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BRIEF SURVEY OF RECENT APPLICATIONS OF AN ORDER PRESERVING OPERATOR INEQUALITY

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Dedicated to Professor Hang-Chin Lai with Respect and Affection

Abstract. This short paper surveys recent several applications of an order preserving operator inequality, especially, logarithmic trace inequalities are presented.

1. Order Preserving Operator Inequality and Its Generalization

A capital letter means a bounded linear operator on a Hilbert space without specification.

Theorem L-H (Löwner-Heinz inequality). *$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$.*

Although Theorem L-H is very useful, the condition " $\alpha \in [0,1]$ " is too restrictive. In fact Theorem L-H does not always hold for $\alpha \notin [0,1]$. The following result was obtained from this point of view.

Theorem F. *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$ and

(ii) $(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$ hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

In Theorem F, (i) is equivalent to (ii). The domain for p,q and r drawn in Figure 1 is the best possible one for Theorem F (Tanahashi [47]).

Theorem F yields the Löwner-Heinz inequality asserting that $A \geq B \geq 0$ *ensures* $A^{\alpha} \geq B^{\alpha}$ *for any* $\alpha \in [0,1]$ when we put $r = 0$ in (i) or (ii) of Theorem F.

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Consider two magic boxes

$$f(\Box) = (B^{\frac{r}{2}}\Box B^{\frac{r}{2}})^{\frac{1}{q}} \text{and} g(\Box) = (A^{\frac{r}{2}}\Box A^{\frac{r}{2}})^{\frac{1}{q}}.$$

Theorem F can be regarded as follows. Although $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for $p > 1$, Theorem F asserts the following two *"order preserving operator inequalities"*

$$f(A^p) \geq f(B^p) \text{and} g(A^p) \geq g(B^p)$$

hold whenever $A \geq B \geq 0$ under the condition on p , q and r in Figure 1.

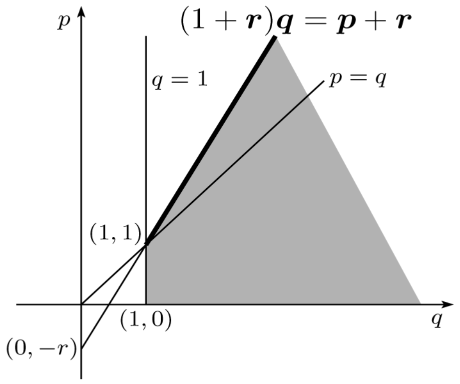


Fig. 1.

We have been finding a lot of applications of Theorem F in several branches according to remarkable achievements of many mathematicians who have been interested in operator inequalities, we divide these branches into the following three branches (A) operator inequalities, (B) norm inequalities, and (C) operator equations.

(A) Operator Inequalities

(A-1) Several characterizations of operators with $logA \geq logB$ and its applications;

(A-2) Applications to the relative operator entropy;

(A-3) Applications to Ando-Hiai log majorization and logarithmic trace inequalities;

(A-4) Generalized Aluthge transformation on p-hyponormal operators;

(A-5) Several classes associated with log-hyponormal and paranormal operators;

(A-6) Operator functions implying order preserving inequalities;

(A-7) Applications to Kantorovich type operator inequalities.

(B) Norm Inequalities

(B-1) Several generalizations of Heinz-Kato theorem;

(B-2) Generalizations of some theorems associated with norms;

(B-3) An extension of Kosaki trace inequality and parallel results.

(C) Operator Equations

(C-1) Generalizations of Pedersen-Takesaki's theorem and related results.

In this short survey, we would like to focus ourselves to state log majorization (A-3), Aluthge transformation (A-4) and order preserving operator functions (A-6).

Lemma A. (Lemma 1, Furuta [18]). *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^\lambda = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

Proof of Lemma A. Let $YX^{\frac{1}{2}} = UH$ be the polar decomposition of $YX^{\frac{1}{2}}$,where U is unitary and $H = |YX^{\frac{1}{2}}|$. Then we have

$$\begin{aligned}(YXY^*)^\lambda &= (UH^2U^*)^\lambda = YX^{\frac{1}{2}}H^{-1}H^{2\lambda}H^{-1}X^{\frac{1}{2}}Y^* \\ &= YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.\end{aligned}$$ ■

It easily turns out that we don't require the invertibility of A and B in the case $\lambda \geq 1$ in Lemma A which is obviously seen in the proof. Lemma A is very simple with its proof stated above, but quite a useful tool in order to treat operator transformation in operator theory.

Proof of Lemma F. At first we prove (ii). In the case $1 \geq p \geq 0$, the result is obvious by Theorem L-H. We have only to consider $p \geq 1$ and $q = \frac{p+r}{1+r}$ since (ii) of Theorem F for values q larger than $\frac{p+r}{1+r}$ follows by Theorem L-H, that is, we have only to prove the following

$$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad for\ any\ p \geq 1\ and\ r \geq 0. \tag{1.1}$$

We may assume that A and B are *invertible* without loss of generality. In the case $r \in [0, 1]$, $A \geq B \geq 0$ ensures $A^r \geq B^r$ holds by Theorem L-H. Then we have

$$\begin{aligned}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} &= A^{\frac{r}{2}}B^{\frac{p}{2}}(B^{\frac{-p}{2}}A^{-r}B^{\frac{-p}{2}})^{\frac{p-1}{p+r}}B^{\frac{p}{2}}A^{\frac{r}{2}} \quad \text{by Lemma A}\\ &\leq A^{\frac{r}{2}}B^{\frac{p}{2}}(B^{\frac{-p}{2}}B^{-r}B^{\frac{-p}{2}})^{\frac{p-1}{p+r}}B^{\frac{p}{2}}A^{\frac{r}{2}}\\ &= A^{\frac{r}{2}}BA^{\frac{r}{2}} \leq A^{1+r},\end{aligned}$$

where the first inequality follows by $B^{-r} \geq A^{-r}$ and Theorem L-H since $\frac{p-1}{p+r} \in [0,1]$ holds, and the last inequality follows by $A \geq B > 0$. So we have the following (1.2)

$$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \text{for } p \geq 1 \text{ and } r \in [0,1]. \tag{1.2}$$

Put $A_1 = A^{1+r}$ and $B_1 = (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ in (1.2). Repeating (1.2) again for $A_1 \geq B_1 \geq 0$, $r_1 \in [0,1]$ and $p_1 \geq 1$,

$$A_1^{1+r_1} \geq (A_1^{\frac{r_1}{2}}B_1^{p_1}A_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}.$$

Put $p_1 = \frac{p+r}{1+r} \geq 1$ and $r_1 = 1$, then

$$A^{2(1+r)} \geq (A^{r+\frac{1}{2}}B^pA^{r+\frac{1}{2}})^{\frac{2(1+r)}{p+2r+1}} \quad \text{for } p \geq 1, \text{ and } r \in [0,1]. \tag{1.3}$$

Put $\frac{s}{2} = r + \frac{1}{2}$ in (1.3). Then $\frac{2(1+r)}{p+2r+1} = \frac{1+s}{p+s}$ since $2(1+r) = 1+s$, so that (1.3) can be rewritten as follows;

$$A^{1+s} \geq (A^{\frac{s}{2}}B^pA^{\frac{s}{2}})^{\frac{1+s}{p+s}} \quad \text{for } p \geq 1 \text{ and } s \in [1,3]. \tag{1.4}$$

Consequently (1.2) and (1.4) ensure that (1.2) holds for *any* $r \in [0,3]$ since $r \in [0,1]$ and $s = 2r+1 \in [1,3]$ and repeating this process proves (1.1) for *any* $r \geq 0$. Hence (ii) is shown.

If $A \geq B > 0$, then $B^{-1} \geq A^{-1} > 0$. Then by (ii), for each $r \geq 0$, $B^{\frac{-(p+r)}{q}} \geq (B^{\frac{-r}{2}}A^{-p}B^{\frac{-r}{2}})^{\frac{1}{q}}$ holds for each p and q such that $p \geq 0$, $q \geq 1$ and $(1+r)q \geq p+r$. Taking inverses gives (i), so the proof of Theorem F is completed. ∎

This one page proof is in Furuta [16] and the original one is in Furuta [14]; other proofs afterward are in M.Fijii [7] and Kamei [34].

Remark 1.1. Recall that the essential assertion of Theorem F is as follows since Theorem F is obvious in case $1 \geq p \geq 0$ by Theorem L-H:

$$A \geq B \geq 0 \Longleftrightarrow A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \text{ for } p \geq 1 \text{ and } r \geq 0.$$

Theorem GF. (Generalization of Theorem F). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$G(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is a decreasing function for $r \geq t$ and $s \geq 1$, and $A^{1-t} = G(A,A,r,s) \geq G(A,B,r,s)$, that is,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{GF}$$

holds for $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$.

The original proof of Theorem GF is in Furuta [18], and an alternative one is in M.Fijii-Kamei [9]. An elementary one-page proof of (GF) is in Furuta [20]. Further extensions of Theorem GF and related results are obtained by many researchers, and some of them are in Furuta [22], Furuta-Hashimoto-Ito [26], Furuta-Yanagida-Yamazaki [28], Lin [41] and Kamei [35]. It was shown in Tanahashi [48] that the exponent value $\dfrac{1-t+r}{(p-t)s+r}$ of the right hand of (GF) is the best possible and alternative proofs of this fact are in Fujii-Matsumoto-Nakamoto [11] and Yamazaki [53]. (GF) interpolates Theorem F and an inequality equivalent to the main result of log majorization in Ando-Hiai [4]. (See Remark 3.1 in §3).

2. Aluthge Transformation on p-Hyponormal Operators

An operator T on a Hilbert space H is said to be *p-hyponormal* if

$$(T^*T)^p \geq (TT^*)^p \text{ for positive number } p.$$

The class of p-hyponormal has been defined as an extension of hyponormal and also it has been studied by many authors.

For a p-hyponormal operator $T = U|T|$, define $\widetilde{T}$ as follows:

$$\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$$

which is called "Aluthge transformation".

Theorem AL. ([1]). *Let $T = U|T|$ be p-hyponormal for $p > 0$ and U be unitary. Then*

(i) $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is $(p+\frac{1}{2})$-hyponormal if $0 < p < \frac{1}{2}$.*

(ii) $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is hyponormal if $\frac{1}{2} \leq p < 1$.*

Proof. (i). Since T is p-hyponormal for $p > 0$, the following (2.1) holds obviously

$$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^* \text{ for any } p > 0, \tag{2.1}$$

Let $A = U^*|T|^{2p}U$, $B = |T|^{2p}$ and $C = U|T|^{2p}U^*$ in (2.1). Then (2.1) means

$$A \geq B \geq C \geq 0. \tag{2.2}$$

As $(1+\frac{1}{2p})\frac{2}{2p+1} = \frac{1}{p} = \frac{1}{2p} + \frac{1}{2p}$ holds, we can apply Theorem F in §1, that is,

$$\begin{aligned}
(\widetilde{T}^*\widetilde{T})^{p+\frac{1}{2}} &= (|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{p+\frac{1}{2}} \\
&= (B^{\frac{1}{4p}}A^{\frac{1}{2p}}B^{\frac{1}{4p}})^{p+\frac{1}{2}} \\
&\geq (B^{\frac{1}{4p}}B^{\frac{1}{2p}}B^{\frac{1}{4p}})^{p+\frac{1}{2}} \\
&\geq (B^{\frac{1}{4p}}C^{\frac{1}{2p}}B^{\frac{1}{4p}})^{p+\frac{1}{2}} \\
&= (|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}})^{p+\frac{1}{2}} \\
&= (\widetilde{T}\widetilde{T}^*)^{p+\frac{1}{2}}
\end{aligned} \tag{2.3}$$

Hence (2.3) ensure $(\widetilde{T}^*\widetilde{T})^{p+\frac{1}{2}} \geq B^{1+\frac{1}{2p}} \geq (\widetilde{T}\widetilde{T}^*)^{p+\frac{1}{2}}$ that is, $\widetilde{T}$ is $p+\frac{1}{2}$ -hyponormal.

(ii). As $|T|^{2p} \geq |T^*|^{2p}$, we have $|T| \geq |T^*|$ by Theorem L-H since $\frac{1}{2p} \in [\frac{1}{2}, 1]$, or equivalently

$$U^*|T|U \geq |T| \geq U|T|U^* \tag{2.4}$$

Then we have

$$\widetilde{T}^*\widetilde{T} - \widetilde{T}\widetilde{T}^* = |T|^{\frac{1}{2}}(U^*|T|U - U|T|U^*)|T|^{\frac{1}{2}} \geq 0 \quad \text{by (2.4)}. \tag{2.5}$$

(2.5) implies $\widetilde{T}^*\widetilde{T} \geq \widetilde{T}\widetilde{T}^*$, that is, $\widetilde{T}$ is hyponormal. ∎

We state the following further extension of Theorem AL.

Theorem HIY. ([31, 32, 57]). *Let $T = U|T|$ be the polar decomposition of p-hyponormal for $p > 0$. Then the following assertions hold*: (i) $\widetilde{T}_{s,t} = |T|^sU|T|^t$ *is* $\dfrac{p+min\{s,t\}}{s+t}$*-hyponormal for any $s > 0$ and $t > 0$ such that $Max\{s,t\} \geq p$.*

(ii) $\widetilde{T}_{s,t} = |T|^sU|T|^t$ *is hyponormal for $s > 0$ and $t > 0$ such that $Max\{s,t\} \leq p$.*

"Aluthge transformation" $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was originally defined in Aluthge [1].

Aluthge transformation $\widetilde{T} = |T|^{\frac{1}{2}}|U|T|^{\frac{1}{2}}$ is quite an interesting and useful idea in order to reaerch operator theory. In fact, $\widetilde{T} = |T|^{\frac{1}{2}}|U|T|^{\frac{1}{2}}$ has the same spectrum as $T = U|T|$ for p-hyponormal operator $T = U|T|$ since $\sigma(ST) - \{0\} = \sigma(TS) - \{0\}$ for any operator S and T and we have to emphasize the following remarkable and surprizing fact that

$$\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \text{ belongs to the class of } (p + \frac{1}{2})\text{−hyponormal}$$

that is, $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ belongs to $(p+\frac{1}{2})$-hyponormal class which is smaller than the original class of p-hyponormal operators.

It is easily checked that the assumption of unitarity of U in Theorem AL is needless.

Further extensions of generalized Aluthge transformation have been considered and now many authors have been publishing a lot of papers which are too many to state them here.

Theorem AL was obtained in Aluthge [1] as a very nice application of Theorem F and Theorem HIY was proved in Huruya [31], Yoshino [57], Ito [32], and Furuta-Yanagida [27] under some condition.

3. Notations and Fundamental Results Associated with Log Majorization

In this section a capital letter means $n \times n$ matrix. Following Ando and Hiai [4], let us define the *log majorization* for positive semidefinite matrices $A, B \geq 0$, denoted by $A \underset{(\log)}{\succ} B$ if

$$\prod_{i=1}^{k} \lambda_i(A) \geq \prod_{i=1}^{k} \lambda_i(B), \qquad k = 1, 2, ..., n-1,$$

and

$$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B), \text{ i.e., } \det A = \det B,$$

where $\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_n(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq ... \geq \lambda_n(B)$ are the eigenvalues of A and B, respectively, arranged in decreasing order. When $0 \leq \alpha \leq 1$, the α-power mean of positive invertible matrices $A, B > 0$ is defined in Kubo-Ando [40] by

$$A \,\#_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\alpha}A^{\frac{1}{2}}.$$

Further, $A\#_\alpha B$ for $A, B \geq 0$ is defined by $A\#_\alpha B = \lim_{\epsilon \downarrow 0}(A+\epsilon I)\#_\alpha(B+\epsilon I)$.

For the sake of notational convenience, we define $A \natural_s B$, for any real number $s \geq 0$ and for $A > 0$ and $B \geq 0$, by the following

$$A \natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^s A^{\frac{1}{2}}.$$

$A \natural_\alpha B$ in the case $0 \leq \alpha \leq 1$ coincides with the usual α-power mean. The following excellent and useful log majorization was shown in Ando and Hiai [4, Theorem 2.1].

Theorem A. *For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$,*

$$(A\#_\alpha B)^r \underset{(\log)}{\succ} A^r \#_\alpha B^r \qquad \textit{for } r \geq 1. \tag{3.1}$$

(3.1) was transformed into the following matrix inequality (3.2) of Theorem B in Ando and Hiai [4,Theorem 3.5]:

Theorem B. *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^p A^{\frac{-1}{2}})^r A^{\frac{r}{2}}\}^{\frac{1}{p}} \qquad \textit{for } r, p \geq 1. \tag{3.2}$$

We obtained the following extension of Theorem A in Furuta [18, Therorem 2.1] applying the method in Ando and Hiai [4] to (GF) of Theorem GF in §1.

Theorem C. *For every $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ and for each $t \in [0, 1]$,*

$$(A\#_\alpha B)^h \underset{(\log)}{\succ} A^{1-t+r}\#_\beta\,(A^{1-t} \natural_s B)$$

holds for $s \geq 1$, and $r \geq t \geq 0$, where $\beta = \dfrac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}$ and $h = \dfrac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$.

Remark 3.1. The inequality (GF) in Theorem GF interpolates Theorem F and Theorem B, in fact, when we put $t = 1$ and $r = s$ in (GF), we have Theorem B, and when we put $t = 0$ and $s = 1$ in (GF), we have Theorem F by Remark 1.1. Also when we put $t = 1$ and $r = s$ in Theorem C which is equivalent to (GF), we have Theorem A.

Next, we state the following result which was shown in Hiai and Petz [30, Theorem 3.5] and, recently, a new proof was given in Bebiano, Lemos and Providencia [5, Theorem 2.2].

Theorem D. *If $A, B \geq 0$, then for every $p > 0$*

$$\frac{1}{p}\ \mathrm{Tr}[A\ \log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] \geq\ \mathrm{Tr}[A\,(\log A + \log B)] \tag{3.3}$$

holds and the left-hand side of (3.3) *converges to the right-hand side as $p \downarrow 0$.*

Theorem E. *If $A \geq 0$, $B > 0$, $0 \leq \alpha \leq 1$ and $p > 0$, then*

$$\frac{1}{p}\mathrm{Tr}\,[A\ \log(A^p \sharp_\alpha B^p)] + \frac{\alpha}{p}\mathrm{Tr}\,[A\ \log(A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}})] \geq \mathrm{Tr}[A \log A] \tag{3.4}$$

holds and the left-hand side of (3.4) *converges to the the right-hand side as $p \downarrow 0$.*

The inequality (3.4) was shown in Ando and Hiai [4, Theorem 5.3], and the convergence of (3.4) was shown in Bebiano, Lemos and Providencia [5, Corollary 2.2].

We shall extend Theorem D and Theorem E.

We shall show a log majorization equivalent to an order preserving operator inequality.

Theorem 3.1. ([23]). *The following* (i) *and* (ii) *hold and are equivalent*:

(i) *If $A, B \geq 0$, then for each $t \in [0, 1]$ and $r \geq t$* $A^{\frac{1}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{(p-tq)s+rq}{2ps}}\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^rB^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{q}{ps}}A^{\frac{(p-tq)s+rq}{2ps}}$ *holds for any $s \geq 1$ and $p \geq q > 0$.*

(ii) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $r \geq t$*

$$A^{\frac{(p-tq)s+rq}{ps}} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{\frac{p}{q}}A^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q}{ps}}.$$

holds for any $s \geq 1$ and $p \geq q > 0$.

Corollary 3.2. ([23]). *The following* (i) *and* (ii) *hold and are equivalent*:

(i) *If $A, B \geq 0$, then for each $r \geq 0$*

$$A^{\frac{1}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1}{2}(1+\frac{r}{p}q)}B^qA^{\frac{1}{2}(1+\frac{r}{p}q)}$$

holds for any $p \geq q > 0$.

(ii) *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$A^{1+\frac{r}{p}q} \geq (A^{\frac{r}{2}}B^{\frac{p}{q}}A^{\frac{r}{2}})^{\frac{q}{p}}$$

holds for any $p \geq q > 0$.

Corollary 3.3. ([23]). *The following* (i) *and* (ii) *hold and are equivalent*:

(i) *If* $A, B \geq 0$*, then for each* $r \geq 1$

$$A^{\frac{1}{2}}(A^{\frac{r-1}{2}}BA^{\frac{r-1}{2}})^{q}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1}{2}}\{B^{\frac{1}{2}}(B^{\frac{1}{2}}A^{r}B^{\frac{1}{2}})^{r-1}B^{\frac{1}{2}}\}^{\frac{q}{r}}A^{\frac{1}{2}}$$

holds for any $1 \geq q > 0$.

(ii) *If* $A \geq B \geq 0$ *with* $A > 0$*, then for each* $r \geq 1$

$$A \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^{\frac{1}{q}}A^{\frac{-1}{2}})^{r}A^{\frac{r}{2}}\}^{\frac{q}{r}}.$$

holds for any $1 \geq q > 0$.

4. Logarithmic Trace Inequalities as an Application of Theorem 3.1

Throughout this section, a capital letter means an $n \times n$ matrix.

For $A, B > 0$, the relative operator entropy $\hat{S}(A, B)$ is defined by

$$\hat{S}(A, B) = A^{\frac{1}{2}}\log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$$

in J.I.Fujii and Kamei [6], and $\hat{S}(A|I) = -A\log A$ is the usual operator entropy (see Nakamura-Umegaki [43]). The Umegaki relative operator entropy $S(A, B)$ is defined by

$$S(A, B) = \mathrm{Tr}[A(\log A - \log B)]$$

(see Umegaki [60]). For $A, B > 0$, let $\Delta(A|B)$ are defined by

$$\Delta(A|B) = -\mathrm{Tr}[\hat{S}(A, B)] - S(A|B).$$

We shall discuss the lower bound of $\Delta(A|B)$ in terms of the trace for A and B and a parameter, and this result implies the well-known inequality $\Delta(A|B) \geq 0$ (for example, Hiai-Petz [30], Bebiano-Lemos-Providência [5]).

Theorem 4.1. ([23]). *If* $A, B \geq 0$*, then, for every* $t \in [0, 1]$ *and* $p \geq 0$*,*

$$\begin{aligned} &\mathrm{Tr}[A\log(A^{\frac{r-t}{2}}B^{p}A^{\frac{r-t}{2}})^{s}] \\ &\geq (r-ts)\mathrm{Tr}[A\log A] + \mathrm{Tr}[A\log\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{r}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}] \end{aligned} \tag{4.1}$$

holds for any $r \geq t$ *and* $s \geq 1$.

Corollary 4.2 [23]. *If* $A, B \geq 0$*, then, for every* $p \geq 0$ *and* $r \geq 0$*,*

$$\mathrm{Tr}[A\log(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{s}] \geq \mathrm{Tr}[A\log A^{r}] + \mathrm{Tr}[A\log\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{r}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}] \tag{4.2}$$

holds for any $s \geq 1$. *In particular,*

$$\mathrm{Tr}[A \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})] \geq \mathrm{Tr}[A \log A^r + A \log B^p] \tag{4.3}$$

and

$$\mathrm{Tr}[A \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^2] \geq \mathrm{Tr}[A \log A^r] + \mathrm{Tr}[A \log(B^p A^r B^p)]. \tag{4.4}$$

The inequality (4.3) of Corollary 4.2 may be considered as the two parameter version of (3.3) in Theorem D. In fact, (4.3) of Corollary 4.2 is equivalent to (3.3) in Theorem D.

Corollary 4.3. ([23]). *If* $A, B \geq 0$*, then*

$$\mathrm{Tr}[A \log(A^{\frac{r-1}{2}} B A^{\frac{r-1}{2}})^r] \geq \mathrm{Tr}[A \log\{B^{\frac{1}{2}}(B^{\frac{1}{2}} A^r B^{\frac{1}{2}})^{r-1} B^{\frac{1}{2}}\}] \tag{4.5}$$

holds for every real number $r \geq 1$. *In particular,*

$$\mathrm{Tr}[A \log(A^{\frac{1}{2}} B A^{\frac{1}{2}})^2] \geq \mathrm{Tr}[A \log(B A^2 B)]. \tag{4.6}$$

Corollary 4.4. ([23]). *If* $A, B > 0$*, then*

$$\begin{aligned} \Delta(A|B) \geq \mathrm{Tr}[A \log B] - \frac{2(s-1)}{s} \mathrm{Tr}[A \log A] \\ + \frac{1}{s} \mathrm{Tr}[A \log\{A^{-1}(AB^{-1}A)^s A^{-1}\}] \end{aligned} \tag{4.7}$$

holds for every real number $s \geq 1$. *In particular,* $\Delta(A|B) \geq 0$ *holds.*

We remark that the right-hand side of (4.7) is zero when $s = 1$, or when A commutes with B.

5. Convergence of Logarithmic Trace Inequalities Via Generalized Lie-trotter Formulae

Throughout this section, a capital letter means an $n \times n$ matrix.

In this section, we shall state the convergence of the logarithmic trace inequalities obtained in §4 via generalized Lie-Trotter formulae shown in [23].

Theorem 5.1. [23]. *If* $A, B \geq 0$*, then, for every* $p > 0$*,*

$$\frac{s}{p} \mathrm{Tr}[A \log(A^{\frac{p}{2}} B^p A^{\frac{p}{2}})] - \frac{1}{p} \mathrm{Tr}[A \log\{B^{\frac{p}{2}}(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{s-1} B^{\frac{p}{2}}\}] \geq \mathrm{Tr}[A \log A] \tag{5.1}$$

holds for any $p > 0$ and $s \geq 1$, and the left-hand side converges to the right-hand side as $p \downarrow 0$.

Theorem 5.1 yields the following Corollary 5.2.

Corollary 5.2. ([23]). (i) *If $A, B \geq 0$, then, for every $p > 0$,*

$$\frac{1}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] \geq \mathrm{Tr}[A\log A + A\log B] \tag{5.2}$$

holds and the left-hand side converges to the right-hand side as $p \downarrow 0$.

(ii) *If $A, B \geq 0$, then, for every $p > 0$,*

$$\frac{2}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] - \frac{1}{p}\mathrm{Tr}[A\log(B^pA^pB^p)] \geq \mathrm{Tr}[A\log A] \tag{5.3}$$

holds and the left-hand side converges to the right-hand side as $p \downarrow 0$.

We remark that (i) of Corollary 5.2 is Theorem D in §3.

Theorem 5.3. ([23]). *If $A > 0$ and $B \geq 0$, then, for every positive number β,*

$$\frac{s}{p}\mathrm{Tr}[A\log(A^p\natural_\beta B^p)] - \frac{1}{p}\mathrm{Tr}[A\log\{A^{\frac{-p}{2}}(A^p\natural_\beta B^p)^sA^{\frac{-p}{2}}\}] \geq \mathrm{Tr}[A\log A] \tag{5.4}$$

holds for any $p > 0$, $s \geq 1$, and the left-hand side converges to the right-hand side as $p \downarrow 0$.

Theorem 5.3 implies the following Corollary 5.4.

Corollary 5.4. ([23]). (i) *If $A, B > 0$, then, for every positive number β,*

$$\frac{1}{p}\mathrm{Tr}[A\log(A^p\natural_\beta B^p)] + \frac{\beta}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}})] \geq \mathrm{Tr}[A\log A] \tag{5.5}$$

holds for any $p > 0$, and the left-hand side converges to the right hand-side as $p \downarrow 0$.

(ii) *If $A, B > 0$, then, for every positive number β,*

$$\begin{aligned} &\frac{2}{p}\mathrm{Tr}[A\log(A^p\natural_\beta B^p)] - \frac{1}{p}\mathrm{Tr}[A\log(A^{\frac{-p}{2}}B^pA^{\frac{-p}{2}})^\beta A^p(A^{\frac{-p}{2}}B^pA^{\frac{-p}{2}})^\beta] \\ &\geq \mathrm{Tr}[A\log A] \end{aligned} \tag{5.6}$$

holds for any $p > 0$ and the left-hand side converges to the right hand-side as $p \downarrow 0$.

We remark that, when $A \geq 0$, $B > 0$ and $\beta \in [0,1]$, (i) of Corollary 5.4 becomes Theorem E in §3.

Theorem 5.5. ([23]). *If $A > 0$ and $B \geq 0$, then for every $\alpha \in [0,1]$*

$$\frac{s}{p}\mathrm{Tr}[A\log(A^p\nabla_\alpha B^p)] - \frac{1}{p}\mathrm{Tr}[A\log\{A^{\frac{-p}{2}}(A^p\nabla_\alpha B^p)^s A^{\frac{-p}{2}}\}] \geq \mathrm{Tr}[A\log A] \tag{5.7}$$

holds for any $p > 0$, $s \geq 1$, and the-left hand side converges to the right-hand side as $p \downarrow 0$, where $S\nabla_\alpha T = (1-\alpha)S + \alpha T$ for $S, T \geq 0$.

Theorem 5.5 implies the following Corollary 5.6.

Corollary 5.6. ([23]). *If $A > 0$ and $B \geq 0$, then*

$$\begin{aligned} &\frac{1}{p}\mathrm{Tr}[A\log((1-\alpha)A^p + \alpha B^p)] \\ &-\frac{1}{p}\mathrm{Tr}[A\log\{(1-\alpha)I + A^{\frac{-p}{2}}B^pA^{\frac{-p}{2}}\}] \geq \mathrm{Tr}[A\log A] \end{aligned} \tag{5.8}$$

holds for any $p > 0$, $\alpha \in [0,1]$, and the left-hand side converges to the right hand-side as $p \downarrow 0$. Moreover,

$$\frac{s}{p}\mathrm{Tr}[A\log\frac{A^p + B^p}{2}] - \frac{1}{p}\mathrm{Tr}[A\log\{A^{\frac{-p}{2}}(\frac{A^p + B^p}{2})^s A^{\frac{-p}{2}}\}] \geq \mathrm{Tr}[A\log A] \tag{5.9}$$

holds for any $p > 0$, $s \geq 1$, and the left-hand side converges to the right hand-side as $p \downarrow 0$.

6. Operator Inequality Implying Generalized Bebiano-lemos-providência One

Throughout this section a capital letter means $n \times n$ matrix.

Let $A, B \geq 0$ and $0 \leq \alpha \leq 1$. The famous Araki-Cordes inequality states that $(A^{\frac{1}{2}}BA^{\frac{1}{2}})^\alpha \underset{(\log)}{\succ} A^{\frac{\alpha}{2}}B^\alpha A^{\frac{\alpha}{2}}$ holds and also Bebiano-Lemos-Providência inequality [5] asserts that

$$A^{\frac{1}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1+t}{2}}B^tA^{\frac{1+t}{2}} \text{ holds for } s \geq t \geq 0.$$

Very recently, Fujii-Nakamoto-Tominaga [13, Theorem 2.1 and Corollary 2.2] have shown the following interesting norm inequality:

Let $A, B \geq 0$. Then

$$||A^{\frac{1}{2}}(A^{\frac{s}{2}}B^{p+s}A^{\frac{s}{2}})A^{\frac{1}{2}}||^{\frac{p(1+s)}{p+s}} \geq ||A^{\frac{1+s}{2}}B^{1+s}A^{\frac{1+s}{2}}||$$

holds for all $p \geq 1$ *and* $s \geq 0$.

In fact, this result is essentially equivalent to the following Theorem FNT, which is essentially shown in Fujii-Nakamoto-Tominaga, as an extension of both Araki-Cordes inequality and Bebiano-Lemos-Providência one:

Theorem FNT. ([13]). *For every* $A, B \geq 0$ *and* $p \geq 1$,

$$\{A^{\frac{1}{2}}(A^{\frac{s}{2}}B^{p+s}A^{\frac{s}{2}})^{\frac{1}{p}}A^{\frac{1}{2}}\}^{\frac{p(1+s)}{p+s}} \underset{(\log)}{\succ} A^{\frac{1+s}{2}}B^{1+s}A^{\frac{1+s}{2}}$$

holds for any $s \geq 0$.

As an application of (GF) of Theorem GF, we shall give an operator inequality implying generalized Bebiano-Lemos-Providência one.

Theorem 6.1 [24]. *The following* (i) *and* (ii) *hold and they are equivalent*: (i) *For every* $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ *and each* $t \in [0, 1]$, *and any real number* $q \neq 0$,

$$\begin{aligned} &\{A^{\frac{q}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\alpha}A^{\frac{q}{2}}\}^{h} \\ \text{(6.1)}\qquad &\underset{(\log)}{\succ} A^{\frac{q(1-t+r)}{2}}\{A^{\frac{-qr}{2}}(A^{\frac{1+qt}{2}}BA^{\frac{1+qt}{2}})^{s}A^{\frac{-qr}{2}}\}^{\beta}A^{\frac{q(1-t+r)}{2}} \end{aligned}$$

holds for $s \geq 1$ *and* $r \geq t$, *where* $\beta = \dfrac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}$ *and* $h = \dfrac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$.

(ii) *If* $A \geq B \geq 0$ *with* $A > 0$, *then for* $t \in [0, 1]$ *and* $p \geq 1$,

$$\text{(6.2)}\qquad A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $s \geq 1$ *and* $r \geq t$.

Remark 6.1. (6.1) in (i) of Theorem 6.1 can be rewritten as follows: For every $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ and each $t \in [0, 1]$, and any real number $q \neq 0$,

$$\text{(6.1')}\qquad \{A^{\frac{1+q}{2}}(A^{-1}\sharp_{\alpha}B)A^{\frac{1+q}{2}}\}^{h} \underset{(\log)}{\succ} A^{\frac{1+q}{2}}\{A^{q(r-t)-1}\sharp_{\beta}(A^{-(1+qt)}\natural_{s}B)\}A^{\frac{1+q}{2}}$$

holds for $s \geq 1$ *and* $r \geq t$, *where* β *and* h *are as in* (6.1).

Remark 6.2. Put $q = -1$ and replace A by A^{-1} in (6.1'), then (i) of Theorem 6.1 yields the following result (a). Moreover, (a) implies (b) by putting $t = 1$ and $r = s$.

(a) *For every* $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ *and each* $t \in [0, 1]$

$$(A \sharp_\alpha B)^h \underset{(\log)}{\succ} A^{1-t+r} \sharp_\beta (A^{1-t} \natural_s B)$$

holds for $s \geq 1$ *and* $r \geq t$, *where* β *and* h *are as in* (6.1).

(b) *For every* $A, B \geq 0$, $0 \leq \alpha \leq 1$

$$(A \sharp_\alpha B)^r \underset{(\log)}{\succ} A^r \sharp_\alpha B^r \quad r \geq 1.$$

In fact (a) is Theorem C itself in §3 and (b) is Theorem A itself in §3, which is a very important result in log majorization.

Corollary 6.2 ([24]). *The following* (i), (ii) *and* (iii) *hold and they are equivalent*:

(i) *For every* $A, B \geq 0$, $0 \leq \alpha \leq 1$ *and any real number* $q \neq 0$,

$$\{A^{\frac{q}{2}} (A^{\frac{1}{2}} B A^{\frac{1}{2}})^\alpha A^{\frac{q}{2}}\}^{\frac{1+r}{1+\alpha r}} \underset{(\log)}{\succ} A^{\frac{q(1+r)}{2}} (A^{\frac{1-qr}{2}} B A^{\frac{1-qr}{2}})^{\frac{\alpha(1+r)}{1+\alpha r}} A^{\frac{q(1+r)}{2}}$$

holds for any $r \geq 0$.

(ii) *If* $A \geq B \geq 0$, *then for* $p \geq 1$,

$$A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

holds for any $r \geq 0$.

(iii) *For every* $A, B \geq 0$, $p \geq 1$ *and any real number* $q \neq 0$,

$$\begin{aligned} &\{A^{\frac{sq}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{sq}{2}}\}^{\frac{p(1+r)}{p+r}} \\ \underset{(\log)}{\succ}\ & A^{\frac{sq(1+r)}{2}} (A^{\frac{s(1-qr)}{2}} B^{p+s} A^{\frac{s(1-qr)}{2}})^{\frac{1+r}{p+r}} A^{\frac{sq(1+r)}{2}} \end{aligned}$$

holds for any $r \geq 0$ *and* $s \geq 0$.

Corollary 6.3 [24]. *The following* (i), (ii) *and* (iii) *hold and they are equivalent*:

(i) *For every* $A, B \geq 0$ *and* $0 \leq \alpha \leq 1$,

$$\{A^{\frac{q}{2}} (A^{\frac{1}{2}} B A^{\frac{1}{2}})^\alpha A^{\frac{q}{2}}\}^{\frac{1+q}{\alpha+q}} \underset{(\log)}{\succ} A^{\frac{1+q}{2}} B^{\frac{\alpha(1+q)}{\alpha+q}} A^{\frac{1+q}{2}}$$

holds for any $q \geq 0$.

(ii) *If $A \geq B \geq 0$, then for $p \geq 1$,*

$$A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

holds for any $r \geq 0$.

(iii) *For every $A, B \geq 0$ and $p \geq 1$,*

$$\{A^{\frac{1}{2}}(A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}}\}^{\frac{p(1+s)}{p+s}} \underset{(\log)}{\succ} A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}}$$

holds for any $s \geq 0$.

Remark 6.3. We remark that (i) of Theorem 6.1 is "*log majorization equivalent to Theorem GF in matrix case* ", and (i), (iii) of Corollary 6.2 and also (i), (iii) of Corollary 6.3 are all considered as "*log majorization equivalent to an essential part of Theorem F in matrix case* ". Needless to say, (iii) of Corollary 6.3 is Theorem FNT itself. And the equivalence between (i) and (iii) in Corollary 6.3 is essentially shown in Mujii-Nakamoto-Tominaga.

7. Decreasing Monotonicity of Order Preserving Operator Functions Associated with (GF) in Theorem GF in §1

In this section, we state the recent results on decreasing monotonicity of order preserving operator functions associated with (GF) and related satellite order preserving operator inequalities associated with (GF).

Theorem 7.1. ([25]). *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p \geq 1$. Then*

$$F(\lambda, u) = A^{\frac{-\lambda}{2}}\{A^{\frac{\lambda}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^u A^{\frac{\lambda}{2}}\}^{\frac{1-t+\lambda}{(p-t)u+\lambda}} A^{\frac{-\lambda}{2}}$$

satisfies the following properties:

(i) $F(r, w) \geq F(r, 1) \geq F(r, s) \geq F(r, s')$
holds for any $s' \geq s \geq 1$, $r \geq t$ and $\dfrac{1-t}{p-t} \leq w \leq 1$.

(ii) $F(q, s) \geq F(t, s) \geq F(r, s) \geq F(r', s)$
holds for any $r' \geq r \geq t$, $s \geq 1$ and $t-1 \leq q \leq t$.

We state several satellite inequalities of (GF) in Theorem GF as applications of Theorem 7.1.

Corollary 7.2. ([25]). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

(i)
$$(A^t \sharp_w B^p)^{\frac{1}{(p-t)w+t}} \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$$
$$\geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$

and

(ii)
$$(A^t \sharp_w B^p)^{\frac{1}{(p-t)w+t}} \geq B \geq A^{\frac{t-r}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}}A^{\frac{t-r}{2}}$$
$$\geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$

hold for $s \geq 1$, $r \geq t$ *and* $\dfrac{1-t}{p-t} \leq w \leq 1$.

Corollary 7.3. ([25]). *If* $A \geq B \geq 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geq 1$,

(i)
$$A^{t-q}\sharp_{\frac{1-t+q}{p-t+q}}B^p \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$$
$$\geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$

and

(ii)
$$A^{t-q}\sharp_{\frac{1-t+q}{p-t+q}}B^p \geq B \geq A^{\frac{t-r}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}}A^{\frac{t-r}{2}}$$
$$\geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$

hold for $s \geq 1$, $r \geq t$ *and* $t-1 \leq q \leq t$.

Very recently, the following Theorem K is shown:

Theorem K. ([36, 10]). *If* $A \geq B \geq 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geq 1$,

$$A^t\sharp_{\frac{1-t}{p-t}}B^p \geq A^{\frac{t}{2}}F(r,s)A^{\frac{t}{2}} \ holds\ for\ r \geq t\ and\ s \geq 1.$$

Since $A^{\frac{t}{2}}F(r,s)A^{\frac{t}{2}} = A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$ holds, (i) or (ii) of Corollary 7.2 implies Theorem K. Also (i) or (ii) of Corollary 7.3 implies Theorem K.

Corollary 7.2 and Corollary 7.3 easily imply the following known satellite inequalities in [22, § 3.2.5, Corollary 2],

If $A \geq B \geq 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geq 1$,

(i) $\{B^{\frac{t}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sB^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}} \geq A \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$

and

(ii) $B^{\frac{t-r}{2}}(B^{\frac{r-t}{2}}A^pB^{\frac{r-t}{2}})^{\frac{1-t+r}{p-t+r}}B^{\frac{t-r}{2}} \geq A \geq B \geq A^{\frac{t-r}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{1-t+r}{p-t+r}}A^{\frac{t-r}{2}}$

hold for $s \geq 1$, $r \geq t$ *and* $t \in [0,1]$.

We list statements (7.1)-(7.4) in the following Remark 7.1 as a concluding remark.

Remark 7.1. Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p \geq 1$. Then the following properties hold.

$$F(r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}} \tag{7.1}$$

is a decreasing function of r and s such that $r \geq t$ and $s \geq 1$.

$$F(r,w) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^wA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)w+r}}A^{\frac{-r}{2}} \tag{7.2}$$

is not a decreasing function of r and w such that $r \geq t$ and $\dfrac{1-t}{p-t} \leq w \leq 1$, but

$$F(r,w) \geq F(r,1)$$

holds for any $r \geq t$ and $\dfrac{1-t}{p-t} \leq w \leq 1$.

$$F(q,s) = A^{\frac{-q}{2}}\{A^{\frac{q}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{q}{2}}\}^{\frac{1-t+q}{(p-t)s+q}}A^{\frac{-q}{2}} \tag{7.3}$$

is not a decreasing function of q and s such that $0 \leq q \leq t$ and $s \geq 1$, but

$$F(q,s) \geq F(t,s)$$

holds for any $0 \leq q \leq t$ and $s \geq 1$.

$$F(q,s) = A^{\frac{-q}{2}}\{A^{\frac{q}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{q}{2}}\}^{\frac{1-t+q}{(p-t)s+q}}A^{\frac{-q}{2}} \tag{7.4}$$

is not a decreasing function of q and is not an increasing of s such that $t-1 \leq q \leq 0$ and $s \geq 1$, but

$$F(q,s) \geq F(t,s)$$

holds for any $t-1 \leq q \leq 0$ and $s \geq 1$. (7.1) is nothing but the former half of Theorem GF in §1.

(See that the coordinate $(r,s) \in \mathcal{D}_{rs}$ in Figure 2).

The latter half of (7.2), that is, $F(r,w) \geq F(r,1)$ holds for $r \geq t$ and $\dfrac{1-t}{p-t} \leq w \leq 1$ is shown by (i) of Theorem 7.1.

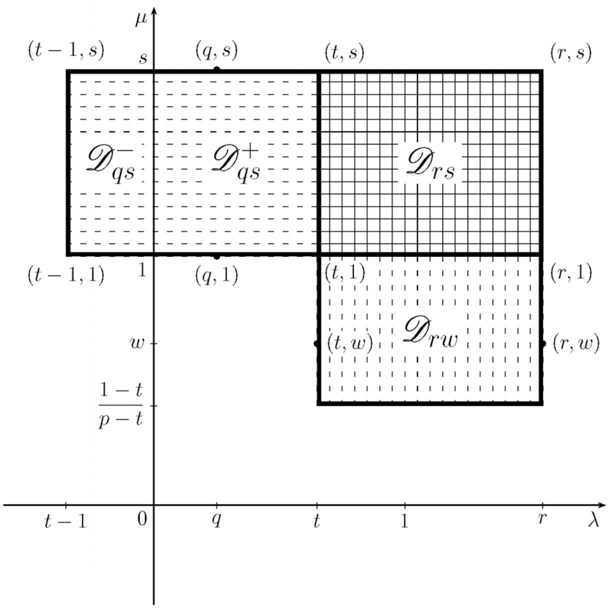
(See that the coordinate $(r,w) \in \mathcal{D}_{rw}$ in Figure 2).

The latter half of (7.3), that is, $F(q,s) \geq F(t,s)$ holds for $0 \leq q \leq t$ and $s \geq 1$ is shown by (ii) of Theorem 7.1.

(See that the coordinate $(q,s) \in \mathcal{D}^+_{qs}$ in Figure 2).

The latter half of (7.4), that is, $F(q,s) \geq F(t,s)$ holds for $t-1 \leq q \leq 0$ and $s \geq 1$ is shown by (ii) of Theorem 7.1.

(See that the coordinate $(q,s) \in \mathcal{D}^-_{qs}$ in Figure 2).



$$t \in [0,1],\ p \geq 1,\ r \geq t,\ s \geq 1,$$
$$\frac{1-t}{p-t} \leq w \leq 1 \text{ and } t-1 \leq q \leq t$$

Fig. 2.

Acknowledgment

We would like to express our cordial thanks to two referees for their useful comments and kind suggestions to improve the first version and the revised one.

We remark that the number at the end of each reference below indicates its subject listed in §1, for example,

[1] A. Aluthge, *p-hyponormal operators for* $0 < p < 1$, Integral Equation Operator Theory, **13** (1990), 307-315. (A-4)

That is, this paper [1] is closely related to (A-4) at **(A) OPERATOR INEQUALITIES** stated in §1.

References

1. A. Aluthge, p-hyponormal operators for $0 < p < 1$, *Integral Equation Operator Theory*, **13** (1990), 307-315. (A-4)
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