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A REPRESENTATION THEOREM FOR NORMS IN HILBERT SPACE

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Abstract. A representation theorem for all norms equivalent to the original norm in a complex Hilbert space is established by using ellipsoids.

Let H be a complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $||\cdot||$. An operator on H means a *bounded linear transformation* from H into itself. We call that an operator A is *positive*, denoted as $A > 0$, if $\langle Ax, x\rangle > 0$ for all nonzero x in H. Note that the condition $\langle Ax, x\rangle \geq 0$ for all x in H implies that $A = A^*$. For the proof, the condition implies that

$$\langle Ax, x\rangle = \langle x, Ax\rangle \quad \text{for all } x \text{ in } H. \tag{1}$$

By an analog of polarization identity, for all x, y in H we have

$$\begin{aligned} &\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle + i\langle A(x+iy), x+iy\rangle \\ &-i\langle A(x-iy), x-iy\rangle \\ =\ & 4\langle Ax, y\rangle \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle + i\langle x+iy, A(x+iy)\rangle \\ &-i\langle x-iy, A(x-iy)\rangle \\ =\ & 4\langle x, Ay\rangle. \end{aligned} \tag{3}$$

Combining (2) and (3) with (1) proves that $A = A^*$.

We will establish the following:

Theorem. *Let* $|||\cdot|||$ *be a norm on* H *equivalent to* $||\cdot||$. *Then*

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$$|||x||| = \sup_{A \in \mathcal{A}} \langle Ax, x\rangle^{1/2},$$

where

$$\mathcal{A} = \{A > 0;\ \langle Ax, x\rangle \le |||x|||^2 \text{ for all } x \in H\}.$$

The theorem is equivalent to saying that every compact convex balanced set in H with center at 0 is the intersection of all ellipsoids containing it with center at 0 (see Figure 1). For an operator $A > 0$ and x_0 in H, the set $\{x \in H;\ \langle A(x - x_0), x - x_0\rangle \le 1\}$ is called an *ellipsoid* with center at x_0.

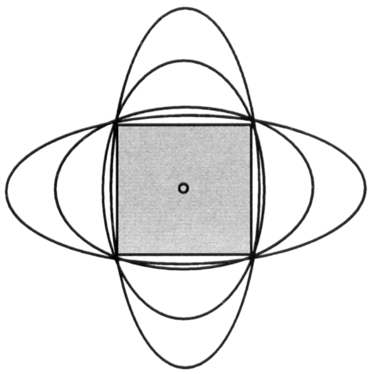

Fig. 1.

Proof of Theorem. Let $x_0 \in H$ and consider the subspace

$$W = \{\alpha x_0;\ \alpha \in \mathbb{C}\}.$$

Define a complex linear functional l on W by

$$l(\alpha x_0) = \alpha|||x_0||| \text{ for all } \alpha \in \mathbb{C}.$$

By the complex version of the Hahn-Banach theorem [1, 2, 3], the complex linear functional l can be extended to H such that

$$l(x_0) = |||x_0||| \quad \text{and} \quad |l(x)| \le |||x||| \text{ for all } x \in H.$$

By the Riesz representation theorem, there exists a unique $y \in H$ such that

$$l(x) = \langle x, y\rangle \text{ for all } x \in H.$$

Define an operator T on H by

$$T(x) = l(x)y \text{ for all } x \in H.$$

Then

$$\langle Tx, x\rangle = |l(x)|^2 \leq |||x|||^2 \text{ for all } x \in H$$

and

$$\langle Tx_0, x_0\rangle^{1/2} = |l(x_0)| = |||x_0|||.$$

Since $|||\cdot|||$ is equivalent to $||\cdot||$, there exists $\alpha > 0$ such that

$$|||x|||^2 \geq \alpha\langle x, x\rangle \text{ for all } x \in H.$$

For $k = 1, 2, \ldots,$ let

$$T_k = (1 + \frac{1}{k})^{-1}(T + \frac{\alpha}{k}I).$$

Then $T_k > 0$ for $k = 1, 2, \ldots,$

$$\langle T_k x, x\rangle \leq |||x|||^2 \text{ for all } x \in H \text{ and } k = 1, 2, \ldots,$$

and

$$||T_k - T|| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Therefore

$$\sup_{A\in\mathcal{A}} \langle Ax_0, x_0\rangle^{1/2} \leq |||x_0||| = \langle Tx_0, x_0\rangle^{1/2} = \lim_{k\to\infty} \langle T_k x_0, x_0\rangle^{1/2} \leq \sup_{A\in\mathcal{A}} \langle Ax_0, x_0\rangle^{1/2},$$

revealing the representation of $|||x_0|||$, proving the theorem. ■

References

1. H. F. Bohnenblust and A. Sobczyk, Extension of functions on complex linear spaces, *Bull. Amer. Math. Soc.*, 44 (1938), 91-93.
2. P. D. Lax, *Functional Analysis*, John Wiley & Sons, Inc., 2002.
3. G. A. Soukhomlinoff, Über Fortsetzung von linearen Funktionalen in linearen komplexen Räumen und linearen Quaternion-räumen, *Sbornik, N. S.*, 3 (1938), 353-358.

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