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THREE STEP ITERATIVE PROCEDURE WITH ERRORS FOR GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce a class of new iterative procedures with errors approximating the common fixed point for three generalized asymptotically quasi-nonexpansive mappings and prove several strong convergence theorems for the iterative procedures with errors in uniformly convex Banach spaces. Our results extend and improve the recent corresponding theorems obtained by Chang-Kim-Jin [6], Ghosh-Debnath [7], Kim-Kim-Kim [14], Li-Kim-Huang [17], Liu ([21, 22]), Nammanee-Noor-Suantai [24], Suantai [27], Tan-Xu [28], Xu-Noor [30], Zeng-Wong-Yao [32], Zeng-Yao [33], Zhou-Cho-Grabiec [34] and Zhou-Guo-Hwang-Cho [35].

1. INTRODUCTION

In 1974, Ishikawa [10] proved the convergence theorem for the two-step iterative sequence: Let C be a convex compact subset of a Hilbert space H, T be a Lipschitzian pseudo-contractive mapping from C into itself, and x_1 be any point in C. Then the sequence $\{x_n\}$

$$(1.1)\qquad \begin{cases} x_{n+1} = \alpha_n T y_n + (1-\alpha_n)x_n \\ y_n = \beta_n T x_n + (1-\beta_n)x_n \end{cases} \quad n = 1, 2, \cdots,$$

converges strongly to a fixed point of T, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ that satisfy the following two conditions:

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(i) $\lim_{n\to\infty} \beta_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.

The iteration (1.1) is said to be a *Ishikawa iterative sequence* [10].

If $\beta_n = 0$ for each $n \geq 1$, then it is said to be a *Mann iterative sequence* [23]. That is,

$$x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n n = 1, 2, \cdots, \tag{1.2}$$

In 1972, Goebel-Kirk [8] introduced the concept of asymptotically nonexpansive mapping. The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically nonexpansive type mapping, asymptotically quasi-nonexpansive mapping, and asymptotically quasi-nonexpansive type mapping were studied extensively by Bai-Kim [1], Goebel-Kirk [8], Li-Kim-Huang [17], Liu ([21, 22]), Chang-Kim-Kang [6] and references therein (see, [3, 11, 12, 16]) in the setting of Hilbert spaces or Banach spaces.

Recently, Xu-Noor [30] introduced and studied the following three-step iterative sequence to approximate fixed points of asymptotically nonexpansive mapping T with some control conditions on $\{a_n\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$:

$$\begin{cases} z_n = a_n x_n + b_n T^n x_n, \\ y_n = \lambda_n x_n + \mu_n T^n z_n, \qquad n = 1, 2, \cdots, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n y_n, \end{cases} \tag{1.3}$$

And, Kim-Kim-Kim [14] proved the convergence theorems of modified three-step iterative sequences with mixed errors for an asymptotically quasi-nonexpansive mapping T with suitable conditions:

$$\begin{cases} z_n = a_n x_n + b_n T^n(x_n) + w_n, \\ y_n = \lambda_n x_n + \mu_n T^n(z_n) + v_n, \qquad n = 1, 2, \cdots, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n(y_n) + u_n, \end{cases} \tag{1.4}$$

Very recently, Zhou-Cho-Grabiec [34] introduced a class of new generalized asymptotically nonexpansive mappings and gave a sufficient and necessary condition for the modified Ishikawa and Mann iterative sequences to converge to fixed points for the class of mappings. Zhou-Gu-Huang-Cho [35] established several strong convergence results for the modified three-step iterative sequences with errors for a class of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces.

On the other hand, Huang-Deng-Fang [9] introduced a new three step iterative procedure for approximating the unique common fixed point of fuzzy strongly pseudo-contractive mappings and show the convergence of the iterative procedure by using Petryshyn's inequality in real uniformly smooth Banach spaces.

The purpose of this paper is to introduce a class of new iterative procedures with errors approximating the common fixed point for three generalized asymptotically quasi-nonexpansive mappings and to give several strong convergence theorems for the iterative procedures with errors in uniformly convex Banach spaces. The results presented in this paper extend, improve and unify the recent corresponding results announced by [6, 7, 14, 17, 18, 19, 20, 21, 23, 24, 25, 27, 28, 29, 30, 32, 33, 34] and [35].

2. Preliminaries

Let X be a normed linear space, C be a nonempty convex subset of X and $T_i : C \to C$ $(i = 1, 2, 3)$ be given mappings. Then, for any given $x_1 \in C$, the procedure $\{x_n\}$ in C defined by

$$(2.1)\qquad \begin{cases} z_n = a_n x_n + b_n T_3^n(x_n) + c_n w_n, \\ y_n = \lambda_n x_n + \mu_n T_2^n(z_n) + \nu_n v_n, \qquad n = 1, 2, \cdots, \\ x_{n+1} = \alpha_n x_n + \beta_n T_1^n(y_n) + \gamma_n u_n, \end{cases}$$

which is called the *generalized modified three step iterative procedure with errors*, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = \lambda_n + \mu_n + \nu_n = a_n + b_n + c_n = 1$, and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are sequences in C satisfying some conditions.

If $T_i = T$ for $i = 1, 2, 3$, then the procedure $\{x_n\}$ defined by (2.1) becomes to the procedure

$$(2.2)\qquad \begin{cases} z_n = a_n x_n + b_n T^n(x_n) + c_n w_n, \\ y_n = \lambda_n x_n + \mu_n T^n(z_n) + \nu_n v_n, \qquad n = 1, 2, \cdots, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n(y_n) + \gamma_n u_n, \end{cases}$$

which is called the *modified three step iterative procedure with errors* and was studied by Zhou-Guo-Huang-Cho [35] and Kim-Kim-Kim [14]. We note that $\{u_n\}, \{v_n\}$, $\{w_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in (2.2) are the same as in (2.1).

If $b_n = c_n \equiv 0$ in (2.1), then the procedure $\{x_n\}$ in C defined by

$$(2.3)\qquad \begin{cases} y_n = \lambda_n x_n + \mu_n T_2^n(x_n) + \nu_n v_n, \\ x_{n+1} = \alpha_n x_n + \beta_n T_1^n(y_n) + \gamma_n u_n, \end{cases} \qquad n = 1, 2, \cdots,$$

is called the *generalized modified Ishikawa iterative procedure with errors*, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n=\lambda_n+\mu_n+\nu_n=1$, and $\{u_n\}$ and $\{v_n\}$ are sequences in C satisfying some conditions (see, [17]). The iterative procedure (2.3) was studied by Zhou-Cho-Grabiec [34] when $\nu_n=\gamma_n\equiv 0$ and $T_1=T_2$.

If $\mu_n=\nu_n\equiv 0$ in (2.3), then the procedure $\{x_n\}$ in C defined by

$$x_{n+1}=\alpha_n x_n+\beta_n T_1^n(x_n)+\gamma_n u_n, \quad n=1,2,\cdots, \tag{2.4}$$

is called the *generalized modified Mann iterative procedure with errors*, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n=1$, and $\{u_n\}$ is a sequence in C satisfying some conditions.

Remark 2.1. The above three step iterative process with errors includes many iterative processes as special cases, such as Liu ([21, 22]) Suzuki [26], Xu-Noor [30], Zhou-Cho-Grabiec [34] and Zhou-Guo-Huang-Cho [35].

In the sequel, we need the following definitions and lemmas for the main results in this paper.

Definition 2.1. A Banach space X is said to be *uniformly convex* if the modulus of convexity of X:

$$\delta_X(\epsilon)=\inf\{1-\frac{1}{2}\|x+y\| : \|x\|=\|y\|=1,\ \|x-y\|\geq\epsilon\}\geq 0$$

for all $0<\epsilon\leq 2$.

Remark 2.2. If X is a real uniformly convex Banach space, then it is reflexive and strictly convex, and so the normalized duality mapping $J: X\to 2^{X^*}$ defined by

$$J(x)=\{f\in X^*, \langle x,f\rangle=\|x\|\|f\|,\ \|x\|=\|f\|\}, \quad \forall x\in X,$$

is single-valued, where X^* is the dual space of X.

Definition 2.2. Let X be a real Banach space, C be a nonempty subset of X and $F(T)$ denote the set of fixed points of T. A mapping $T: C\to C$ is said to be

(1) *asymptotically nonexpansive* if there exists a sequence $\{r_n\}$ of positive real numbers with $r_n\to 0$ as $n\to\infty$ such that

$$\|T^n(x)-T^n(y)\|\leq(1+r_n)\|x-y\| \tag{2.5}$$

for all $x,y\in C$ and $n\geq 1$;

(2) *asymptotically quasi-nonexpansive* if

$$\|T^n(x) - y\| \le (1 + r_n)\|x - y\| \tag{2.6}$$

holds for all $x \in C$ and $y \in F(T)$;

(3) *generalized asymptotically quasi-nonexpansive* if there exist two sequences $\{r_n\}$ and $\{s_n\} \subset [0, 1)$ with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$ such that

$$\|T^n(x) - p\| \le (1 + r_n)\|x - p\| + s_n\|x - T^n(x)\| \tag{2.7}$$

for all $x \in C$, $p \in F(T)$ and $n \ge 1$;

(4) *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n(x) - T^n(y)\| \le L\|x - y\|$$

for all $x, y \in C$ and $n \ge 1$;

(5) *uniformly L-Hölder continuous* if there exist positive constants L and α such that

$$\|T^n(x) - T^n(y)\| \le L\|x - y\|^\alpha$$

for all $x, y \in C$ and $n \ge 1$;

(6) *uniformly equi-continuous* if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|T^n(x) - T^n(y)\| \le \epsilon$$

whenever $\|x-y\| \le \delta$ for all $x, y \in C$ and $n \ge 1$ equivalently, T is uniformly equi-continuous if and only if

$$\|T^n(x_n) - T^n(y_n)\| \to 0$$

whenever $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Remark 2.3. It is easy to see that

(1) if $s_n \equiv 0$ for all $n \ge 1$, then the generalized asymptotically quasi-nonexpansive mapping reduces to the asymptotically quasi-nonex-pansive mapping;

(2) if $r_n = s_n$ for all $n \ge 1$, then the generalized asymptotically quasi-nonexpansive mapping becomes to the quasi-nonexpansive mapping;

(3) if T is asymptotically nonexpansive, then it is uniformly $L-$Lip-schitzian;

(4) if T is uniformly $L-$Lipschitzian, then it is L-uniformly Hölder continuous;

(5) if T is uniformly L-Hölder continuous, then it is uniformly equi-continuous.

However, their converses are not true.

Example 2.1. Let $X = R^1$ and $C = [0, 1]$. Define a mapping $T : C \to C$ by

$$Tx = (1 - x^{\frac{3}{2}})^{\frac{2}{3}}$$

for all $c \in C$. Then T is uniformly equi-continuous, but it is not uniformly $L-$Lipschitzian.

Lemma 2.1. *Let C be a nonempty subset of X and for $i = 1, 2, 3$, $T_i : C \to C$ be generalized asymptotically quasi-nonexpansive mappings, the real sequences $\{r_{in}\}$ and $\{s_{in}\}$ be as in (2.7). Then there exist two sequences $\{r_n\}$ and $\{s_n\} \subset [0, 1)$ converging to 0 as $n \to \infty$ such that, for any $x \in C$, $p \in F(T)$ and $i = 1, 2, 3$,*

$$\|T_i^n(x) - p\| \leq (1 + r_n)\|x - p\| + s_n\|x - T_i^n(x)\|, \quad n \geq 1.$$

Proof. Setting

$$r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}, \quad s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}, \quad n \geq 1,$$

then from (2.7), we have $r_n, s_n \subset [0, 1)$, $r_n, s_n \to 0$ $(n \to \infty)$ and

$$\begin{aligned}\|T_i^n(x) - p\| &\leq (1 + r_{in})\|x - p\| + s_{in}\|x - T_i^n(x)\| \\ &\leq (1 + r_n)\|x - p\| + s_n\|x - T_i^n(x)\|, \quad n \geq 1,\end{aligned}$$

for all $x \in C$, $p \in F(T)$ and $i = 1, 2, 3$. ■

Lemma 2.2. ([27]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

By Lemma 2.2, we can prove the following import lemma for the our main theorems.

Lemma 2.3. *Let X be a normed linear space and C be a nonempty convex subset of X. For $i = 1, 2, 3$, let $T_i : C \to C$ be generalized asymptotically quasi-nonexpansive mappings and two sequences $\{r_{in}\}$ and $\{s_{in}\}$ be as in (2.7). Assume that*

$$F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset,$$

$r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}$, $s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}$, $n \geq 1$ and $\{x_n\}$ is a sequence defined by (2.1) with the conditions that

(1) $\{v_n\}$ *and* $\{w_n\}$ *are bounded,* $u_n = u'_n + u''_n, \ \ n \geq 1$, *and*

$$\sum_{n=1}^{\infty} \|u'_n\| < \infty, \qquad \|u''_n\| = o(\alpha_n);$$

(2) $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} \leq \infty$;

(3) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then we have the following.

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for any* $p \in F$.

(ii) $\lim_{n\to\infty} D(x_n, F)$ *exists,*

where $D(x, F)$ *denotes the distance from* x *to the set* F, *that is,*

$$D(x, F) = \inf_{y \in F} \|x - y\|.$$

Proof. Let $p \in F = F(T_1) \cap F(T_2) \cap F(T_3)$,

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\}, \quad M_2 = \sup\{\|v_n - p\| : n \geq 1\}\},$$
$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, \quad M = \max\{M_i : i = 1, 2, 3\},$$

and for all $n \geq 1$,

$$r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}, \quad s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}, \quad \sigma_n = \frac{2 + r_n}{1 - s_n}.$$

By the condition (1), we have $\|u_n - p\| \leq \|u'_n - p\| + \|u''_n\|$ with $\|u''_n\| = o(\alpha_n)$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \|u'_n\| < \infty$, and so there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n > 0$ and $\epsilon_n \to 0$ such that $\|u''_n\| = \epsilon_n \alpha_n$, i.e.,

$$\|u_n - p\| \leq \|u'_n - p\| + \epsilon_n \alpha_n$$

and

$$\begin{aligned}\|u'_n - p\| &= \|u_n - p - u''_n\| \leq \|u_n - p\| + \|u''_n\| \\ &\leq M + \epsilon_n \alpha_n.\end{aligned}$$

It follows from (2.1) and Lemma 2.1 that

$$\begin{aligned}\|x_n - T_3^n(x_n)\| &\leq \|x_n - p\| + \|T_3^n(x_n) - p\| \\ &\leq (2 + r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n(x_n)\| \\ &\leq (2 + r_n)\|x_n - p\| + s_n\|x_n - T_3^n(x_n)\|,\end{aligned}$$

that is,

$$\|x_n - T_3^n(x_n)\| \le \frac{2+r_n}{1-s_n}\|x_n - p\| = \sigma_n\|x_n - p\|.$$

Similarly, we have,

$$\|z_n - T_2^n(z_n)\| \le \sigma_n\|z_n - p\|,$$

and

$$\|y_n - T_1^n(y_n)\| \le \sigma_n\|y_n - p\|,$$

$$\begin{aligned}
\|z_n - p\| &\le a_n\|x_n - p\| + b_n\|T_3^n(x_n) - p\| + c_n\|w_n - p\| \\
&\le a_n\|x_n - p\| + b_n[(1+r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n(x_n)\|] + Mc_n \\
&= (a_n + b_n + b_n r_{3n})\|x_n - p\| + b_n s_{3n}\|x_n - T_3^n(x_n)\| + Mc_n \\
&\le (a_n + b_n + r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n(x_n)\| + Mc_n \\
&\le (1 + r_n + s_n\sigma_n)\|x_n - p\| + Mc_n,
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\| &\le \lambda_n\|x_n - p\| + \mu_n\|T_2^n(z_n) - p\| + \nu_n\|v_n - p\| \\
&\le \lambda_n\|x_n - p\| + \mu_n[(1+r_{2n})\|z_n - p\| + s_{2n}\|z_n - T_2^n(z_n)\|] + M\nu_n \\
&\le \lambda_n\|x_n - p\| + \mu_n(1 + r_n + s_n\sigma_n)\|z_n - p\| + M\nu_n \\
&\le \lambda_n\|x_n - p\| + \mu_n(1 + r_n + s_n\sigma_n)(1 + r_n + s_n\sigma_n)\|x_n - p\| \\
&\quad +\mu_n(1 + r_n + s_n\sigma_n)Mc_n + M\nu_n \\
&\le [\lambda_n + \mu_n + \mu_n(r_n + s_n\sigma_n)(2 + r_n + s_n\sigma_n)]\|x_n - p\| \\
&\quad +Mc_n\mu_n(1 + r_n + s_n\sigma_n) + M\nu_n \\
&\le [1 + (r_n + s_n\sigma_n)(2 + r_n + s_n\sigma_n)]\|x_n - p\| \\
&\quad +M\nu_n + Mc_n\mu_n(1 + r_n + s_n\sigma_n),
\end{aligned}$$

and so

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\le \alpha_n\|x_n - p\| + \beta_n[(1+r_{1n})\|y_n - p\| + s_{1n}\|y_n - T_1^n(y_n)\|] + \gamma_n\|u_n - p\| \\
&\le \alpha_n\|x_n - p\| + \beta_n(1 + r_n + s_n\sigma_n)\|y_n - p\| + M\gamma_n + 2\gamma_n\alpha_n\epsilon_n \\
&\le \{\alpha_n + \beta_n(1 + r_n + s_n\sigma_n)[1 + (r_n + s_n\sigma_n)(2 + r_n + s_n\sigma_n)]\}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times\|x_n - p\| + M\nu_n\beta_n(1 + r_n + s_n\sigma_n)\\
&\quad +Mc_n\beta_n\mu_n(1 + r_n + s_n\sigma_n)^2 + M\gamma_n + 2\gamma_n\alpha_n\epsilon_n\\
&= \{1 + [(1 + r_n + s_n\sigma_n)^3 - 1]\}\|x_n - p\| + M\nu_n\beta_n(1 + r_n + s_n\sigma_n)\\
&\quad +Mc_n\beta_n\mu_n(1 + r_n + s_n\sigma_n)^2 + M\gamma_n + 2\gamma_n\alpha_n\epsilon_n\\
&\le [1 + (k_n^3 - 1)]\|x_n - p\| + \Gamma(\gamma_n + \nu_n + c_n),
\end{aligned}
$$

where $k_n = 1 + r_n + s_n\sigma_n = \frac{1+r_n+s_n}{1-s_n}$ for all $n \geq 1$ and Γ is a positive constant.

Note that

$$\sum_{n=1}^{\infty} \frac{r_n + 2s_n}{1 - s_n} = \sum_{n=1}^{\infty}(k_n - 1) < \infty$$

is equivalent to

$$\sum_{n=1}^{\infty}({k_n}^3 - 1) < \infty,$$

$\lim_{n\to\infty} \epsilon_n = 0$ and sequences $\{r_{in}\}$ and $\{s_{in}\}$ in (2.7) converge to 0 as $n \to \infty$ for $i = 1, 2, 3$. Therefore, the conclusions of the lemma follows from Lemma 2.2. This completes the proof. ■

Lemma 2.4. ([31]). *Let $p > 1$ and $r > 0$ be two any real numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|)$$

for all $x,\ y \in B_r(0) = \{x \in X : \|x\| \leq r\}$ and all $\lambda \in [0, 1]$, where

$$w_p(\lambda) = \lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p.$$

As an immediate consequence of Lemma 2.4, we have the following:

Lemma 2.5. *Let X be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x,\ y,\ z \in B_r(0)$ and $\lambda,\ \mu,\ \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Proof. We first observe that $\frac{\lambda}{1-\gamma}x + \frac{\mu}{1-\gamma}y \in B_r(0)$ for $x,\ y \in B_r(0)$ and $\lambda,\ \mu,\ \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$. It follows from Lemma 2.4 that

$$\begin{aligned}
\|\lambda x + \mu y + \gamma z\|^2 &= \Big\|(1-\gamma)\Big[\frac{\lambda}{1-\gamma}x + \frac{\mu}{1-\gamma}y\Big] + \gamma z\Big\|^2 \\
&\le (1-\gamma)\Big\|\frac{\lambda}{1-\gamma}x + \frac{\mu}{1-\gamma}y\Big\|^2 + \gamma\|z\|^2 \\
&\le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - (1-\gamma)w_2\Big(\frac{\lambda}{1-\gamma}\Big)g(\|x-y\|) \\
&\le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \frac{\lambda\mu}{1-\gamma}g(\|x-y\|) \\
&\le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x-y\|).
\end{aligned}$$

This completes the proof. ■

3. Main Results

Now, we are in a position to prove the main theorems.

Proposition 3.1. *Let X be a uniformly convex Banach space, C be a nonempty convex subset of X, and for $i = 1, 2, 3$, $T_i : C \to C$ be uniformly equi-continuous and generalized asymptotically quasi-nonexpansive mappings with*

$$F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$$

and two sequences $\{r_{in}\}$ and $\{s_{in}\} \subset [0,1)$ such that $r_{in} \to 0$ and $s_{in} \to 0$. Suppose that $r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}$, $s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}$, $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} < \infty$ and $\{x_n\}$ is a procedure defined by (2.1) with the following conditions:

(*i*) $0 \le \alpha_{n+1} \le \alpha_n < \xi$, $0 \le \lambda_{n+1} \le \lambda_n < \eta$, $0 \le a_{n+1} \le a_n < \zeta$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} a_n = \infty$;

(*ii*) $\beta_n \to 0$, $\mu_n \to 0$ *as* $n \to \infty$;

(*iii*) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(*iv*) $\{v_n\}$ *and* $\{w_n\}$ *are bounded,* $u_n = u'_n + u''_n, n \ge 1$, *and*

$$\sum_{n=1}^{\infty} \|u'_n\| < \infty, \qquad \|u''_n\| = o(\alpha_n).$$

Then we have

$$\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0,$$

for each i=1,2,3.

Proof. By Lemma 2.3, we know that $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F$. It follows that $\{x_n - p\}$, $\{T_1^n(y_n) - y_n\}$, $\{T_2^n(z_n) - z_n\}$, $\{T_3^n(x_n) - x_n\}$, $\{z_n - p\}$, $\{y_n - p\}$ are all bounded. Also, $\{T_1^n(y_n) - p\}$, $\{T_2^n(z_n) - p\}$ and $\{T_3^n(x_n) - p\}$ are bounded by the assumptions of T_i $(i = 1, 2, 3)$. Now, we set

$$r_1 = \sup\{\|x_n - p\| : n \geq 1\}, \quad r_2 = \sup\{\|T_3^n(x_n) - p\| : n \geq 1\},$$

$$r_3 = \sup\{\|y_n - p\| : n \geq 1\}, \quad r_4 = \sup\{\|T_1^n(y_n) - p\| : n \geq 1\},$$

$$r_5 = \sup\{\|z_n - p\| : n \geq 1\}, \quad r_6 = \sup\{\|T_2^n(z_n) - p\| : n \geq 1\},$$

$$r_7 = \sup\{\|u_n - p\| : n \geq 1\}, \quad r_8 = \sup\{\|v_n - p\| : n \geq 1\},$$

$$r_9 = \sup\{\|w_n - p\| : n \geq 1\}, \quad r = \max\{r_i : 1 \leq i \leq 9\}.$$

By using Lemma 2.5, (2.1) and condition (iv), we have

$$\|u_n - p\| \leq \|u_n' - p\| + \|u_n''\|$$

with $\|u_n''\| = o(\alpha_n)$ for all $n \geq 1$ and $\sum_{n=1}^{\infty}\|u_n'\| < \infty$, and so there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n > 0$ and $\epsilon_n \to 0$ such that $\|u_n''\| = \epsilon_n\alpha_n$, that is,

$$\begin{aligned}\|u_n - p\| &\leq \|u_n' - p\| + \epsilon_n\alpha_n,\\ \|u_n' - p\| &= \|u_n - p - u_n''\| \leq \|u_n - p\| + \|u_n''\|\\ &\leq M + \epsilon_n\alpha_n,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + \beta_n\|T_1^n(y_n) - p\|^2\\ &\quad +\gamma_n\|u_n - p\|^2 - \alpha_n\beta_n g(\|x_n - T_1^n(y_n)\|)\\ &\leq \alpha_n\|x_n - p\|^2 + \beta_n\|T_1^n(y_n) - p\|^2\\ &\quad +\gamma_n(r + 2\alpha_n\epsilon_n)^2 - \alpha_n\beta_n g(\|x_n - T_1^n(y_n)\|)\end{aligned} \tag{3.1}$$

It follows from the assumption of T_1 that

$$\begin{aligned}\|y_n - T_1^n(y_n)\| &\leq \|y_n - p\| + \|T_1^n(y_n) - p\|\\ &\leq (2 + r_{1n})\|y_n - p\| + s_{1n}\|y_n - T_1^n(y_n)\|\\ &\leq (2 + r_n)\|y_n - p\| + s_n\|y_n - T_1^n(y_n)\|,\end{aligned}$$

that is,

$$\|y_n - T_1^n(y_n)\| \leq \sigma_n\|y_n - p\|,$$

where $\sigma_n = \frac{2+r_n}{1-s_n}$, and so

$$
\begin{aligned}
\|T_1^n(y_n) - p\|^2 &\leq [(1+r_{1n})\|y_n - p\| + s_{1n}\|y_n - T_1^n(y_n)\|]^2 \\
&= (1+r_{1n})^2\|y_n - p\|^2 + s_{1n}^2\|y_n - T_1^n(y_n)\|^2 \\
&\quad +2s_{1n}(1+r_{1n})\|y_n - p\|\|y_n - T_1^n(y_n)\| \\
&\leq [(1+r_{1n})^2 + s_{1n}^2\sigma_n^2 + 2s_{1n}(1+r_{1n})\sigma_n]\|y_n - p\|^2 \\
&\leq k_n^2\|y_n - p\|^2,
\end{aligned} \tag{3.2}
$$

where $k_n = 1 + r_n + s_n\sigma_n = \frac{1+r_n+s_n}{1-s_n} > 1$ for all $n \geq 1$.

Similarly, we have

$$\|z_n - T_2^n(z_n)\| \leq \sigma_n\|z_n - p\|,$$

which implies that

$$
\begin{aligned}
\|y_n - p\|^2 &\leq \lambda_n\|x_n - p\|^2 + \mu_n\|T_2^n(z_n) - p\|^2 + \nu_n r^2 \\
&\quad -\lambda_n\mu_n g(\|x_n - T_2^n(z_n)\|) \\
&\leq \lambda_n\|x_n - p\|^2 + \mu_n[(1+r_n)\|z_n - p\| + s_n\|z_n - T_2(z_n)\|]^2 \\
&\quad +\nu_n r^2 - \lambda_n\mu_n g(\|x_n - T_2^n(z_n)\|) \\
&\leq \lambda_n\|x_n - p\|^2 + \mu_n k_n^2\|z_n - p\|^2 + \nu_n r^2 \\
&\quad -\lambda_n\mu_n g(\|x_n - T_2^n(z_n)\|)
\end{aligned} \tag{3.3}
$$

and

$$\|x_n - T_3^n(x_n)\| \leq \sigma_n\|x_n - p\|,$$

which implies that

$$
\begin{aligned}
\|z_n - p\|^2 &\leq a_n\|x_n - p\|^2 + b_n\|T_2^n(z_n) - p\|^2 + c_n\|w_n - p\|^2 \\
&\quad -a_n b_n g(\|x_n - T_3^n(x_n)\|) \\
&\leq a_n\|x_n - p\|^2 + b_n[(1+r_n)\|x_n - p\| + s_n\|x_n - T_3(x_n)\|]^2 \\
&\quad +c_n r^2 - a_n b_n g(\|x_n - T_3^n(x_n)\|) \\
&\leq a_n\|x_n - p\|^2 + b_n k_n^2\|x_n - p\|^2 + c_n r^2 \\
&\quad -a_n b_n g(\|x_n - T_3^n(x_n)\|) \\
&= (a_n + b_n k_n^2)\|x_n - p\|^2 + c_n r^2 - a_n b_n g(\|x_n - T_3^n(x_n)\|)
\end{aligned} \tag{3.4}
$$

$$\begin{aligned}
&\leq (1-b_n+b_nk_n^2)\|x_n-p\|^2+c_nr^2-a_nb_ng(\|x_n-T_3^n(x_n)\|)\\
&= [1+b_n(k_n^2-1)]\|x_n-p\|^2+c_nr^2-a_nb_ng(\|x_n-T_3^n(x_n)\|)\\
&\leq [1+(k_n^2-1)]\|x_n-p\|^2+c_nr^2-a_nb_ng(\|x_n-T_3^n(x_n)\|)\\
&\leq k_n^2\|x_n-p\|^2+c_nr^2-a_nb_ng(\|x_n-T_3^n(x_n)\|).
\end{aligned}$$

Thus, it follows from (3.4) and (3.3) that

$$\begin{aligned}
\|y_n-p\|^2 &\leq \lambda_n\|x_n-p\|^2+\mu_nk_n^2[k_n^2\|x_n-p\|^2+c_nr^2\\
&\quad -a_nb_ng(\|x_n-T_3^n(x_n)\|)]+\nu_nr^2-\lambda_n\mu_ng(\|x_n-T_2^n(z_n)\|)\\
&\leq (\lambda_n+\mu_nk_n^4)\|x_n-p\|^2+c_n\mu_nk_n^2r^2+\nu_nr^2\\
&\quad -a_nb_n\mu_nk_n^2g(\|x_n-T_3^n(x_n)\|)-\lambda_n\mu_ng(\|x_n-T_2^n(z_n)\|)\\
&\leq (1-\mu_n+\mu_nk_n^4)\|x_n-p\|^2+k_n^2r^2c_n+r^2\nu_n\\
&\quad -a_nb_n\mu_nk_n^2g(\|x_n-T_3^n(x_n)\|)-\lambda_n\mu_ng(\|x_n-T_2^n(z_n)\|)\\
&\leq [1+\mu_n(k_n^4-1)]\|x_n-p\|^2+k_n^2r^2c_n+r^2\nu_n\\
&\quad -a_nb_n\mu_nk_n^2g(\|x_n-T_3^n(x_n)\|)-\lambda_n\mu_ng(\|x_n-T_2^n(z_n)\|)\\
&\leq [1+(k_n^4-1)]\|x_n-p\|^2+k_n^2r^2c_n+r^2\nu_n\\
&\quad -a_nb_n\mu_nk_n^2g(\|x_n-T_3^n(x_n)\|)-\lambda_n\mu_ng(\|x_n-T_2^n(z_n)\|)\\
&= k_n^4\|x_n-p\|^2+k_n^2r^2c_n+r^2\nu_n\\
&\quad -a_nb_n\mu_nk_n^2g(\|x_n-T_3^n(x_n)\|)-\lambda_n\mu_ng(\|x_n-T_2^n(z_n)\|)
\end{aligned} \tag{3.5}$$

Substituting (3.2) and (3.5) into (3.1), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leq \alpha_n\|x_n-p\|^2+\beta_nk_n^2\|y_n-p\|^2+\gamma_n(r+2\alpha_n\epsilon_n)^2\\
&\quad -\alpha_n\beta_ng(\|x_n-T_1^n(y_n)\|)\\
&\leq \alpha_n\|x_n-p\|^2+\beta_nk_n^2\{k_n^4\|x_n-p\|^2+k_n^2r^2c_n+r^2\nu_n\\
&\quad -a_nb_n\mu_nk_n^2g(\|x_n-T_3^n(x_n)\|)-\lambda_n\mu_ng(\|x_n-T_2^n(z_n)\|)\}\\
&\quad +\gamma_n(r+2\alpha_n\epsilon_n)^2-\alpha_n\beta_ng(\|x_n-T_1^n(y_n)\|)\\
&= (\alpha_n+\beta_nk_n^6)\|x_n-p\|^2+k_n^4r^2c_n+k_n^2r^2\nu_n+\gamma_n(r+2\alpha_n\epsilon_n)^2
\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&-\alpha_n\beta_n g(\|x_n - T_1^n(y_n)\|) - a_n b_n \mu_n \beta_n k_n^4 g(\|x_n - T_3^n(x_n)\|)\\
&-\lambda_n \mu_n \beta_n k_n^2 g(\|x_n - T_2^n(z_n)\|)\\
\le\ & (1-\beta_n+\beta_n k_n^6)\|x_n - p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n + \gamma_n (r + 2\alpha_n \epsilon_n)^2\\
&-\alpha_n\beta_n g(\|x_n - T_1^n(y_n)\|) - a_n b_n \mu_n \beta_n k_n^4 g(\|x_n - T_3^n(x_n)\|)\\
&-\lambda_n \mu_n \beta_n k_n^2 g(\|x_n - T_2^n(z_n)\|)\\
=\ & [1+\beta_n(k_n^6 - 1)]\|x_n - p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n + (r + 2\alpha_n \epsilon_n)^2\gamma_n\\
&-\alpha_n\beta_n g(\|x_n - T_1^n(y_n)\|) - a_n b_n \mu_n \beta_n k_n^4 g(\|x_n - T_3^n(x_n)\|)\\
&-\lambda_n \mu_n \beta_n k_n^2 g(\|x_n - T_2^n(z_n)\|)\\
\le\ & [1+(k_n^6 - 1)]\|x_n - p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n + (r + 2\alpha_n \epsilon_n)^2\gamma_n\\
&-\alpha_n\beta_n g(\|x_n - T_1^n(y_n)\|) - \lambda_n \mu_n \beta_n k_n^2 g(\|x_n - T_2^n(z_n)\|)\\
&-a_n b_n \mu_n \beta_n k_n^4 g(\|x_n - T_3^n(x_n)\|).
\end{aligned}
$$

Since $k_n - 1 = \frac{1+r_n+s_n}{1-s_n} - 1 = \frac{r_n+2s_n}{1-s_n}$, the assumption $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} < \infty$ implies that $\lim_{n\to\infty} k_n = 1$. Thus, there exists a constant $\varepsilon \in (0,1)$ such that $k_n \ge 1-\varepsilon$ for all $n \ge 1$. Therefore, it follows from the assumption (i) and (3.6) that

$$
\begin{aligned}
&\|x_{n+1} - p\|^2\\
\le\ & [1+(k_n^6 - 1)]\|x_n - p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n + (r + 2\alpha_n \epsilon_n)^2\gamma_n\\
&-\alpha_n(1-\alpha_n-\gamma_n) g(\|x_n - T_1^n(y_n)\|)\\
&-\lambda_n(1-\lambda_n-\nu_n)(1-\alpha_n-\gamma_n) k_n^2 g(\|x_n - T_2^n(z_n)\|)\\
&-a_n(1-a_n-c_n)(1-\lambda_n-\nu_n)(1-\alpha_n-\gamma_n) k_n^4 g(\|x_n - T_3^n(x_n)\|) \qquad (3.7)\\
\le\ & [1+(k_n^6 - 1)]\|x_n - p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n + (r + 2\alpha_n \epsilon_n)^2\gamma_n\\
&-\alpha_n(1-\xi) g(\|x_n - T_1^n(y_n)\|)\\
&-\lambda_n(1-\eta)(1-\xi)(1-\varepsilon)^2 g(\|x_n - T_2^n(z_n)\|)\\
&-a_n(1-\zeta)(1-\eta)(1-\xi)(1-\varepsilon)^4 g(\|x_n - T_3^n(x_n)\|).
\end{aligned}
$$

From (3.7), we have

$$
\begin{aligned}
&\|x_{n+1} - p\|^2\\
\le\ & [1+(k_n^6 - 1)]\|x_n - p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n + (r + 2\alpha_n \epsilon_n)^2\gamma_n\\
&-\alpha_n(1-\xi) g(\|x_n - T_1^n(y_n)\|)
\end{aligned}
$$

which leads to

$$\begin{aligned}
&\alpha_n(1-\xi)g(\|x_n - T_1^n(y_n)\|) \\
&\leq [1+(k_n^6-1)]\|x_n-p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n \\
&+\gamma_n(r+2\alpha_n\epsilon_n)^2 - \|x_{n+1}-p\|^2 \\
&\leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \delta_n + M(c_n+\nu_n+\gamma_n),
\end{aligned} \tag{3.8}$$

where $\delta_n = r^2(k_n^6-1)$ and M is a positive constant.

Notice that $\sum_{n=1}^{\infty}(k_n-1)<\infty$ is equivalent to $\sum_{n=1}^{\infty}({k_n}^6-1)<\infty$. Therefore, we have $\sum_{n=1}^{\infty}\delta_n<\infty$. It follows from (3.8) that

$$\begin{aligned}
&\alpha_{n+1}(1-\xi)g(\|x_{n+1} - T_1^{n+1}(y_{n+1})\|) \\
&\leq \|x_{n+1}-p\|^2 - \|x_{n+2}-p\|^2 + \sigma_{n+1} \\
&+M(\lambda_{n+1}+\mu_{n+1}+\nu_{n+1}).
\end{aligned} \tag{3.9}$$

Adding the both sides of (3.8) and (3.9) and using the assumption (i), then we have

$$\begin{aligned}
&(1-\xi)\sum_{n=1}^{\infty} a_{n+1}[g(\|x_n - T_1^n(y_n)\|) + g(\|x_{n+1} - T_1^{n+1}(y_{n+1})\|)] \\
&\leq \|x_1-p\|^2 + \|x_2-p\|^2 + 2\sum_{n=1}^{\infty}\sigma_n + 2M\sum_{n=1}^{\infty}(\lambda_n+\mu_n+\nu_n),
\end{aligned}$$

which implies that

$$\liminf_{n\to\infty}[g(\|x_n - T_1^n(y_n)\|) + g(\|x_{n+1} - T_1^{n+1}(y_{n+1})\|)] = 0.$$

By virtue of the properties of g, we conclude that

$$\lim_{j\to\infty}\|x_{n_j} - T_1^{n_j}(y_{n_j})\| = 0, \quad \lim_{j\to\infty}\|x_{n_j+1} - T_1^{n_j+1}(y_{n_j+1})\| = 0.$$

Observe that

$$\begin{aligned}
\|x_n - T_1^n(x_n)\| &\leq \|x_n - T_1^n(y_n)\| + \|T_1^n(y_n) - T_1^n(x_n)\|, \\
\|y_n - x_n\| &\leq \mu_n\|x_n - T_2^n(z_n)\| + \nu_n\|v_n - x_n\| \\
&\leq \mu_n(\|x_n-p\| + \|T_2^n(z_n)-p\|) + \nu_n\|v_n-x_n\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. It follows from the uniform equi-continuity of T that $\|T^n x_n - T^n y_n\| \to 0$ as $n \to \infty$. This yields that

$$\|x_{n_j} - T_1^{n_j}(x_{n_j})\| \to 0, \quad \|x_{n_j+1} - T_1^{n_j+1}(x_{n_j+1})\| \to 0$$

as $j \to \infty$. Observe that

$$\begin{aligned} \|x_n - T_1(x_n)\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_1^{n+1}(x_{n+1})\| \\ &\quad + \|T_1^{n+1}(x_{n+1}) - T_1^{n+1}(x_n)\| + \|T_1^{n+1}(x_n) - T_1(x_n)\| \end{aligned}$$

and

$$\|x_{n_j+1} - x_{n_j}\| \leq \beta_{n_j} \|T_1^{n_j}(y_{n_j}) - x_{n_j}\| + \gamma_{n_j} \|u_{n_j} - x_{n_j}\| \to 0$$

as $j \to \infty$. Again, by using the uniform equi-continuity of T, we conclude that

$$\|T_1^{n_j+1}(x_{n_j}) - T_1^{n_j+1}(x_{n_j+1})\| \to 0$$

as $j \to \infty$, which proves that

$$\|x_{n_j} - T_1(x_{n_j})\| \to 0 \quad \text{as} \quad j \to \infty. \tag{3.10}$$

Similarly, by (3.7), we now know that

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\leq [1 + (k_n^6 - 1)]\|x_n - p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n + (r + 2\alpha_n \epsilon_n)^2 \gamma_n \\ &\quad - \lambda_n (1 - \eta)(1 - \xi)(1 - \varepsilon)^2 g(\|x_n - T_2^n(z_n)\|), \end{aligned}$$

that is,

$$\begin{aligned} &\lambda_n (1 - \eta)(1 - \xi)(1 - \varepsilon)^2 g(\|x_n - T_2^n(z_n)\|) \\ &\leq [1 + (k_n^6 - 1)]\|x_n - p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n \\ &\quad + (r + 2\alpha_n \epsilon_n)^2 \gamma_n - \|x_{n+1} - p\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \delta_n + M(c_n + \nu_n + \gamma_n) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\leq [1 + (k_n^6 - 1)]\|x_n - p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n + (r + 2\alpha_n \epsilon_n)^2 \gamma_n \\ &\quad - a_n (1 - \zeta)(1 - \eta)(1 - \xi)(1 - \varepsilon)^4 g(\|x_n - T_3^n(x_n)\|), \end{aligned}$$

that is,

$$
\begin{aligned}
&a_n(1-\zeta)(1-\eta)(1-\xi)(1-\varepsilon)^4 g(\|x_n - T_3^n(x_n)\|)\\
&\le [1+(k_n^6-1)]\|x_n-p\|^2 + k_n^4 r^2 c_n + k_n^2 r^2 \nu_n\\
&\quad +(r+2\alpha_n\epsilon_n)^2\gamma_n - \|x_{n+1}-p\|^2\\
&\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \delta_n + M(c_n+\nu_n+\gamma_n),
\end{aligned}
\tag{3.12}
$$

where δ_n and M are the same as in (3.8). By virtue of the properties of g, the assumption (i), (3.11) and (3.12), we have

$$\lim_{j\to\infty}\|x_{n_j} - T_2^{n_j}(z_{n_j})\| = 0, \quad \lim_{j\to\infty}\|x_{n_j+1} - T_2^{n_j+1}(z_{n_j+1})\| = 0,$$

and

$$\lim_{j\to\infty}\|x_{n_j} - T_3^{n_j}(x_{n_j})\| = 0, \quad \lim_{j\to\infty}\|x_{n_j+1} - T_3^{n_j+1}(x_{n_j+1})\| = 0.$$

Since for $i = 2, 3$,

$$
\begin{aligned}
\|x_n - T_i(x_n)\| \le\ & \|x_{n+1}-x_n\| + \|x_{n+1} - T_i^{n+1}(x_{n+1})\|\\
&+\|T_i^{n+1}(x_{n+1}) - T_i^{n+1}(x_n)\| + \|T_i^{n+1}(x_n) - T_i(x_n)\|
\end{aligned}
$$

and

$$\|x_{n_j+1} - x_{n_j}\| \le \beta_{n_j}\|T_1^{n_j}(y_{n_j}) - x_{n_j}\| + \gamma_{n_j}\|u_{n_j} - x_{n_j}\| \to 0$$

as $j\to\infty$. By the proof of (3.10), we get

$$\|x_{n_j} - T_i(x_{n_j})\| \to 0 \quad \text{as} \quad j\to\infty, \tag{3.13}$$

for each $i = 2, 3$.
It follows from (3.10) and (3.13) that

$$\|x_{n_j} - T_i(x_{n_j})\| \to 0 \quad \text{as} \quad j\to\infty, \quad i = 1, 2, 3.$$

This completes the proof. ■

Let $\{\varpi_n\}$ be a given sequence in C. A mapping $T : C \to C$ is said to satisfy *Condition* (I):

If $F(T) \neq \emptyset$, then there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\|\varpi_n - T(\varpi_n)\| \ge f(D(\varpi_n, F(T)))$$

for all $n \geq 1$.

By using Proposition 3.1, we have the following result.

Theorem 3.2. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T_i : C \to C$ $(i = 1, 2, 3)$ uniformly equi-continuous and generalized asymptotically quasi-nonexpansive mappings with*

$$F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$$

and two sequences $\{r_{in}\}$ and $\{s_{in}\} \subset [0, 1)$ such that $r_{in} \to 0$ and $s_{in} \to 0$. Let $r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}$, $s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}$, $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} < \infty$ and $\{x_n\}$ be a procedure defined by (2.1) with the following conditions:

- (i) $0 \leq \alpha_{n+1} \leq \alpha_n < \xi$, $0 \leq \lambda_{n+1} \leq \lambda_n < \eta$, $0 \leq a_{n+1} \leq a_n < \zeta$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} a_n = \infty$;
- (ii) $\beta_n \to 0$, $\mu_n \to 0$ *as* $n \to \infty$;
- (iii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;
- (iv) $\{v_n\}$ *and* $\{w_n\}$ *are bounded,* $u_n = u'_n + u''_n, n \geq 1$, *and*

$$\sum_{n=1}^{\infty} \|u'_n\| < \infty, \qquad \|u''_n\| = o(\alpha_n).$$

If $T_i (i = 1, 2, 3)$ satisfy Condition (I), *then the procedure $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. It follows from Proposition 3.1 that

$$\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad i = 1, 2, 3$$

and so, by Condition (I), we see that

$$\liminf_{n\to\infty} f(D(x_n, F)) = 0.$$

By virtue of the property of f, we assert that

$$\liminf_{n\to\infty} D(x_n, F) = 0.$$

Therefore, It follows from Lemma 2.3 that

$$\lim_{n\to\infty} D(x_n, F) = 0.$$

Now, we can take an infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F$ such that $\|x_{n_j} - p_j\| \leq 2^{-j}$. Set $M = \exp\{\sum_{n=1}^{\infty}(k_n - 1)\}$ and write $n_{j+1} = n_j + l$ for some $l \geq 1$. Then we have

$$\begin{aligned}\|x_{n_{j+1}} - p_j\| &= \|x_{n_j+l} - p_j\| \\ &\leq [1 + (k_{n_j+l-1} - 1)]\|x_{n_j+l-1} - p_j\| \\ &\leq \exp\Big\{\sum_{m=0}^{l-1}(k_{n_j+m} - 1)\Big\}\|x_{n_j} - p_j\| \\ &\leq \frac{M}{2^j},\end{aligned}$$

which implies that

$$\|p_{j+1} - p_j\| \leq \frac{2M+1}{2^{j+1}}.$$

Hence $\{p_j\}$ is a Cauchy sequence. Assume that $p_j \to p$ as $j \to \infty$. Then $p \in F$ from the closedness of $F(T)$, which implies that $x_j \to p$ as $j \to \infty$. This completes the proof. ■

If $T_i = T$ for all $i = 1, 2, 3$ in Theorem 3.2, we have the following theorem.

Theorem 3.3. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T : C \to C$ be a uniformly equi-continuous and generalized asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and two sequences $\{r_n\}$ and $\{s_n\} \subset [0, 1)$ such that $r_n \to 0$ and $s_n \to 0$. Let $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} < \infty$ and $\{x_n\}$ be a procedure defined by (2.2) with the following conditions:*

(*i*) $0 \leq \alpha_{n+1} \leq \alpha_n < \xi$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(*ii*) $\mu_n \to 0$ *as* $n \to \infty$;

(*iii*) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(*iv*) $\{v_n\}$ *and* $\{w_n\}$ *are bounded,* $u_n = u_n' + u_n''$, $n \geq 1$, *and*

$$\sum_{n=1}^{\infty} \|u_n'\| < \infty, \qquad \|u_n''\| = o(\alpha_n).$$

If T satisfy Condition (I), *then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Remark 3.1. Theorem 3.3 reduces to Theorem 6 in Zhou-Guo-Huang-Cho [35] when $s_n \equiv 0$ for all $n \geq 1$.

If $b_n = c_n \equiv 0$ for all $n \geq 1$ in Theorem 3.2, we can obtain the following conclusion ([17]).

Theorem 3.4. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T_i : C \to C$ $(i = 1, 2)$ uniformly equi-continuous and generalized asymptotically quasi-nonexpansive mappings with*

$$F = F(T_1) \cap F(T_2) \neq \emptyset$$

and two sequences $\{r_{in}\}$ and $\{s_{in}\} \subset [0, 1)$ such that $r_{in} \to 0$ and $s_{in} \to 0$ for $i = 1, 2$. Let $r_n = \max\{r_{1n}, r_{2n}\}$, $\quad s_n = \max\{s_{1n}, s_{2n}\}$, $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} < \infty$ and $\{x_n\}$ be a procedure defined by (2.3) with the following conditions:

(*i*) $0 \le \alpha_{n+1} \le \alpha_n < \xi$, $0 \le \lambda_{n+1} \le \lambda_n < \eta$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(*ii*) $\beta_n \to 0$, $\mu_n \to 0$ *as* $n \to \infty$;

(*iii*) $\sum_{n=1}^{\infty} \nu_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(*iv*) $\{v_n\}$ *is bounded,* $u_n = u'_n + u''_n, n \ge 1$, *and*

$$\sum_{n=1}^{\infty} \|u'_n\| < \infty, \qquad \|u''_n\| = o(\alpha_n).$$

If T_1 and T_2 satisfy Condition (I)*, then the procedure $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

If $\mu_n = \nu_n = b_n = c_n \equiv 0$ in Theorem 3.2 for all $n \ge 1$, then we can obtain the following result.

Theorem 3.5. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T_1 : C \to C$ uniformly equi-continuous and generalized asymptotically quasi-nonexpansive mapping with $F(T_1) \neq \emptyset$ and two sequences $\{r_{1n}\}$ and $\{s_{1n}\} \subset [0, 1)$ such that $r_{1n} \to 0$ and $s_{1n} \to 0$. Let $\sum_{n=1}^{\infty} \frac{r_{1n}+2s_{1n}}{1-s_{1n}} < \infty$ and $\{x_n\}$ be a sequence defined by (2.4) with the following conditions:*

(*i*) $0 \le \alpha_{n+1} \le \alpha_n < \xi$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(*ii*) $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(*iii*) $u_n = u'_n + u''_n, n \ge 1$, *and*

$$\sum_{n=1}^{\infty} \|u'_n\| < \infty, \qquad \|u''_n\| = o(\alpha_n).$$

If T_1 satisfies Condition (I)*, then the procedure $\{x_n\}$ converges strongly to a fixed point of T_1.*

Now We introduce an another convergence theorem of the three step iterative procedure for the generalized asymptotically quasi-nonexpansive mappings without the Condition (I).

Theorem 3.6. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X, and for $i = 1, 2, 3$, $T_i : C \to C$ uniformly L-Hölder continuous and generalized asymptotically quasi-nonexpansive mappings with*

$$F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$$

and two sequences $\{r_{in}\}$ and $\{s_{in}\} \subset [0, 1)$ such that $r_{in} \to 0$ and $s_{in} \to 0$. Let $r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}$, $s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}$, $n \geq 1$, $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} < \infty$ and $\{x_n\}$ be a procedure defined by (2.1) with the following conditions:

(*i*) $0 \leq \alpha_{n+1} \leq \alpha_n < \xi$, $0 \leq \lambda_{n+1} \leq \lambda_n < \eta$, $0 \leq a_{n+1} \leq a_n < \zeta$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} a_n = \infty$;

(*ii*) $\beta_n \to 0$, $\mu_n \to 0$ *as* $n \to \infty$;

(*iii*) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(*iv*) $\{v_n\}$ *and* $\{w_n\}$ *are bounded,* $u_n = u'_n + u''_n, n \geq 1$, *and*

$$\sum_{n=1}^{\infty} \|u'_n\| < \infty, \qquad \|u''_n\| = o(\alpha_n).$$

If for some integer $m \geq 1$ and $i = 1, 2, 3$, T_i^m are completely continuous, then $\{x_n\}$ converges strongly to some common fixed point of T_1, T_2 and T_3.

Proof. Since for $i = 1, 2, 3$, T_i are uniformly L-Hölder continuous and $\|T_i x_{n_j} - x_{n_j}\| \to 0$ as $j \to \infty$, we have

$$\begin{aligned} \|T_i^m x_{n_j} - x_{n_j}\| &\leq \|T_i^m x_{n_j} - T_i x_{n_j}\| + \|T_i x_{n_j} - x_{n_j}\| \\ &\leq \|T_i^m x_{n_j} - T_i^{m-1} x_{n_j}\| + \cdots + \|T_i x_{n_j} - x_{n_j}\| \\ &\leq L(m-1)\|T_i x_{n_j} - x_{n_j}\|^{\alpha} + \|T_i x_{n_j} - x_{n_j}\| \\ &\quad \to 0 \quad \text{as} \quad j \to \infty. \end{aligned}$$

Since T_i^m are completely continuous, we conclude that there exists a subsequence $\{x_{n_i}\}$ of $\{x_{n_j}\}$ such that $T_i^m x_{n_i} \to p$ as $i \to \infty$. Hence $x_{n_i} \to p$ as $i \to \infty$. It follows from Proposition 3.1 that $p \in F(T)$. Now the conclusion of the theorem follows from Lemma 2.3. This completes the proof. ∎

Remark 3.2. The results presented in this paper extend, improve and unify the corresponding results in [1, 6, 7, 14, 17-22, 24, 25, 27-30, 32-35].

Question. In recently, Beg-Abbas-Kim [2], Chang-Kim [4], Chang-Kim-Jin [5], Kim-Kim-Nam [13], Kim-Kim-Kim [15] studied the convergence theorems for the iterative processes generated by many kind of nonlinear operators in Convex metric spaces.

From the Theorem 3.2 and Theorem 3.6, following Question arises quite naturally and is so much interesting.

Can the Theorem 3.2 and Theorem 3.6 be extended to metric spaces and metric spaces with weak-metrics ?

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