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ESTIMATION OF DELAY ON SYNCHRONIZATION STABILITY IN A CLASS OF COMPLEX SYSTEMS WITH COUPLING DELAYS

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Abstract. Nowadays one of the most concerned topics in the field of complex networks is to find how the synchronizability depends on various parameters of networks. Complex networks with coupling delays have gained increasing attention in various fields of science and engineering in the past decade. One interesting problem is to investigate the effect of delay on dynamical behaviors and to determine the range of delay, in which, the synchronization stability can be achieved. In this paper, based on the qualitative theory of linear time-delay systems, the synchronization stability in complex dynamical networks with coupling delays is considered and some stability criteria of synchronization state are obtained. It is shown that by virtue of these obtained criteria, the range of delay on synchronization stability of complex networks with coupling delays can be analytically estimated. Finally, a couple of examples are illustrated which agree well with our theoretical results.

1. Introduction

It is noted that an increasing interest has been focused on complex networks with different topologies in the past decade [1-6]. A complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. There are various networks, which we often meet or hear of in the real world. Typical examples of complex networks include the Internet, the World Wide Web,

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electrical power grids, food webs, cellular and metabolic networks, etc. From the human brain to the Internet, and to the human society, complex networks are prominent candidates to describe sophisticated collaborative dynamics in many sciences [1, 7]. Dynamics of complex networks have been extensively investigated, with a emphasis on the interplay between complexity in the overall topology and local dynamical properties of the coupled nodes. As a particular kind of dynamics, synchronization in complex networks has been an important subject. The dependence of emergent collective phenomena on the coupling strength and on the topology was unveiled for homogeneous and heterogeneous complex networks in [8]. Synchronization in complex networks of Kuramoto phase oscillators has been studied. The results revealed that the route towards full synchronization strongly depends on whether one deals with homogeneous or heterogenous topologies [9]. This study has extended existing results on paths towards synchronization in complex networks [8, 9]. Based on a stochastic optimization technique, it was found that the value of degree mixing for providing optimal conditions of synchronization depends on the weighted coupling scheme [10]. Motivated by the abundance of directed synaptic couplings in a real biological neuronal network, the synchronization behavior of the Hodgkin-Huxley model in a directed network tells us that directedness of complex networks usually plays an important role in emerging dynamical behaviors. Synchronization in weighted complex networks shows that the synchronizability of random networks with a large minimum degree is determined by two leading parameters: the mean degree and the heterogeneity of the distribution of nodes intensity [12].

It is well known that time-delayed systems are ubiquitous in nature, technology, and society because of finite signal transmission times, switching speeds, and memory effects [13]. Recently, synchronization of complex networks with delayed coupling has received considerable attention. For example, Synchronization in oscillator networks with small-world interactions and coupling delays was investigated and a stability criterion for the network synchronized state was derived [14]. Results showed that the stability of the synchronized state is independent of the network topology. Continuous complex dynamical networks with coupling delays in the whole networks were studied in [15] and a stability theorem of synchronization is established by constructing a Lyapunov–Krasovskii functional. In terms of the linear matrix inequality and the stability theory of delay systems, some criteria of synchronization stability in symmetric networks with coupling delays were obtained for both delay-independent and delay-dependent cases [16-18]. However, by means of existing criteria of synchronization, in many cases it is still very difficult to estimate the range of delay analytically, in which synchronization of complex networks is realized.

In the present paper, based on the theory of asymptotic stability of linear time-delay systems, we are trying to explore some novel criteria of synchronization which enable us to estimate the range of delay effectively. The rest of the paper is organized as follows. In Section 2, stability criteria of synchronization for complex dynamical networks with coupling delays are presented. A couple of numerical examples are illustrated in Section 3 and a brief conclusion is given in Section 4.

2. Criteria of Synchronization Stability

In this section we consider a complex dynamical network consisting of N identically coupled nodes with each node being an n-dimensional dynamical system, and introduce the coupling delays in this network. The resultant dynamical system can be described as:

$$\dot{x_i} = f(x_i) + c\sum_{j=1}^{n} C_{ij}\Gamma(x_j(t-\tau)), \qquad i = 1, 2, \ldots, n, \tag{1}$$

where $f = (f_1, f_2, \ldots, f_n)$: $\mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable vector-valued function, which determines the dynamical behavior of nodes. $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n$ denotes the state variables of node i and N is the number of network nodes.. The constant c represents the coupling strength, $\Gamma = \mathrm{diag}\{r_1, r_2, \ldots, r_n\} \in \mathbb{R}^{n\times n}$ represents a constant 0–1 matrix linking the coupled variables. $C = (C_{ij})_{N\times N}$ represents the coupling configuration between nodes of the whole network (it is often assumed that there is at most one connection between node i and a different node j, and that there are no isolated clusters.), whose entries C_{ij} are defined as follows: if there is a connection between node i and node j $(j \neq i)$, then $C_{ij} = C_{ji} = 1$; otherwise, $C_{ij} = C_{ji} = 0$, and the diagonal elements of matrix C are defined by

$$C_{ii} = -\sum_{j=1,\neq i}^{n} C_{ij} = -\sum_{j=1,\neq i}^{n} C_{ji}, \qquad (i = 1, 2, \ldots, n). \tag{2}$$

Thus, we have the following technical lemmas immediately

Lemma 1. ([17]). *If the matrix C satisfies the above condition* (2)*, then we have*

(1) *0 is an eigenvalue of C with multiplicity 1, associated with the eigenvector (1 1, . . ., 1);*

(2) *all the other eigenvalues μ_i of C are less than 0 and μ_i $(i = 1, 2, \ldots, n)$ can be ordered as: $0 = \mu_1 > \mu_2 \geq \mu_3 \geq \cdots \geq \mu_n$.*

(3) *there exists a unitary matrix, $\Phi = (\phi_1, \phi_2, \ldots, \phi_N)$ such that $C^T\phi_k = \mu_k\phi_k$, where μ_k $(k = 1, 2, \ldots, n)$ are the eigenvalues of C.*

The synchronization state of the complex network (1) is defined as:

$$x_1(t) \longrightarrow x_2(t) \longrightarrow \ldots \longrightarrow x_n(t) = s(t), \qquad \text{as } t \to +\infty, \tag{3}$$

where synchronization state $s(t)$ of the network (1) satisfies the differential system:

$$\dot{s}(t) = f(s(t)).$$

This paper mainly aims to this case when the synchronous state is a stable equilibrium point. Hence, synchronization manifold can be set as $s(t) = e$, where e is the stable equilibrium state. Clearly, the stability of synchronized states (3) of the network (1) is determined by the coupling strength c, the inner-coupling matrix C, the outer-coupling matrix Γ, and the time-delay constant τ.

Lemma 2. *Consider the delayed dynamical network (1), whose synchronization manifold is a stable equilibrium state $s(t) = e$. If the other $N-1$ pieces of n-dimensional linear delayed differential equations are asymptotically stable about the zero solutions:*

$$\dot{w}(t) = J(e)w(t) + c\mu_i \Gamma w(t-\tau), \quad i = 2, \ldots, n, \tag{4}$$

where $J(e) = Df(e)$ and $Df(e)$ is the Jacobian of $f(x(t))$ at e, then the synchronized states (3) are asymptotically stable.

Proof of Lemma 2. Set

$$x_i = e + \eta_i(t), \quad i = 2, \ldots, n,$$

then we have

$$\begin{aligned} \dot{\eta}_i(t) &= \left(f(e+\eta_i(t)) + c\sum_{j=1}^{n} C_{ij}\Gamma(e+\eta_j(t-\tau)) \right) \\ &\quad - \left(f(e) + c\sum_{j=1}^{n} C_{ij}\Gamma e \right). \end{aligned} \tag{5}$$

Since f is continuously differentiable, it is easy to see that the origin of nonlinear system (5) is an asymptotically stable equilibrium point if it is an asymptotically stable equilibrium point of the linear system:

$$\begin{aligned} \dot{\eta}_i(t) &= J(e)\eta_i(t) + c\sum_{j=1}^{n} C_{ij}\Gamma\eta_j(t-\tau), \\ &= J(e)\eta_i(t) + c\Gamma\eta(t-\tau)(C_{i1}, C_{i2}, \ldots, C_{in})^T, \end{aligned}$$

where $J(e) = Df(e)$ and $Df(e)$ is the Jacobian of $f(x(t))$ at e.

Letting $\eta(t) = (\eta_1(t), \eta_2(t), \ldots, \eta_n(t)) \in \mathbb{R}^{n\times n}$, we have

$$\dot{\eta}(t) = J(e)\eta(t) + c\Gamma\eta(t-\tau)C^T.$$

According to Lemma 1, there exists a nonsingular matrix, $\Phi = (\phi_1, \phi_2, \ldots, \phi_n)$ such that $C^T\Phi = \Lambda\Phi$ with $\Lambda = diag(\mu_1, \mu_2, \ldots, \mu_n)$, where μ_k $(k = 1, 2, \ldots, n)$ are the eigenvalues of C. Using the nonsingular transform $\eta(t)\Phi = w(t) = (v_1(t), \ldots, v_n(t)) \in \mathbb{R}^{n\times n}$, we have

$$\dot{w}(t) = J(e)w(t) + c\Gamma w(t-\tau)\Lambda,$$

that is,

$$\dot{v}_i(t) = J(e)v_i(t) + c\mu_i\Gamma v_i(t-\tau), \quad i = 1, 2, \ldots, n. \tag{6}$$

Hence, through this way we convert the stability problem of the synchronized states (3) into the stability problem of the N pieces of n-dimensional linear delayed differential equations (6). Since $\mu_1 = 0$ corresponds to the synchronizing state e, the synchronized states (3) are asymptotically stable when the $N-1$ pieces of the n-dimensional linear delayed differential system (4) are asymptotically stable about the zero solution. Therefore, the proof is completed. ∎

In what follows, some stability criteria of system (4) are formulated. To do these, we firstly consider a linear time–delay dynamical system:

$$\dot{x} = Ax + Bx(t-\tau) \tag{7}$$

where $x \in \mathbb{R}^n$, A, $B \in \mathbb{R}^{n\times n}$ and $\tau > 0$ is the time delay. In order to present our results in a straightforward manner, here we introduce several notations

$$\begin{aligned}
b_1 &= \max_{\theta\in[0,2\pi]} \frac{1}{2}\lambda_{max}\{A_s + B_s\cos(\theta) + jB_\alpha\sin(\theta)\},\\
l_1 &= \max\{\frac{1}{2}\lambda_{max}(A_s), b_1\},\\
b_2 &= \max_{\theta\in[0,2\pi]} \frac{1}{2}\lambda_{max}\{j(A_\alpha + B_\alpha\cos(\theta)) - B_s\sin(\theta)\},\\
l_2 &= \max\{\frac{1}{2}\lambda_{max}(jA_\alpha), b_2\},
\end{aligned}$$

where

$$\begin{aligned}
A_\alpha &= A^T - A, \quad A_s = A^T + A,\\
B_\alpha &= B^T - B, \quad B_s = B^T + B.
\end{aligned}$$

The following two fundamental results which give the conditions for the asymptotic stability of system (7) are presented in [19]:

Lemma 3. *If $l_1 < 0$ holds, then the zero solution of system (7) is asymptotically stable.*

Assume that system (7) is asymptotically stable when $\tau = 0$. The following lemma provides us a way to estimate how large the time delay τ can be such that system (7) keeps its asymptotic stability.

Lemma 4. *Let $\psi(\theta) = \det[(A+Be^{-j\theta}) \bigotimes I_n + I_n \bigotimes (A+Be^{j\theta})]$. If there is no positive real number θ in $[0, \pi]$ such that $\psi(\theta) = 0$, then system (7) is asymptotically stable independent of delay. Otherwise, system (7) keeps its asymptotic stability if*

$$\tau < \begin{cases} \dfrac{\underline{\theta}}{l_2}, & \text{if} \quad l_2 \neq 0 \\ \infty, & \text{if} \quad l_2 = 0 \end{cases}$$

where $\underline{\theta}$ is the least positive real number in $[0, \pi]$ such that $\psi(\theta) = 0$.

The above lemmas sufficiently enable us to formulate stability criteria of synchronization for complex networks with coupling delays.

Theorem 1. (Delay-Independent Criterion). *If the matrix $J_s(e)$ is stable, then when $\max\limits_{\theta \in [0,2\pi]} \lambda_{max}(J_s(e) + 2c\mu_i \Gamma \cos(\theta)) < 0$ holds for all μ_i, the synchronization state (3) is asymptotically stable for any delay τ, where $J_s(e) = J^T(e) + J(e)$.*

Proof of Theorem 1. For the linear time-delay system (4), since $B_\alpha = (c\mu_i\Gamma)^T - c\mu_i\Gamma = 0$, we have

$$\begin{aligned} b_1 &= \max_{\theta \in [0,2\pi]} \frac{1}{2} \lambda_{max}\{A_s + B_s \cos(\theta) + jB_\alpha \sin(\theta)\}, \\ &= \max_{\theta \in [0,2\pi]} \lambda_{max}(J_s(e) + 2c\mu_i\Gamma \cos(\theta)). \end{aligned}$$

Moreover, since $J_s(e)$ is stable, it is inferred that $\lambda_{max}(J_s(e)) < 0$. Hence, $l_1 = \max\{\frac{1}{2}\lambda_{max}(J_s(e)), b_1\} = \max\limits_{\theta \in [0,2\pi]} \lambda_{max}(J_s(e) + 2c\mu_i\Gamma \cos(\theta))$. By virtue of lemma 3, when $l_1 = \max\limits_{\theta \in [0,2\pi]} \lambda_{max}(J_s(e) + 2c\mu_i\Gamma \cos(\theta)) < 0$ for all μ_i, the synchronization state (3) is asymptotically stable for any delay. ■

In particular, when $\Gamma = I_n$, the following remark can be easily seen:

Remark 1. If the matrix $J_s(e)$ is stable and $-2c\mu_N + \lambda_{max}(J_s(e)) < 0$ holds, then the synchronization state (3) is asymptotically stale for any delay.

Theorem 1. (Criterion of Estimating Delay). *Let* $\psi(\theta) = \det[(J(e)+c\mu_i\Gamma e^{-j\theta}) \bigotimes I_n + I_n \bigotimes (J(e)+c\mu_i\Gamma e^{j\theta})]$. *If there is no positive real number* θ *in* $[0, \pi]$ *for any* μ_i *such that* $\psi(\theta) = 0$*, then the synchronization state (3) is asymptotically stable independent of delay. Otherwise, the synchronization state (3) keeps its asymptotic stability if*

$$\tau < \begin{cases} \dfrac{\underline{\theta}}{l_2}, & \text{if} \quad l_2 \neq 0, \\ \infty, & \text{if} \quad l_2 = 0, \end{cases}$$

where $\underline{\theta}$ *is the least positive real number in* $[0, \pi]$ *such that* $\psi(\theta) = 0$*, and*

$$b_2 = \max_{\theta \in [0, 2\pi]} \frac{1}{2} \lambda_{max}\{j(J^T(e) - J(e)) - 2c\mu_i\Gamma \sin(\theta)\},$$

$$l_2 = \max\{\frac{1}{2}\lambda_{max}(j(J^T(e) - J(e))), b_2\}.$$

Theorem 2 can be proved immediately by means of Lemmas 2 and 4. Apparently, the obtained stability criterion presents a way by which the delay can be estimated for the synchronization stability of complex networks. It is certainly important for testing synchronization of complex networks with the coupling delays since the range of delay on the synchronization stability can be analytically derived.

3. Examples

Example 1. The synchronization criteria presented in the preceding section can be applied to networks with different topologies and different sizes. To better understand the criteria, we consider a network model of five nodes with numerical simulations, in which each node is a simple three-dimensional stable linear system as described in [15]:

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -2x_2 \\ \dot{x}_3 &= -3x_3 \end{aligned}$$

whose zero solution is asymptotically stable since its Jacobian matrix is

$$J(e) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Assume that the inner-coupling matrix is $\Gamma = \mathrm{diag}\{1, 1, 1\}$, and the outer-coupling matrix is

$$C = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix},$$

with five eigenvalues 0, -1.382, -2.382, -3.168 and -4.168.

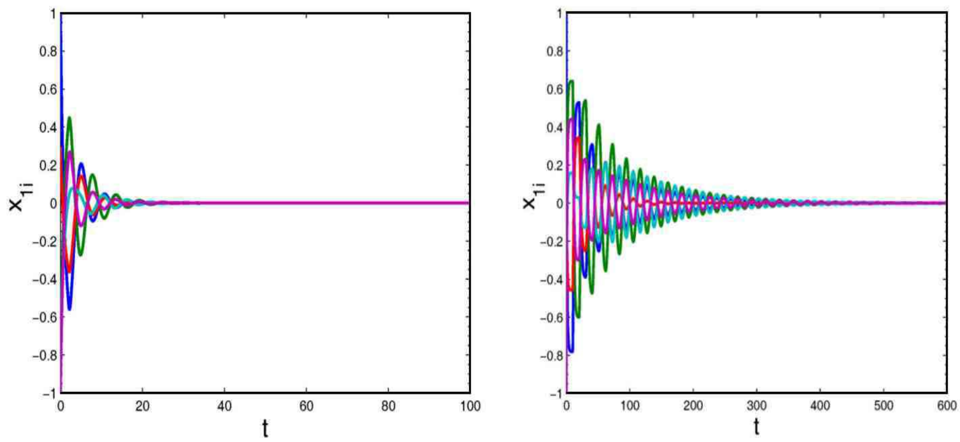

Fig. 1. (a) Time series of the first variable x_{1i} of node i for the time delay $\tau = 2$. (b) Time series of the first variable x_{1i} of node i for the time delay $\tau = 20$. Here, the coupling strength is $c = 0.2$.

Since $J_s(e)$ is stable, by Remark 1, if $-2c\mu_N + \lambda_{max}(J_s(e)) < 0$, then it is inferred that for any delay, synchronization of the complex network can be achieved. After a simple calculation, we can derive that when $c < 1/4.168 \approx 0.2399$, synchronization of the complex network can be achieved for any delay. For clearer visions, we take the coupling strength $c = 0.2 < 0.2399$ and the time delay $\tau = 2$, 20, respectively. Numerical simulations in Fig. 1 show that networks can eventually achieve synchronization state at $s(t) = 0$ irrespectively of the size of the time delay. This result is in agreement with that described in [16, 17], which is numerically tested by the linear matrix inequalities and the matrix measure. Hence, our approach provides a simple and effective alternative way to test the synchronization of complex networks with coupling delays.

However, if $l_1 \geq 0$ for a certain μ_i, we can resort to Theorem 2 to estimate the range of the time delay τ for the fixed value c, where the stability of synchronization can be achieved. According to Theorem 2, $\psi(\theta)$ is given by

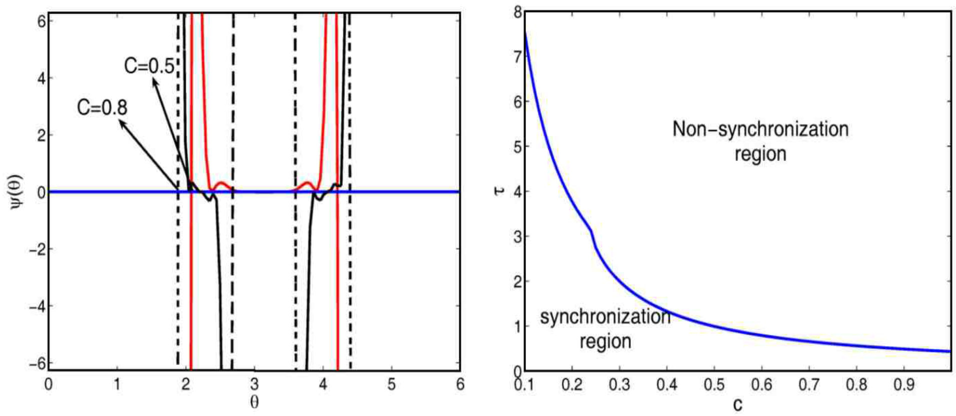

Fig. 2. (a) Variations of $\psi(\theta)$ with respect to θ for the fixed μ_N when the coupling strength is chosen as $c = 0.5$ and 0.8, respectively. (b) Synchronization and non-synchronization region in the plane (c, τ).

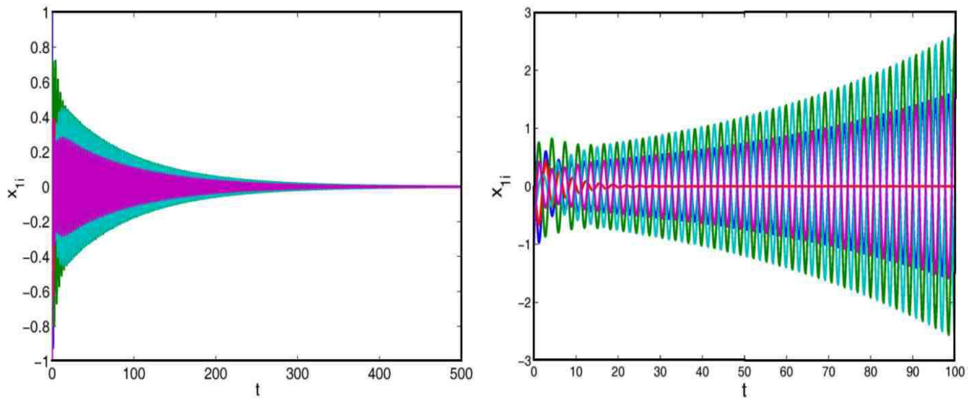

Fig. 3. (a) Time series of the first variable x_{1i} of node i for the time delay $\tau = 0.95$. (b) Time series of the first variable x_{1i} of node i for the time delay $\tau = 1$. Here, the coupling strength is $c = 0.5$.

$$
\begin{aligned}
\psi(\theta) &= \det[(J(e) + c\mu_i\Gamma e^{-j\theta}) \bigotimes I_3 + I_3 \bigotimes (J(e) + c\mu_i\Gamma e^{j\theta})], \\
&= (-2 + 2c\mu_i\cos(\theta))(-3 + 2c\mu_i\cos(\theta))^2(-4 + 2c\mu_i\cos(\theta))^3 \\
&\quad (-5 + 2c\mu_i\cos(\theta))^2(-6 + 2c\mu_i\cos(\theta)), \\
b_2 &= \max_{\theta\in[0,2\pi]} \frac{1}{2}\lambda_{max}\{2c\mu_i\sin(\theta)\} = -c\mu_i, \\
l_2 &= \max\{\frac{1}{2}\lambda_{max}(j(J^T(e) - J(e))), b_2\}, \\
&= -c\mu_i.
\end{aligned}
$$

From the above calculations, one can see that when $(-2 + 2c\mu_i\cos(\theta)) > 0$, there exists at least one θ such that $\psi(\theta) = 0$. For example, if we take $c = 0.5$, we have $\underline{\theta} = 2.0808$. Variations of $\psi(\theta)$ with the respect to θ are presented in Fig. 2. At the same time, a simple calculation gives $l_2 = 2.0840$. By virtue of Theorem 2, we know that system (1) keeps its synchronization stability if $\tau < \frac{\underline{\theta}}{l_2} = 0.9985$. The corresponding numerical simulation is illustrated in Fig. 3 (a) and (b), respectively. It is easy to observe that when $\tau = 0.95$ and $c = 0.5$, time series x_{1i} (i=1, 2, 3, 4, 5) eventually synchronize to the stability state of the zero solution; when $\tau = 1$ and $c = 0.5$, the network can not synchronize. This agrees well with the theoretical analysis established in Section 2.

Furthermore, note that from $(-2+2c\mu_N\cos(\theta)) = 0$, when the coupling strength increases, $\underline{\theta}$ in Theorem 2 will decrease, and then the upper boundary of τ is lowered. The corresponding numerical results are shown in Fig. 2 (a), which exhibits the decreasing of $\underline{\theta}$. In Fig. 2 (b) it presents variations of the upper boundary of τ when the coupling strength increases, which also verify our theoretical analysis.

Example 2. Consider the network consisting of the third-order smooth Chua's circuits [20], in which each node equation is given as

$$\begin{aligned} \dot{x}_{i1} &= -k\alpha x_{i1} + k\alpha x_{i2} - k\alpha(ax_{i1}^3 + bx_{i1}), \\ \dot{x}_{i2} &= kx_{i1} - kx_{i2} + kx_{i3}, \\ \dot{x}_{i3} &= -k\beta x_{i2} - k\gamma x_{i3}. \end{aligned} \tag{8}$$

Making a linearization for (8) at its zero equilibrium point yields

$$J(e) = \begin{pmatrix} -k\alpha - k\alpha b & k\alpha & 0 \\ k & -k & k \\ 0 & -k\beta & -k\gamma \end{pmatrix}.$$

If we take $k = 1$, $\alpha = -0.1$, $\beta = -1$, $\gamma = 1$, $a = 1$ and $b = -25$, then $J(e)$ is stable, namely, system (8) is locally stable at zero.

The inner-coupling matrix is $\Gamma = \mathrm{diag}\{1, 1, 1\}$. The outer-coupling matrix is

$$C = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix},$$

with five eigenvalues -3.6180, -3.6180, -1.3820, -1.3820 and 0.000.

Now we will estimate the delay where the synchronization stability can be achieved. Using Theorem 2, we have

$$\begin{aligned}\psi(\theta) &= \det[(J(e)+c\mu_i\Gamma e^{-j\theta})\bigotimes I_3 + I_3\bigotimes(J(e)+c\mu_i\Gamma e^{j\theta})]\\ &= \det[J(e)\bigotimes I_3 + (c\mu_i\Gamma e^{-j\theta})\bigotimes I_3 + I_3\bigotimes J(e) + I_3\bigotimes(c\mu_i\Gamma e^{j\theta})]\\ &= \det[J(e)\bigotimes I_3 + I_3\bigotimes J(e) + (c\mu_i\Gamma e^{j\theta} + c\mu_i\Gamma e^{-j\theta})\bigotimes I_3]\\ &= \det[J(e)\bigotimes I_3 + I_3\bigotimes J(e) + 2c\mu_i\cos\theta\Gamma\bigotimes I_3]\\ &= (-0.0416+c\mu_i\cos\theta)(-4.3792+c\mu_i\cos\theta)^2(19.2357-8.7584c\mu_i\cos\theta\\ &\quad +c^2\mu_i^2\cos^2\theta)(4.9004-4.4208c\mu_i\cos\theta+c^2\mu_i^2\cos^2\theta)^2.\end{aligned}$$

If we take $c = 0.1$, a simple computation shows that only in the case where $-0.0416 + c\mu_i\cos\theta = 0$, there exists a $\theta \in [0, \pi]$ such that $\psi(\theta) = 0$. Hence, we can have $\underline{\theta} = \arccos(-\frac{0.0416}{0.3618}) = 1.6861$, which corresponds to the eigenvalue $\mu_N = -3.618$. Variations of $|\psi(\theta)|$ for different eigenvalues μ_i are shown in Fig. 4 (a). It is clear to see that there exists a minimal value $\theta = 1.6861$ such that $|\psi(\theta)| = 0$, which is in a good agreement with our theoretical analysis. From Theorem 2, we can find values of b_2 and l_2 as:

$$\begin{aligned}b_2 &= \max_{\theta\in[0,2\pi]}\frac{1}{2}\lambda_{max}\{j(J^T(e)-J(e)) - 2c\mu_N\Gamma\sin(\theta)\}\\ &= \max_{\theta\in[0,2\pi]}\{\frac{1809}{5000}\sin(\theta)+\frac{11}{20}\} = 0.9118,\end{aligned}$$

$$l_2 = \max\left(\frac{1}{2}\lambda_{max}(j(J^T(e)-J(e))), b_2\right) = \max\{0.5500,\ 0.9118\} = 0.9118.$$

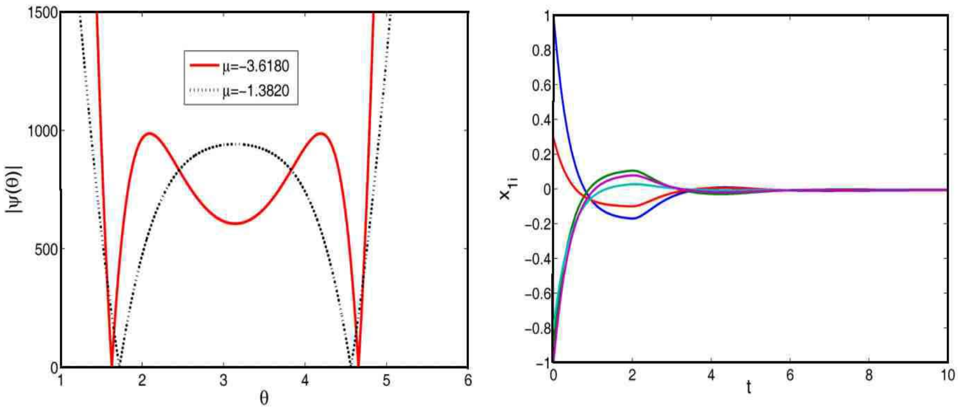

Fig. 4. (a) Variations of $|\psi(\theta)|$ with respect to θ for two different eigenvalues. (b) Time series of the first variable x_{1i} of node i for the time delay $\tau = 1.8$. with the coupling strength $c = 0.1$.

Using Theorem 2, if the time delay $\tau < 1.6861/0.9118 = 1.8492$, the stability of synchronization state is achieved for the network consisting of the third-order smooth Chua's circuits. When we take $\tau = 1.8$, Fig. 4(b) shows that synchronization is eventually realized as the time is evolving.

4. Conclusion

In this work synchronization of complex dynamical networks with coupling delays was investigated. Based on the stability theory of the linear time-delay system, we have obtained new stability criteria of synchronization state in complex dynamical networks with coupling delays. By means of these criteria, we can estimate the range of delay, in which, synchronization stability can be realized. Moreover, one advantage of these criteria is that it is easy to be verified by means of combination of the theoretical analysis and numerical simulations. These synchronization conditions are also applicable to networks with different topologies and different sizes. As illustrations, two examples are given to support our theoretical analysis.

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