \int_0^t q(s)ds$$

for all $t \in [0,T]$.

Let $\mathcal{E}, \mathcal{E}'$ be Banach spaces with MNCs β and β' respectively, $\mathcal{J} : \mathcal{E} \to \mathcal{E}'$ a bounded linear operator.

Definition 2. (cf. [1]). The value

$$\|\mathcal{J}\|^{(\beta',\beta)} = \inf\{C : \beta'\left(\mathcal{J}\Omega\right) \leq C\beta\left(\Omega\right); \Omega \subset \mathcal{E} \text{ is a bounded set}\}$$

is called the (β',β)-norm of $\mathcal{J}$.

In particular, if β, β' are the Hausdorff MNCs χ, χ', the value $\|\mathcal{J}\|^{(\chi',\chi)}$ is denoted by $\|\mathcal{J}\|^{(\chi)}$ and is called the χ-norm of $\mathcal{J}$. The χ-norm may be evaluated by the formula ([1])

$$\|\mathcal{J}\|^{(\chi)} = \chi'\left(\mathcal{J}S\right) = \chi'\left(\mathcal{J}B\right)$$

where S is an unit sphere and B is an unit ball in $\mathcal{E}$. The above formula easily implies

$$\|\mathcal{J}\|^{(x)} \leq \|\mathcal{J}\|.$$

Definition 4. A multimap $\mathcal{F} : X \subseteq \mathcal{E} \to K(\mathcal{E})$ is called condensing with respect to a MNC β (or β-condensing) if for every bounded set $\Omega \subseteq X$ that is not relatively compact we have

$$\beta(\mathcal{F}(\Omega)) \ngeq \beta(\Omega).$$

Let $D \subseteq \mathcal{E}$ be a nonempty convex closed set; $V \subset D$ be an bounded relatively open set, β a monotone nonsingular MNC in $\mathcal{E}$ and $\mathcal{F} : \overline{V} \to Kv(D)$ an u.s.c. β-condensing multimap such that $x \notin \mathcal{F}(x)$ for all $x \in \partial V$, where $\overline{V}$ and ∂V denote the relative closure and the boundary of the set V.

In such a setting, *the relative topological degree*

$$\deg_D\left(i - \mathcal{F}, \overline{V}\right)$$

of the corresponding multivalued vector field $i - \mathcal{F}$ satisfying the standard properties is defined (see, for example, [17, 8, 15]). In particular, the condition

$$\deg_D\left(i - \mathcal{F}, \overline{V}\right) \neq 0$$

implies that *the fixed points set* $Fix\mathcal{F} = \{x : x \in \mathcal{F}(x)\}$ is a nonempty subset of V.

The application of the topological degree theory yields the following fixed point principles which we will use in the sequel.

Theorem 1. (cf. [15], Corollary 3.3.1). *Let $\mathcal{Q}$ be a bounded convex closed subset of $\mathcal{E}$ and $\mathcal{F} : \mathcal{Q} \to Kv(\mathcal{Q})$ an u.s.c. β-condensing multimap. Then $Fix\mathcal{F} \neq \emptyset$.*

Theorem 2. (cf. [15], Theorem 3.3.4). *Let $a \in V$ be an interior point and $\mathcal{F} : \overline{V} \to Kv(D)$ an u.s.c. β-condensing multimap satisfying the boundary condition*

$$x - a \notin \lambda\left(\mathcal{F}(x) - a\right)$$

for all $x \in \partial V$ and $0 < \lambda \leq 1$. Then $Fix\mathcal{F} \neq \emptyset$.

2.2. Multivalued linear operators

We begin with some necessary definitions and results from the theory of multivalued linear operators. Details can be found in [4, 5], and [11].

Let E be a complex Banach space.

Definition 5. A multivalued map (multimap) $A : E \to 2^E$ is said to be a multivalued linear operator (MLO) on E if:

(1) $D(A) = \{x \in E : Ax \neq \emptyset\}$ is a linear subspace of E;

(2)
$$\begin{cases} Ax + Ay \subset A(x+y), & \forall x, y \in D(A) ; \\ \lambda Ax \subseteq A(\lambda x), & \forall \lambda \in \mathbb{C}, x \in D(A) . \end{cases}$$

It is an easy consequence of the definition to note that $Ax + Ay = A(x+y)$ for all $x, y \in D(A)$ and $\lambda Ax = A(\lambda x)$ for all $x \in D(A)$, $\lambda \neq 0$. It is also clear that A is a MLO on E if and only if its graph G_A is a linear subspace of $E \times E$. A MLO A is said to be *closed* if G_A is the closed subspace of $E \times E$. The collection of all closed MLO's in E will be denoted by $ML(E)$.

Definition 6. The inverse A^{-1} of a MLO is defined as:

(1) $D\left(A^{-1}\right) = R(A)$;

(2) $A^{-1}y = \{x \in D(A) : y = Ax\}$.

It is obvious that $(y, x) \in G_{A^{-1}}$ if and only if $(x, y) \in G_A$ and hence $A^{-1} \in ML(E)$ if $A \in ML(E)$.

Denote by $\mathcal{L}(E)$ the space of all single-valued bounded operators on E.

Definition 7. The resolvent set $\rho(A)$ of a MLO A is defined as the collection of all $\lambda \in \mathbb{C}$ for which:

(1) $R(\lambda I - A) = D\left((\lambda I - A)^{-1}\right) = E$;

(2) $(\lambda I - A)^{-1} \in \mathcal{L}(E)$.

Definition 8. The operator-valued function $R(\cdot, A) : \rho(A) \to \mathcal{L}(E)$

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

is called the *resolvent* of a MLO A.

Remark 1. If E is a real Banach space and A is a MLO on E, we may consider the *complexification* $\widetilde{E} = E + iE$ and $\widetilde{A}$ defined by

$$G_{\widetilde{A}} = \{(x, y_1) + i(x, y_2) : x \in D(A), y_1, y_2 \in Ax\}.$$

Then we set, by definition, $\rho(A) = \rho(\widetilde{A})$.

Let $U : \mathbb{R}_+ = [0, +\infty) \to \mathcal{L}(E)$ be a C_0-semigroup of operators, i.e., we suppose the following conditions:

(i) $U(t+s) = U(t)U(s)$, $\forall t, s \in \mathbb{R}_+$;

(ii) for each $x \in E$, the function $t \to U(t)x$ is continuous on $\mathbb{R}_+$.

Notice that the usual condition $U(0) = I$ is absent here. From assumption (i) it follows that $U(0) = P \in \mathcal{L}(E)$ is the projector. In case $P \neq I$ the semigroup U is called *generalized* (or *degenerate*).

It is easy to verify that there exist constants $C \geq 1$ and $\gamma \geq 0$ such that

$$\|U(t)\|_{\mathcal{L}(E)} \leq Ce^{\gamma t}, \quad t \in \mathbb{R}_+. \tag{0.1}$$

Therefore, for each $\lambda \in \mathbb{C}_\gamma = \{\mu \in \mathbb{C} : Re\,\mu > \gamma\}$ the bounded linear operator $R(\lambda)$ may be defined by the following Laplace transformation:

$$R(\lambda)x = \int_0^\infty U(\tau)xe^{-\lambda\tau}d\tau.$$

The function $R : \mathbb{C}_\gamma \to \mathcal{L}(E)$ satisfies Hilbert equality and it is the resolvent of a certain (unique) $A \in ML(E)$. This MLO A is called the *generator* of the generalized semigroup U.

Let E^* be the dual space of E. For $A \in ML(E)$, we denote by A^* a MLO on E^* defined in the following way: for $h, g \in E^*$, the relation $h \in A^*(g)$ means that $g(y) = h(x)$ for all pairs $(x, y) \in G_A$. It is easy to verify that $A^*0^* = \{h \in E^* : \overline{D(A)} \subset Kerh\} = \overline{D(A)}^\perp$.

Consider the following assumptions on $A \in ML(E)$.

(A_1) functionals from A^*0^* are separated by vectors of $A0$, i.e., for each $h \in A^*0^*$, $h \neq 0^*$ there exists $y \in A0$ such that $h(y) \neq 0$;

(A_2) the Hille–Yosida condition: there exist a constant $C > 0$ and $\gamma \in \mathbb{R}$ such that $\mathbb{C}_\gamma \subset \rho(A)$ and

$$\|R(\lambda, A)^n\|_{\mathcal{L}(E)} \leq \frac{C}{(Re\,\lambda - \gamma)^n}, \quad n = 1, 2, ... \quad \lambda \in \mathbb{C}_\gamma.$$

Remark 2. In [4] it was shown that each of the following conditions implies (A_1): (i) the space E is reflexive; (ii) $dim\,A0 = dimA^*0^* < \infty$.

The following result holds true (cfr. [4, 11]).

Theorem 3. *Conditions (A_1) and (A_2) are necessary and sufficient for $A \in ML(E)$ to be the generator of a C_0-semigroup U. Moreover, the semigroup U is generalized iff A is not single-valued. In this case the space E may be represented as $E = E_1 \oplus E_1$, where $E_0 = \overline{D(A)}$, $E_1 = A0$ and the restriction of $U(t)$ on E_0 defines the usual C_0-semigroup on E_0 whereas the restriction on E_1 vanishes.*

2.3. Phase space

We will employ the axiomatical definition of the *phase space* $\mathcal{B}$, introduced by J. K. Hale and J. Kato (see [13, 14]). The space $\mathcal{B}$ will be considered as a linear topological space of functions mapping $(-\infty, 0]$ into a Banach space E endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$.

For any function $y_t : (-\infty; T] \to E$ and for every $t \in (-\infty; T]$, y_t represents the function from $(-\infty, 0]$ into E defined by

$$y_t(\theta) = y(t + \theta),\ \theta \in (-\infty; 0].$$

We will assume that $\mathcal{B}$ satisfies the following axioms.

($\mathcal{B}$) If $y : (-\infty; T] \to E$ is continuous on $[0; T]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0; T]$ we have

(i) $y_t \in \mathcal{B}$;

(ii) function $t \mapsto y_t$ is continuous;

(iii) $\|y_t\|_{\mathcal{B}} \leq K(t) \sup\limits_{0 \leq \tau \leq t} \|y(\tau)\| + N(t)\|y_0\|_{\mathcal{B}}$, where $K(\cdot), N(\cdot) : [0; \infty) \to [0; \infty)$ are independent on y, $K(\cdot)$ is strictly positive and continuous, and $N(\cdot)$ is bounded.

We may consider the following examples of phase spaces satisfying all above properties.

(1) For $\nu > 0$ let $\mathcal{B} = C_\nu$ be a space of continuous functions $\psi : (-\infty; 0] \to E$ having a limit $\lim\limits_{\theta \to -\infty} e^{\nu\theta}\psi(\theta)$ with

$$\|\psi\|_{\mathcal{B}} = \sup_{-\infty < \theta \leq 0} e^{\nu\theta}\|\psi(\theta)\|.$$

(2) (*Spaces of "fading memory"*). Let $\mathcal{B} = C_\rho$ be a space of functions $\psi : (-\infty; 0] \to E$ such that

(a) ψ is continuous on $[-r; 0], r > 0$;

(b) ψ is Lebesgue measurable on $(-\infty; r)$ and there exists a positive Lebesgue integrable function $\rho : (-\infty; -r) \to \mathbb{R}^+$ such that $\rho\psi$ is Lebesgue integrable on $(-\infty; r)$; moreover, there exists a locally bounded function $P : (-\infty; 0] \to \mathbb{R}^+$ such that, for all $\xi \leq 0$, $\rho(\xi + \theta) \leq P(\xi)\rho(\theta)$ a.e. $\theta \in (-\infty; -r)$. Then,

$$\|\psi\|_{\mathcal{B}} = \sup_{-r \leq \theta \leq 0} \|\psi(\theta)\| + \int_{-\infty}^{-r} \rho(\theta)\|\psi(\theta)\|d\theta$$

.

A simple example of such a space is given by $\rho(\theta) = e^{\mu\theta}$, $\mu \in \mathbb{R}$.

3. Controllability Problem

Let $M : D(M) \subseteq E \to E$ be a bounded linear operator and $L : D(L) \subseteq E \to E$ a closed linear operator in a real separable Banach space E satisfying the condition

$$(ML)\, D(L) \subseteq D(M) \text{ and } \overline{M(D(L))} \subseteq R(M).$$

We will consider the nonlinear control system governed by a degenerate (Sobolev type) functional differential inclusion in E of the form

$$\frac{dMx(t)}{dt} \in Lx(t) + F(t, Mx_t) + Bu(t), \; t \in [0,T] := J \tag{3.1}$$

where the function $x : (-\infty, T] \to E$ satisfies the initial condition

$$Mx_0 = \widetilde{\psi} \in \mathcal{B}\,. \tag{3.2}$$

With the change $y(t) = Mx(t)$ we can rewrite system (3.1) - (3.2) into the following form

$$\frac{dy(t)}{dt} \in Ay(t) + F(t, y_t) + Bu(t), \qquad t \in J, \tag{3.3}$$

$$y_0 = \widetilde{\psi} \in \mathcal{B}\,, \tag{3.4}$$

where $A = LM^{-1}$. It is clear that $A \in ML(E)$ if M is not invertible and that $D(A) = M(D(L))$.

It will be supposed that:

(A) $A = LM^{-1}$ satisfies conditions (A_1), (A_2) of Section 2.2

and whence A is the generator of a C_0-semigroup U. It should be mentioned that to guarantee condition (A_2), it is sufficient to assume that:

(i) $[Ly, My] \leq \gamma \|My\|^2, \forall y \in D(L)$ for some $\gamma \in \mathbb{R}$, where $[,]$ is a semi-scalar product in E
and

(ii) (ii) $R(\lambda_0 M - L) = E$ for some $\lambda_0 > \gamma$.
(See [4]).

We will denote

$$\mathcal{U} = \sup_{t \in J} \|U(t)\| \ .$$

In the sequel, we consider the phase space $\mathcal{B}$ of functions $\psi : (-\infty, 0] \to E_0$, with $E_0 = \overline{D\,(A)} = \overline{M\,(D\,(L))}$, satisfying all axioms of Section 2.3.

We will assume that the multivalued nonlinearity $F : J \times \mathcal{B} \to Kv(E)$ obeys the following conditions:

(F1) for each $\psi \in \mathcal{B}$, the multifunction $F(\cdot, \psi) : J \to Kv(E)$ admits a measurable selection;

(F2) for a.e. $t \in J$, the multimap $F(t, \cdot) : \mathcal{B} \to Kv(E)$ is u.s.c.;

(F3) for each nonempty, bounded set $\Omega \subset \mathcal{B}$, there exists a function $\alpha_\Omega \in L^1_+(J)$ such that

$$\|F(t, \psi)\|_E := \sup\{\|z\|_E : z \in F(t, \psi)\} \leq \alpha_\Omega(t)$$

for a.e. $t \in J$, $\psi \in \Omega$;

(F4) there exists a function $k \in L^1_+(J)$ such that for each nonempty bounded set $\Omega \subset \mathcal{B}$

$$\chi(F(t, \Omega)) \leq k(t)\varphi(\Omega)$$

for a.e. $t \in J$, where χ is the Hausdorff MNC in E and $\varphi(\Omega)$ is the modulus of fiber noncompactness of the set Ω.

Remark 3. It is known (see, e.g., [9, 15]) that condition $(F1)$ is fulfilled if the multifunction $F\,(\cdot, \psi)$ is measurable for each $\psi \in \mathcal{B}$.

Remark 4. Under conditions $(F1) - (F3)$ for every continuous function $v : J \to \mathcal{B}$ the multifunction $F(t, v(t))$ is integrable (the proof is analogous to the one of Theorem 1.3.5 in [15]).

By the symbol $\mathcal{C}((-\infty, T]; E_0)$ we will denote the linear topological space of functions $y : (-\infty, T] \to E_0$ such that $y_0 \in \mathcal{B}$ and the restriction $y|_J$ is continuous, endowed with a seminorm

$$\|y\|_{\mathcal{C}} = \|y_0\|_{\mathcal{B}} + \|y\|_C,$$

where the last norm is the usual sup-norm in the space $C(J; E_0)$.

For $\widetilde{\psi} \in \mathcal{B}$ from initial condition (3.2), we consider the set

$$D = \{y \in C(J; E_0) : y(0) = \widetilde{\psi}(0)\}$$

which is obviously convex and closed.

Further, for any $y \in D$ we define the function $y[\widetilde{\psi}] \in \mathcal{C}((-\infty, T]; E_0)$:

$$y[\widetilde{\psi}](t) = \begin{cases} \widetilde{\psi}(t), & -\infty < t < 0, \\ y(t), & 0 \le t \le T\,. \end{cases}$$

Then, clearly for $t \in J$:

$$y[\widetilde{\psi}]_t(\theta) = \begin{cases} \widetilde{\psi}(t+\theta), & -\infty < \theta < -t, \\ y(t+\theta), & -t \le \theta \le 0\,. \end{cases}$$

We consider a map $\pi : J \times D \to \mathcal{B}$ defined by

$$\pi(t, y) = y[\widetilde{\psi}]_t.$$

Notice that $\pi(\cdot, y)$ is continuous by axiom $(\mathcal{B})(ii)$.

Moreover, we claim that $\pi(t, \cdot)$ is Lipschitz continuous in the seminorm $\|\cdot\|_{\mathcal{B}}$ uniformly with respect to $t \in J$. In fact, denoting

$$K = \max_{t \in J} K(t) \tag{3.5}$$

for the function $K(\cdot)$ from axiom $(\mathcal{B})(iii)$, we obtain for any $y, y' \in D$ by the same axiom

$$\begin{aligned} \|\pi(t,y) - \pi(t,y')\|_{\mathcal{B}} = \|y[\widetilde{\psi}]_t - y'[\widetilde{\psi}]_t\|_{\mathcal{B}} &\le K\|y - y'\|_C + N(t)\|y[\widetilde{\psi}]_0 - y'[\widetilde{\psi}]_0\|_{\mathcal{B}} \\ &= K\|y - y'\|_C. \end{aligned}$$

Now we may consider the superposition multioperator $\mathcal{P}_F : D \multimap L^1(J; E)$:

$$\begin{aligned} &\mathcal{P}_F(y) = \mathcal{S}_{F(\cdot, \pi(\cdot, y))} \\ &= \left\{ f \in L^1(J; E) : f(t) \in F(t, \pi(t, y)) = F(t, y[\widetilde{\psi}]_t) \;\; a.e.\ t \in J \right\}. \end{aligned} \tag{3.6}$$

By applying Remark , $\mathcal{P}_F$ is well defined. Moreover, using the uniform Lipschitz continuity of $\pi(t, \cdot)$ and applying Lemma 5.1.1 of [15] we have the following property of weak closedness of $\mathcal{P}_F$.

Lemma 1. *Let $\{y_n\}$ be a sequence in D converging to $y_0 \in D$ and suppose that a sequence $\{f_n\} \subset L^1([0,T]; E)$, $f_n \in \mathcal{P}_F(y_n)$, $n \ge 1$ weakly converges to a function f_0. Then $f_0 \in \mathcal{P}_F(y_0)$.*

Further, we suppose that the control function $u(\cdot)$ is given in $L^2(J; \mathbb{U})$, where $\mathbb{U}$ is the Banach space of controls.

At last, $B : \mathbb{U} \to E$ is a bounded linear operator.

Definition 9. A function $x : (-\infty, T] \to E$ is called a mild solution to problem (3.1) - (3.2) if the function $y(t) = Mx(t)$ satisfies initial condition (3.2) and on interval J it has the form

$$y(t) = U(t)y(0) + \int_0^t U(t-s) f(s) \, ds + \int_0^t U(t-s) Bu(s) \, ds \ ,$$

where $f \in \mathcal{P}_F(y)$ and $u \in L^2(J; \mathbb{U})$.

We will consider the controllability problem for the above system, i.e., assuming that an initial function $\widetilde{\psi} \in \mathcal{B}$ and a point $x_* \in E_0$ are given, we will study conditions which guarantee the existence of a mild solution x to problem (3.1) - (3.2) satisfying

$$Mx(T) = x_* \ . \tag{3.7}$$

A pair (x, u) satisfying $(3.1), (3.2), (3.7)$ will be called *a solution of controllability problem* $(3.1), (3.2), (3.7)$.

Toward this goal we will suppose the standard assumption on the controllability of the corresponding linear problem. More exactly, we assume that the linear controllability operator $W : L^2(J; \mathbb{U}) \to E_0$ given by

$$Wu = \int_0^T U(T-s) Bu(s) \, ds$$

has a bounded inverse

$$W^{-1} : E_0 \to L^2(J; \mathbb{U}) / KerW \ .$$

Let us mention, that we may suppose, w.l.o.g., that

$$W^{-1} : E_0 \to L^2(J; \mathbb{U}) \ ,$$

(see [3, 19]).

Let $\mathcal{M}_1, \mathcal{M}_2$ be positive constants such that

$$\|B\| \leq \mathcal{M}_1$$

and

$$\left\|W^{-1}\right\| \leq \mathcal{M}_2 \ .$$

Let us consider the multivalued operator $\Gamma : D \multimap D$ of the following form

$$\Gamma(y) = \Big\{ z \in D : z(t) = U(t)\widetilde{\psi}(0)$$
$$+ \int_0^t U(t-s) \left[f(s) + BW^{-1} \left(x_* - U(T)\widetilde{\psi}(0) - \int_0^T U(T-\tau) f(\tau) \, d\tau \right)(s) \right] ds :$$
$$: f \in \mathcal{P}_F(y) \} \ .$$

It is easy to see that each fixed point $y \in \Gamma(y)$ of the multioperator Γ naturally generates a solution (x, u) of controllability problem $(3.1), (3.2), (3.7)$.

To investigate the properties of the multioperator Γ, we will need some notions and results.

Proposition 2. (cf. Theorem 5.1.2 and Corollary 5.1.2 of [15]). *Let* $\mathbb{S} : L^1(J;E) \to C(J;E)$ *be an abstract operator satisfying the following conditions:*
$(\mathbb{S}1')$ *there exists* $\Delta \geq 0$ *such that*

$$\|\mathbb{S}f - \mathbb{S}g\|_C \leq \Delta \|f - g\|_{L^1} \ , \ \forall f, g \in L^1(J;E) \ ;$$

$(\mathbb{S}2)$ *for each compact set* $\mathcal{K} \subset E$ *and sequence* $\{f_n\} \subset L^1(J;E)$ *such that* $\{f_n(t)\} \subset \mathcal{K}$ *for a.e.* $t \in J$*, the weak convergence* $f_n \rightharpoonup f_0$ *implies* $\mathbb{S}f_n \to \mathbb{S}f_0$.

If $\mathcal{P}_F : D \multimap L^1(J;E)$ *is the superposition multioperator given by* (3.6) *then the composition* $\mathbb{S} \circ \mathcal{P}_F : D \multimap C(J;E)$ *is an u.s.c. multimap with compact values.*

Definition 10. The sequence $\{f_n\} \subset L^1(J;E)$ is said to be semicompact if it is integrably bounded and the set $\{f_n(t)\}$ is relatively compact in E for a.e. $t \in J$.

Proposition 3. (Theorem 5.1.1 of [15]). *Let* $\mathbb{S} : L^1(J;E) \to C(J;E)$ *be an operator satisfying conditions* $(\mathbb{S}1')$ *and* $(\mathbb{S}2)$*. Then for every semicompact sequence* $\{f_n\} \subset L^1(J;E)$ *the sequence* $\{\mathbb{S}f_n\}$ *is relatively compact in* $C(J;E)$.

Definition 11. The bounded linear operator $G : L^1(J;E) \to C(J;E_0)$ defined as

$$(Gf)(t) = \int_0^t U(t-s) f(s)\, ds$$

is called Cauchy operator.

Following Lemma 4.2.1. of [15] one may verify the following assertion.

Proposition 4. *The Cauchy operator G satisfies properties* $(\mathbb{S}1') - (\mathbb{S}2)$.

Consider now the operator $S : L^1(J;E) \to D$ defined by

$$\begin{aligned} (Sf)(t) &= U(t)\widetilde{\psi}(0) \\ &\quad + \int_0^t U(t-s)\Big[f(s) + BW^{-1}\Big(x_* - U(T)\widetilde{\psi}(0) \\ &\quad - \int_0^T U(T-\tau) f(\tau)\, d\tau\Big)(s)\Big]\, ds \ . \end{aligned} \tag{3.8}$$

Lemma 2. *The operator S satisfies properties* $(\mathbb{S}1') - (\mathbb{S}2)$.

Proof. Since S can be represented as

$$(Sf)(t) = U(t)\widetilde{\psi}(0) + (Gf)(t) + (S_1 f)(t),$$

where

$$(S_1 f)(t) = \int_0^t U(t-s)\, BW^{-1}\left(x_* - U(T)\widetilde{\psi}(0) - \int_0^T U(T-\tau) f(\tau)\, d\tau\right)(s)\, ds$$

it is sufficient, by Proposition 4, to prove the assertion only for the operator S_1. To verify property $(\mathbb{S}1')$ let us take any functions $f, g \in L^1(J; E)$. Then we have for $t \in J$

$$\begin{aligned}
&\|(S_1 f)(t) - (S_1 g)(t)\|_E \\
&= \left\| \int_0^t U(t-s)\, BW^{-1}\left(\int_0^T U(T-\tau)(g(\tau) - f(\tau))\, d\tau\right)(s)\, ds \right\| \\
&\leq \mathcal{U}\mathcal{M}_1 \int_0^t \left\| W^{-1}\left(\int_0^T U(T-\tau)(g(\tau) - f(\tau))\, d\tau\right)(s)\right\| ds \\
&\leq \mathcal{U}\mathcal{M}_1 \left\| W^{-1}\left(\int_0^T U(T-\tau)(g(\tau) - f(\tau))\, d\tau\right)\right\|_{L^1(J;\mathbb{U})} \\
&\leq \mathcal{U}\mathcal{M}_1 \sqrt{T} \left\| W^{-1}\left(\int_0^T U(T-\tau)(g(\tau) - f(\tau))\, d\tau\right)\right\|_{L^2(J;\mathbb{U})} \\
&\leq \mathcal{U}\mathcal{M}_1 \mathcal{M}_2 \sqrt{T} \left\| \int_0^T U(T-\tau)(g(\tau) - f(\tau))\, d\tau \right\|_E \\
&\leq \mathcal{U}^2 \mathcal{M}_1 \mathcal{M}_2 \sqrt{T}\, \|f - g\|_{L^1}.
\end{aligned}$$

So we have

$$\|S_1 f - S_1 g\|_C \leq \mathcal{U}^2 \mathcal{M}_1 \mathcal{M}_2 \sqrt{T}\, \|f - g\|_{L^1}.$$

To check up property $(\mathbb{S}2)$ let us represent the operator S_1 in the form

$$S_1 f = G\left(BW^{-1}\left(x_* - U(T)\widetilde{\psi}(0) - \theta G f\right)\right)$$

where $\theta : C(J; E_0) \to E_0$, $\theta y = y(T)$ is a bounded linear operator. Then the assertion follows from Proposition 4 and the boundedness of the linear operators W^{-1}, B and G. ■

Now, as an immediate consequence of Proposition 2 and Lemma 2 we have the following assertion.

Proposition 5. *The multioperator* Γ *is u.s.c. and has compact convex values.*

Now our goal is to give conditions under which the multioperator Γ is condensing. Let

$$\mathcal{N} = \sup_{0<t\leq T} \|U(t)\|^{(\chi)} \tag{3.9}$$

It is clear that $\mathcal{N} \leq \mathcal{U}$. Let $\mathcal{N}_1 \geq 0$ be a constant such that

$$\|B\|^{(\chi)} \leq \mathcal{N}_1 \tag{3.10}$$

(obviously we may assume $\mathcal{N}_1 \leq \mathcal{M}_1$). At last, denoting by $\chi_{\mathbb{U}}$ the Hausdorff MNC in the space $\mathbb{U}$, we suppose that there exists a function $\varkappa(\cdot) \in L^1_+(J)$ such that for each bounded set $\Omega \subset E_0$ we have

$$\chi_{\mathbb{U}}\left(W^{-1}(\Omega)(t)\right) \leq \varkappa(t)\,\chi_E(\Omega) \quad a.e.\ t \in J\ .$$

Now, let us assume that the following condition holds:

$$\left(\mathcal{N} + \mathcal{N}^2\mathcal{N}_1 \int_0^T \varkappa(s)\,ds\right)\int_0^T k(\tau)\,d\tau < 1\ , \tag{C}$$

where $k(\cdot)$ is the function from condition $(F4)$.

Remark 5. Notice that condition (C) is trivially satisfied when the nonlinearity F is compact in the second argument, i.e., $k(t) = 0$ for a.e. $t \in J$.

Consider the MNC

$$\nu(\Omega) = (\varphi(\Omega), \mathrm{mod}_C(\Omega)) \tag{3.11}$$

in the space $C(J; E_0)$ with values in the cone $\mathbb{R}^2_+$, where φ is the modulus of fiber noncompactness and mod_C is the modulus of equicontinuity (see Section 2.1). The MNC ν is monotone, nonsingular and regular.

Proposition 6. *Under condition (C) the multioperator Γ is ν-condensing.*

Proof. Let $\Omega \subset D$ be a bounded set such that

$$\nu(\Gamma(\Omega)) \geq \nu(\Omega) \tag{3.12}$$

in the sense of order generated by the cone $\mathbb{R}^2_+$. We will show that (3.12) implies that Ω is relatively compact.

Let us estimate the value $\varphi(\Gamma(\Omega))$. For any $t \in J$ we have

$$\Gamma(\Omega)(t) \subset U(t)\widetilde{\psi}(0) + G \circ \mathcal{P}_F(\Omega)(t) + S_1 \circ \mathcal{P}_F(\Omega)(t) \tag{3.13}$$

Then, applying (3.9) and $(F4)$ we obtain

$$\chi_E\left(\{U(t-s)\,f(s) : f \in \mathcal{P}_F(\Omega)\}\right) \leq \\ \leq \mathcal{N}k(s)\,\chi_E(\Omega(s)) \leq \mathcal{N}k(s)\,\varphi(\Omega) \ .$$

By Proposition 1

$$\begin{aligned} \chi_E\left(G \circ \mathcal{P}_F(\Omega)(t)\right) &\leq \mathcal{N}\varphi(\Omega)\int_0^t k(s)\,ds \\ &\leq \mathcal{N}\int_0^T k(s)\,ds \cdot \varphi(\Omega) \ . \end{aligned} \tag{3.14}$$

Applying (3.14) we can estimate $\chi_E(S_1 \circ \mathcal{P}_F(\Omega)(t))$ for $t \in J$. Indeed,we have

$$\chi_E\left(\left\{U(t-s)\,BW^{-1}\left(x_* - U(t)\widetilde{\psi}(0) - \int_0^T U(T-\tau)\,f(\tau)\,d\tau\right)(s) : f \in \mathcal{P}_F(\Omega)\right\}\right)$$
$$\leq \mathcal{N}\mathcal{N}_1\varkappa(s)\,\chi_E\left(\left\{\int_0^T U(T-\tau)\,f(\tau)\,d\tau : f \in \mathcal{P}_F(\Omega)\right\}\right)$$
$$\leq \mathcal{N}^2\mathcal{N}_1\int_0^T k(s)\,ds \cdot \varphi(\Omega) \cdot \varkappa(s) \ .$$

So, by Proposition 1

$$\begin{aligned} \chi_E\left(S_1 \circ \mathcal{P}_F(\Omega)(t)\right) &\leq \mathcal{N}^2\mathcal{N}_1\int_0^T k(s)\,ds \cdot \varphi(\Omega) \cdot \int_0^t \varkappa(s)\,ds \\ &\leq \mathcal{N}^2\mathcal{N}_1\int_0^T k(s)\,ds \cdot \int_0^T \varkappa(s)\,ds \cdot \varphi(\Omega) \ . \end{aligned}$$

Therefore, by (3.13), for each $t \in J$ we have

$$\chi_E\left(\Gamma(\Omega)(t)\right) \leq \chi_E\left(G \circ \mathcal{P}_F(\Omega)(t)\right) + \chi_E\left(S_1 \circ \mathcal{P}_F(\Omega)(t)\right)$$
$$\leq \left(\mathcal{N} + \mathcal{N}^2\mathcal{N}_1\int_0^T \varkappa(s)\,ds\right)\int_0^T k(s)\,ds \cdot \varphi(\Omega) \ .$$

Hence

$$\varphi(\Gamma(\Omega)) \leq q\varphi(\Omega) \ , \tag{3.15}$$

where, by (C),

$$q = \left(\mathcal{N} + \mathcal{N}^2\mathcal{N}_1\int_0^T \varkappa(s)\,ds\right)\int_0^T k(s)\,ds < 1 \ .$$

Comparing (3.15) with (3.12) we come to the conclusion that

$$\varphi(\Omega) = 0 \ . \tag{3.16}$$

Now we show that $\text{mod}_C(\Omega) = 0$, i.e., the set Ω is equicontinuous. Notice that from

$$\text{mod}_C(\Gamma(\Omega)) \geq \text{mod}_C(\Omega)$$

it follows that it is sufficient to verify that the set $\Gamma(\Omega)$ is equicontinuous. This is equivalent to show that every sequence $\{z_n\} \subset \Gamma(\Omega)$ satisfies this property. Given a sequence $\{z_n\}$ there exists a sequence $\{y_n\} \subset \Omega$ and a sequence of selections $\{f_n\}$, $f_n \in \mathcal{P}_F(y_n)$ such that

$$z_n(t) = U(t)\widetilde{\psi}(0) + (Gf_n)(t) + (S_1 f_n)(t) \ , \quad t \in J \ .$$

Condition $(F3)$ implies that the sequence of functions $\{f_n\}$ is integrably bounded. By (3.16), the sequence $\{y_n\}$ satisfies the equality

$$\chi_E(\{y_n(t)\}) = 0 \ , \quad \forall t \in J \ ,$$

hence, by condition $(F4)$, we have

$$\chi_E(\{f_n(t)\}) = 0 \text{ for } a.e.\ t \in J$$

and so the sequence $\{f_n\}$ is semicompact. Applying Propositions 3 and 4 and Lemma 2 we come to the conclusion that the sequence $\{z_n\}$ is relatively compact and, hence, equicontinuous. ∎

We can now observe that the topological degree theory described in Section 2.1 can be applied to the multioperator Γ. We can formulate the following general existence principle.

Theorem 4. *Let $V \subset D$ be a bounded open set such that $y \notin \Gamma(y)$ for all $y \in \partial V$. If $\deg_D\left(i - \Gamma, \overline{V}\right) \neq 0$ then problem $(3.1), (3.2), (3.7)$ has a solution (x, u) such that the function $y(t) = Mx(t)$, $t \in J$ is conained in V.*

We will consider several realizations of this general principle. We will need the following assertion.

Lemma 3. *Let $z \in \lambda\Gamma(y)$ for some $0 < \lambda \leq 1$, i.e., $z = \lambda S(f)$, where $f \in \mathcal{P}_F(y)$. Then for each $t \in J$ we have the following estimate:*

$$\|z(t)\| \leq \mathcal{U}\|\widetilde{\psi}(0)\| + \mathcal{U}\mathcal{M}_1\mathcal{M}_2\sqrt{T}\left(\|x_*\| + \mathcal{U}\|\widetilde{\psi}(0)\|\right) + \mathcal{U}\int_0^t \|f(s)\|ds + \mathcal{U}^2\mathcal{M}_1\mathcal{M}_2\sqrt{T}\int_0^T \|f(\tau)\|d\tau.$$

Proof. We have

$$z(t) = \lambda U(t)\widetilde{\psi}(0) + \lambda \int_0^t U(t-s) f(s)\, ds + \lambda \int_0^t U(t-s) BW^{-1}\left(x_* - U(T)\widetilde{\psi}(0) - \int_0^T U(T-\tau) f(\tau)\, d\tau\right)(s)\, ds$$

and hence

$$\begin{aligned}
\|z(t)\| &\le \mathcal{U}\left\|\widetilde{\psi}(0)\right\| + \mathcal{U}\int_0^t \|f(s)\|\, ds \\
&\quad + \mathcal{U}\mathcal{M}_1 \int_0^t \left\|W^{-1}\left(x_* - U(T)\widetilde{\psi}(0) - \int_0^T U(T-\tau) f(\tau)\, d\tau\right)(s)\right\| ds \\
&\le \mathcal{U}\left\|\widetilde{\psi}(0)\right\| + \mathcal{U}\int_0^t \|f(s)\|\, ds \\
&\quad + \mathcal{U}\mathcal{M}_1 \left\|W^{-1}\left(x_* - U(T)\widetilde{\psi}(0) - \int_0^T U(T-\tau) f(\tau)\, d\tau\right)\right\|_{L^1(J;\mathbb{U})} \\
&\le \mathcal{U}\left\|\widetilde{\psi}(0)\right\| + \mathcal{U}\int_0^t \|f(s)\|\, ds \\
&\quad + \mathcal{U}\mathcal{M}_1\sqrt{T} \left\|W^{-1}\left(x_* - U(T)\widetilde{\psi}(0) - \int_0^T U(T-\tau) f(\tau)\, d\tau\right)\right\|_{L^2(J;\mathbb{U})} \\
&\le \mathcal{U}\left\|\widetilde{\psi}(0)\right\| + \mathcal{U}\int_0^t \|f(s)\|\, ds \\
&\quad + \mathcal{U}\mathcal{M}_1\mathcal{M}_2\sqrt{T}\left(\|x_*\| + \mathcal{U}\left\|\widetilde{\psi}(0)\right\| + \mathcal{U}\int_0^T \|f(s)\|(\tau)\, d\tau\right).
\end{aligned}$$

■

We begin with the following situation.

Theorem 5. *Under assumptions* $(A), (F1), (F2), (F4)$, *and* (C), *suppose that condition* $(F3)$ *takes the following form:*
$(F3')$ *there exists a sequence of functions* $\{\omega_n\} \subset L^1_+(J)$, $n = 1, 2, \ldots$ *such that*

$$\sup_{\|\psi\|_{\mathcal{B}} \le n} \|F(t,\psi)\| \le \omega_n(t) \ \text{for a.e. } t \in J,\ n = 1, 2, \ldots$$

If

$$\text{(Q)} \qquad \liminf_{n\to\infty} \frac{1}{\xi(n)} \int_0^T \omega_n(s)\, ds = 0,$$

where

$$\xi(n) = n - N\|\widetilde{\psi}\|_{\mathcal{B}}$$

and N is a upper bound of the function $N(\cdot)$ from axiom $\mathcal{B}(iii)$, then controllability problem $(3.1), (3.2), (3.7)$ *has a solution.*

Proof. We will prove that there exists a closed ball $B_R \subset C(J; E_0)$ such that $\Gamma(D \cap B_R) \subseteq D \cap B_R$.
Supposing the contrary, we will have sequences $\{y_n\}, \{z_n\} \subset D$ such that

$$z_n \in \Gamma(y_n), \ \|y_n\|_C \leq \frac{1}{K}\xi(n), \ \|z_n\|_C > \frac{1}{K}\xi(n)$$

for all $n \geq N\|\widetilde{\psi}\|_{\mathcal{B}}$, where K is defined by (3.5). Then

$$z_n = S(f_n)$$

for some $f_n \in \mathcal{P}_F(y_n)$, $n \geq N\|\widetilde{\psi}\|_{\mathcal{B}}$.
From Lemma 3 we obtain the estimate

$$\|z_n\|_C \leq C_1 + C_2 \int_0^T \|f_n(\tau)\| d\tau$$

where

$$C_1 = \mathcal{U}\left\|\widetilde{\psi}(0)\right\| + \mathcal{U}\mathcal{M}_1\sqrt{T}\left(\|x_*\| + \mathcal{U}\left\|\widetilde{\psi}(0)\right\|\right) \tag{3.17}$$

$$C_2 = \mathcal{U}\left(1 + \mathcal{U}\mathcal{M}_1\mathcal{M}_2\sqrt{T}\right) . \tag{3.18}$$

Further, for $n \geq N\|\widetilde{\psi}\|_{\mathcal{B}}$ and $\tau \in J$ from axiom $\mathcal{B}(iii)$ we have the estimate

$$\|y_n[\widetilde{\psi}]_\tau\|_{\mathcal{B}} \leq K\|y_n\|_C + N\|\widetilde{\psi}\|_{\mathcal{B}} \leq \xi(n) + N\|\widetilde{\psi}\|_{\mathcal{B}} = n$$

and whence from $f_n(\tau) \in F(\tau, y_n[\widetilde{\psi}]_\tau)$ and condition of the theorem it follows that

$$\|f_n(\tau)\| \leq \omega_n(\tau), \quad n \geq N\|\widetilde{\psi}\|_{\mathcal{B}}$$

But then

$$\|z_n\|_C \leq C_1 + C_2 \int_0^T \omega_n(\tau)\, d\tau$$

and

$$1 < \frac{K\|z_n\|_C}{\xi(n)} \leq \frac{KC_1}{\xi(n)} + \frac{KC_2}{\xi(n)} \int_0^T \omega_n(\tau)\, d\tau \ ,$$

giving the contradiction.
It remains only to apply Theorem 1 to the restriction $\Gamma : D \cap B_R \multimap D \cap B_R$. ■

In our second example, let us assume that condition $(F3)$ has the following form:

$(F3'')$ *there exist a function* $p(\cdot) \in L^1_+(J)$ *and a non-decreasing function* $\zeta : R^+ \to R^+$ *such that*

$$\|F(t,\psi)\| \le p(t)\,\zeta(\|\psi\|_{\mathcal{B}}) \quad \text{for } a.e.\ t \in J\ .$$

Theorem 6. *Under conditions* $(A), (F1), (F2), (F3''), (F4)$, *and (C) suppose that there exists a constant* $L > 0$ *such that*

$$\text{(L)} \qquad \frac{L}{C_1 + C_2\zeta\left(KL + N\|\widetilde{\psi}\|_{\mathcal{B}}\right)\int_0^T p(\tau)\,d\tau + \|\widetilde{\psi}(0)\|} > 1\ ,$$

where the constants C_1 *and* C_2 *are those given by (3.17), (3.18) and constants* K *and* N *are the same as above.*
Then controllability problem $(3.1), (3.2), (3.7)$ *has a solution.*

Proof. Denote by $a \in D$ the function identically equal to $\widetilde{\psi}(0)$. Let us demonstrate that there exists an open bounded neighbourhood V of a in D with the property that

$$y - a \notin \lambda(\Gamma(y) - a) \tag{3.19}$$

for all $y \in \partial V$ and $0 < \lambda \le 1$.
Suppose that $y - a \in \lambda(\Gamma(y) - a)$ for some $y \in D$ and $0 < \lambda \le 1$, then $y = \lambda S(f) + (1-\lambda)a,\ f \in \mathcal{P}_F(y)$. Applying Lemma 3 and using the fact that the unction $\zeta(\cdot)$ is nondecreasing, we obtain the estimate

$$\begin{aligned}\|y\|_C &\le C_1 + C_2\int_0^T \|f(\tau)\|\,d\tau + \|a\|_C \\ &\le C_1 + C_2\zeta\left(\left\|y[\widetilde{\psi}]\right\|_{\mathcal{B}}\right)\int_0^T p(\tau)\,d\tau \ + \|\widetilde{\psi}(0)\| \\ &\le C_1 + C_2\zeta\left(K\|y\|_C + N\left\|\widetilde{\psi}\right\|_{\mathcal{B}}\right)\int_0^T p(\tau)\,d\tau \ + \|\widetilde{\psi}(0)\|\end{aligned}$$

or

$$\frac{\|y\|_C}{C_1 + C_2\zeta\left(K\|y\|_C + N\left\|\widetilde{\psi}\right\|_{\mathcal{B}}\right)\int_0^T p(\tau)\,d\tau \ + \|\widetilde{\psi}(0)\|} \le 1\ .$$

So, $\|y\|_C$ does not equal to the constant L, appearing in condition (L). Now, let us take an relatively open set

$$V = \{y \in D : \|y\|_C < L\}\,.$$

Notice that condition (L) implies $a \in V$. We see that condition (3.19) is fulfilled and it remains only to apply Theorem 2. ■

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Y. C. Liou
Department of Information Management,
Cheng Shiu University,
Kaohsiung 833, Taiwan
E-mail: simplex_liou@hotmail.com

V. Obukhovskii
Faculty of Mathematics,
Voronezh State University,
Universitetskaya pl., 1 (394 006),
Voronezh,
Russia
E-mail: valerio@math.vsu.ru

J. C. Yao
Department of Applied Mathematics,
National Sun Yat-sen University,
Kaohsiung 804, Taiwan
E-mail: yaojc@math.nsysu.edu.tw