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OPTIMAL ADJUSTMENT OF COMPETENCE SET WITH LINEAR PROGRAMMING

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Abstract. Management by objectives (MBO) is an effective way for enterprise management. By setting the targets of the productivity the companies try their best, including adjustment of resource allocation and competence, to reach the targets. Within the same framework of productivity and of resources the targets may not be attainable. However, by stretching a little bit, human capacity, resources, the production coefficients, and other relevant parameters may be adjusted so as to make the target feasible. In this article, we formulate the program into linear programming model and study how to optimally adjust the relevant coefficients so that the target solution could be attainable. In case the target is unattainable, we may either utilize the bisection method or the fuzzy linear programming techniques to revise the target as to make it a reachable one.

1. Introduction

Management by objectives (MBO) is an efficient and effective managerial system. Goal setting is the first crucial step in the system of MBO. At this step the participants identify the targets to be achieved. The company then mobilizes all resources and competence, including their reallocation, to reach the targets, or to move toward the targets as close as possible. Therefore, achieving the targets becomes one of the most important criterion in the system of MBO. In order to achieve the targets some relevant parameters, such as the constraint coefficients and the right hand sided resource level in linear programming (LP) problems, need to be adjusted and/or expanded.

One of the well-known researches on the adjustment of parameters is the *inverse LP optimization*. In the class of inverse LP problems, the parameters of the objective

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function with the minimum deviation from the original ones are sought so that a given feasible solution x^0 becomes an optimal one [7]. Zhang and Liu [13] studied inverse assignment and minimum cost flow problems under L_1-norm based on optimality conditions for LP problems. Zhang and Liu [14] further took L_∞-norm into account and investigated inverse 0-1 programming and network programming problems. Ahuja and Orlin [1] considered more general inverse LP problems under both L_1- and L_∞-norms. In addition, Troutt et al. [8] investigated a so-called *linear programming system identification problem* in which both objective function coefficients and constraint matrix are evaluated to best fit a set of historical decisions and its corresponding used resources.

A *competence set* is a collection of ideas, knowledge, information, resources, and skills for satisfactorily solving a given decision problem [10, 11]. By using mathematical programming, a number of researchers have focused on searching for the optimal expansion process from an already acquired competence set to a needed one [4, 6, 12]. Feng and Yu [3] proposed a minimum spanning table algorithm to find the optimal competence set expansion process without formulating the related mathematical program. However, the competence set so far has been assumed to be discrete and finite so as to represent its elements by nodes of a graph. This makes the applications of the competence set expansion in these studies somehow limited, because the number of feasible solutions of a linear system might not be discrete and finite.

In this article, we focus on linear systems. While the literature on inverse LP optimization treats only a feasible target, we intend to determine the optimal adjustment of constraint coefficients in a linear system so that a given target, originally unattainable, can be achieved. Given a target solution, we set up a *competence set adjustment model* (CSA model) to study the optimal adjustment of the related competence sets. The model will enable us to find the optimal adjustment whenever the target is reachable.

In case the target is unattainable, we utilize the bisection method or the fuzzy linear programming techniques to help the DM revise the target as to make it an achievable one. The former is to find a solution which is as close as possible to the target and the latter is to interactively select an achievable target. Then the optimal adjustment could be derived from the aforementioned CSA model with the revised target.

The rest of the paper is structured as follows. Section 2 states our problem more formally. Section 3 proposes the CSA model for finding the optimal adjustment so that x^0 becomes a feasible solution. Section 4 applies the bisection algorithm to find a best compromise between the target and the best solution before the adjustment if the target is incapable of being attained. Section 5 utilizes fuzzy linear programming technique to help the DM select an achievable target. Some examples and applications are also provided. Finally, Section 6 contains some concluding remarks.

2. Problem Statement

Consider a standard LP problem as follows.

$$\begin{aligned} \max \quad z(x) &= cx \\ \text{s.t.} \quad Ax &\le b, \\ x &\ge 0, \end{aligned} \tag{1}$$

where $c = [c_i]$ is the $1 \times n$ objective coefficient vector, $x = [x_j]$ denotes the $n \times 1$ decision vector, $A = [a_{ij}]$ is the $m \times n$ consumption (or productivity) matrix, and $b = [b_i]$ is the $m \times 1$ resource availability vector.

Suppose that x^0 is a target solution set by decision maker (DM). Let D be a parameter matrix whose element, δ_{ij}, denotes the deviation from a_{ij}, and γ be a parameter vector whose component, γ_i, denotes the deviation from b_i. By changing D and γ, we tried to construct $X^0(D, \gamma)$, where $X^0(D, \gamma) = \{x|(A+D)x \le b+\gamma\}$. Since $a_{ij} = 0$ implies that the resource i has no impact on the product j. Thus, a_{ij} is not subject to adjustment. Consequently, we have $\delta_{ij} = 0$ if $a_{ij} = 0$.

Definition 2.1. Given a target x^0, a feasible adjustment is a pair (D, γ) such that $(A + D)x^0 \le b + \gamma$. Thus, $x^0 \in X^0(D, \gamma)$.

Let $\Psi = \{(D, \gamma)|x^0 \in X^0(D, \gamma)\}$ be the set of all feasible adjustments, and $\Phi = \{(i, j)|a_{ij} \ne 0\}$,

Definition 2.2. Given a target solution x^0, and $(D^0, \gamma^0) \in \Psi$, define the relative adjustment measure of (D^0, γ^0) by

$$\Re(D^0, \gamma^0) = \sum_{(i,j)\in\Phi} r_{ij}(D^0) + \sum_{i=1}^{m} s_i(\gamma^0),$$

where

$$r_{ij}(D^0) = |\delta^0_{ij}|/|a_{ij}|, a_{ij} \ne 0,$$

and

$$s_i(\gamma^0) = |\gamma^0_i|/h_i,$$

where

$$h_i = \begin{cases} |b_i|, & \text{if } b_i \ne 0, \\ |M_i|, & \text{if } b_i = 0. \end{cases}$$

Note that $r_{ij}(D^0)$, $a_{ij} \neq 0$, is a relative adjustment measure with respect to the parameter a_{ij}, while $s_i(\gamma^0)$ is that with respect to b_i. Note, when $b_i = 0$ $|\gamma_i^0|/|b_i|$ is not defined. The positive number M_i needs to be chosen properly to reflect the impact of the adjustment on b_i.

Remark 2.1. When needed, $\Re(D^0, \gamma^0)$, r_{ij} and s_i can be changed into other forms of cost functions to fit the cost of adjustment.

Definition 2.3. A feasible adjustment alternative (D^*, γ^*) is optimal if (D^*, γ^*) minimizes the relative adjustment measure over Ψ. That is,

$$\Re(D^*, \gamma^*) = \min\{\Re(D, \gamma) | (D, \gamma) \in \Psi\}.$$

The adjustment deviation measure as defined in Definition 2.2 is not a linear form because of ”absolute value”. To eliminate the sign of the absolute value in Definition 2.2, the following Lemma 2.1 is useful.

Lemma 2.1. *Given* $D = [\delta_{ij}]$ *and* $\gamma = [\gamma_i]$, *let* $\bar{a}_{ij} = a_{ij} + \delta_{ij}$ *and* $\bar{b}_i = b_i + \gamma_i$. *Define* $D^+ = [\delta_{ij}^+]$, $D^- = [\delta_{ij}^-]$, $\gamma^+ = (\gamma_1^+, \ldots, \gamma_m^+)$, *and* $\gamma^- = (\gamma_1^-, \ldots, \gamma_m^-)$ *with*

$$\delta_{ij}^+ = \begin{cases} \bar{a}_{ij} - a_{ij}, & \textit{if } \bar{a}_{ij} > a_{ij}, \\ 0, & \textit{otherwise}; \end{cases} \tag{2}$$

$$\delta_{ij}^- = \begin{cases} a_{ij} - \bar{a}_{ij}, & \textit{if } a_{ij} > \bar{a}_{ij}, \\ 0, & \textit{otherwise}; \end{cases} \tag{3}$$

$$\gamma_i^+ = \begin{cases} \bar{b}_i - b_i, & \textit{if } \bar{b}_i > b_i, \\ 0, & \textit{otherwise}; \end{cases} \tag{4}$$

$$\gamma_i^- = \begin{cases} b_i - \bar{b}_i, & \textit{if } b_i > \bar{b}_i, \\ 0, & \textit{otherwise}. \end{cases} \tag{5}$$

Then:

(*i*) $\delta_{ij} = \delta_{ij}^+ - \delta_{ij}^-$ *and* $\gamma_i = \gamma_i^+ - \gamma_i^-$, *or* $D = D^+ - D^-$ *and* $\gamma = \gamma^+ - \gamma^-$.

(*ii*) $|\delta_{ij}| = \delta_{ij}^+ + \delta_{ij}^-$ *and* $|\gamma_i| = \gamma_i^+ + \gamma_i^-$.

(*iii*) δ_{ij}^+, δ_{ij}^-, γ_i^+, $\gamma_i^- \geq 0$.

Proof.

(i) Since $\bar{a}_{ij} = a_{ij} + \delta_{ij}$, we have $\delta_{ij} = \bar{a}_{ij} - a_{ij}$. We may replace (3) by (6) as follows.

$$(6) \qquad \delta_{ij}^{-} = \begin{cases} 0, & \text{if } \bar{a}_{ij} > a_{ij}, \\ a_{ij} - \bar{a}_{ij}, & \text{otherwise.} \end{cases}$$

By subtracting (6) from (2) on both sides, we have

$$(7) \qquad \delta_{ij}^{+} - \delta_{ij}^{-} = \begin{cases} \bar{a}_{ij} - a_{ij}, & \text{if } \bar{a}_{ij} > a_{ij}, \\ \bar{a}_{ij} - a_{ij}, & \text{otherwise.} \end{cases}$$

By (7), we have $\delta_{ij} = \delta_{ij}^{+} - \delta_{ij}^{-}$. That $\gamma_i = \gamma_i^{+} - \gamma_i^{-}$ could be proved in a similar way.

(ii) By definition,

$$(8) \qquad |\delta_{ij}| = \begin{cases} \delta_{ij}, & \text{if } \delta_{ij} > 0, \\ -\delta_{ij}, & \text{otherwise.} \end{cases}$$

We may rewrite (8) as follows.

$$(9) \qquad |\delta_{ij}| = \begin{cases} \bar{a}_{ij} - a_{ij}, & \text{if } \bar{a}_{ij} > a_{ij}, \\ a_{ij} - \bar{a}_{ij}, & \text{otherwise.} \end{cases}$$

Observe that (9) could be obtained by adding (6) to (2). Thus, $|\delta_{ij}| = \delta_{ij}^{+} + \delta_{ij}^{-}$. That $|\gamma_i| = \gamma_i^{+} + \gamma_i^{-}$ can be proved similarly.

(iii) It is obviously from (2)-(5). ∎

Note that δ_{ij}^{+} is the value of $\bar{a}_{ij}$ exceeding a_{ij} and δ_{ij}^{-} is that of $\bar{a}_{ij}$ below a_{ij}, while γ_i^{+} is the value of $\bar{b}_i$ exceeding b_i and γ_i^{-} is that of $\bar{b}_i$ below b_i.

3. Optimal Adjustment of Competence Set

Given a target x^0, we try to identify the optimal adjustment alternative (D^*, γ^*) by minimizing the relative adjustment measure $\Re(D^*, \gamma^*)$ over Ψ. By using the aforementioned definitions and lemma, the optimal adjustment of competence set for reaching x^0 can be formulated as the following *competence set adjustment model* (CSA model).

Program 3.1.

$$
(10)\qquad \begin{aligned} z^0 = \min \quad & \sum_{(i,j)\in\Phi} \{(\delta_{ij}^+ + \delta_{ij}^-)/|a_{ij}|\} + \sum_{i=1}^{m} \{(\gamma_i^+ + \gamma_i^-)/h_i\} \\ \text{s.t.} \quad & \sum_{(i,j)\in\Phi} (a_{ij} + \delta_{ij}^+ - \delta_{ij}^-)x_j^0 \le b_i + \gamma_i^+ - \gamma_i^-, i = 1, 2, \ldots, m, \\ & \delta_{ij}^+ \ge 0, \delta_{ij}^- \ge 0, \gamma_i^+ \ge 0, \gamma_i^- \ge 0. \end{aligned}
$$

Note that when $z^0 = 0$, there is no need for adjustment. That is, the original system can produce the target solution x^0.

Lemma 3.1. *The optimal solution* $(D^{+*}, D^{-*}, \gamma^{+*}, \gamma^{-*})$ *to Program 3.1 has the property that* $\delta_{ij}^{+*} \cdot \delta_{ij}^{-*} = 0$, $\gamma_i^{+*} \cdot \gamma_i^{-*} = 0$, *for all* i, j.

Proof.

(i) if $\delta_{ij}^{+*} > \delta_{ij}^{-*} > 0$, set $\delta_{ij}^{+0} = \delta_{ij}^{+*} - \delta_{ij}^{-*}$ and $\delta_{ij}^{-0} = 0$;

(ii) if $\delta_{ij}^{-*} > \delta_{ij}^{+*} > 0$, set $\delta_{ij}^{-0} = \delta_{ij}^{-*} - \delta_{ij}^{+*}$ and $\delta_{ij}^{+0} = 0$;

(iii) if $\gamma_i^{+*} > \gamma_i^{-*} > 0$, set $\gamma_i^{+0} = \gamma_i^{+*} - \gamma_i^{-*}$ and $\gamma_i^{-0} = 0$;

(iv) if $\gamma_i^{-*} > \gamma_i^{+*} > 0$, set $\gamma_i^{-0} = \gamma_i^{-*} - \gamma_i^{+*}$ and $\gamma_i^{+0} = 0$.

Then, $(D^{+0}, D^{-0},\gamma^{+0},\gamma^{-0})$ is a better solution which leads to a contradiction. ∎

In order to make Program 3.1 more effective in computing, we derive the following proposition.

Proposition 3.1. *Given a target solution* x^0*, the optimal solution* $(D^{+*}, D^{-*}, \gamma^{+*}, \gamma^{-*})$ *to Program 3.1 has the property that* $D^{+*} = 0$*,and* $\gamma^{-*} = 0$.

Proof. Given a target solution x^0, consider two possible cases.

Case 1. $\sum_{j=1}^{n} a_{ij}x_j^0 \le b_i, \forall i \in \{1, 2, \ldots, m\}$. Then $\delta_{ij}^{+*} = \delta_{ij}^{-*} = \gamma_i^{+*} = \gamma_i^{-*} = 0$, for all i, j, is the optimal solution. The property obviously holds.

Case 2. $\exists i \in \{1, 2, \ldots, m\}$ such that

$$
(11)\qquad \sum_{j=1}^{n} a_{ij}x_j^0 > b_i.
$$

(i) Assume $\delta_{ij}^{+*} > 0$. By Lemma , we have $\delta_{ij}^{-*} = 0$. Thus, from (11), we have $\sum_{j=1}^{n} (a_{ij} + \delta_{ij}^{+*})x_j^0 > b_i$. In order to satisfy (10), $\gamma_i^{+*} > 0$, and $\gamma_i^{-*} = 0$ (by Lemma). Indeed,

$$\gamma_i^{+*} = \sum_{j=1}^{n} (a_{ij} + \delta_{ij}^{+*})x_j^0 - b_i. \tag{12}$$

Choose $\delta_{ij}^{+0} = \delta_{ij}^{-0} = 0$,

$$\gamma_i^{+0} = \sum_{j=1}^{n} a_{ij}x_j^0 - b_i, \tag{13}$$

and $\gamma_i^{-0} = 0$. Note, (12)-(13) and $\delta_{ij}^{+*} > 0$ implies that $\gamma_i^{+*} > \gamma_i^{+0}$. Thus, $(D^{+0}, D^{-0}, \gamma^{+0}, \gamma^{-0})$ is a better solution than $(D^{+*}, D^{-*}, \gamma^{+*}, \gamma^{-*})$, which leads to a contradiction.

(ii) Assume $\gamma_i^{-*} > 0$. Then, by Lemma , $\gamma_i^{+*} = 0$. From 1, $\delta_{ij}^{+*} = 0$. Thus,

$$\sum_{j=1}^{n} (a_{ij} - \delta_{ij}^{-*})x_j^0 \leq b_i - \gamma_i^{-*}.$$

Choose $\delta_{ij}^{-0} = \delta_{ij}^{-*}$, $\delta_{ij}^{+0} = \gamma_i^{+0} = \gamma_i^{-0} = 0$. Then $(D^{+0}, D^{-0}, \gamma^{+0}, \gamma^{-0})$ is a feasible solution better than $(D^{+*}, D^{-*}, \gamma^{+*}, \gamma^{-*})$, which leads to a contradiction. ∎

According to Proposition 3.1 we could reduce the number of adjustment variables and obtain the following simplified CSA model.

Program 3.2.

$$\begin{aligned} \min \quad & \sum_{(i,j)\in\Phi} \{\delta_{ij}^{-}/|a_{ij}|\} + \sum_{i=1}^{m} \{\gamma_i^{+}/h_i\} \\ \text{s.t.} \quad & \sum_{(i,j)\in\Phi} (a_{ij} - \delta_{ij}^{-})x_j^0 \leq b_i + \gamma_i^{+}, i = 1, 2, \ldots, m, \\ & \delta_{ij}^{-} \geq 0, \gamma_i^{+} \geq 0. \end{aligned} \tag{14}$$

The model we have described considered no adjustment bounds and no costs incurred by adjusting the constraint coefficients. Practically, the degrees of adjustment may be bounded in a certain range as follows.

(15) $$\delta_{ij}^{-} \leq l_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n,$$

(16) $$\gamma_i^{+} \leq u_i, i = 1, 2, \ldots, m,$$

where l_{ij} and u_i denote the upper bounds for adjusting a_{ij} and b_i respectively. In addition, the budget constraint could be written as follows.

(17) $$\sum_{i=1}^{m} \left[\left(\sum_{j=1}^{n} w_{ij}\delta_{ij}^{-} \right) + p_i\gamma_i^{+} \right] \leq G,$$

where the cost for adjusting a_{ij} and b_i is denoted respectively by w_{ij} and p_i, and G denotes the available budget for adjustment.

By combing (14)-(17), we have a more practical and general CSA model as follows.

Program3.3.

(18) $$\begin{aligned} \min \quad & \sum_{(i,j)\in\Phi} \{\delta_{ij}^{-}/|a_{ij}|\} + \sum_{i=1}^{m} \{\gamma_i^{+}/h_i\} \\ \text{s.t.} \quad & \sum_{(i,j)\in\Phi} (a_{ij} - \delta_{ij}^{-})x_j^0 \leq b_i + \gamma_i^{+}, i = 1, 2, \ldots, m, \\ & \delta_{ij}^{-} \leq l_{ij}, (i,j) \in \Phi, \\ & \gamma_i^{+} \leq u_i, i = 1, 2, \ldots, m, \\ & \sum_{i=1}^{m} \left[\left(\sum_{j=1}^{n} w_{ij}\delta_{ij}^{-} \right) + p_i\gamma_i^{+} \right] \leq G, \\ & \delta_{ij}^{-} \geq 0, \gamma_i^{+} \geq 0. \end{aligned}$$

Example 3.1. Consider the following LP problem.

$$\begin{aligned} \max \quad & 90x_1 + 70x_2 \\ \text{s.t.} \quad & 4x_1 + 2x_2 \leq 200, \\ & 2x_1 + 3x_2 \leq 240, \\ & x_1 \geq 0, x_2 \geq 0, \end{aligned}$$

where the optimal solution $x^* = (15, 70)$. Suppose the target solution $x^0 = (20, 90)$ and the available budget for adjustment $G = 2,000$ are set by decision maker. $L = \begin{bmatrix} 0.8 & 0.8 \\ 0.6 & 0.75 \end{bmatrix}$ denotes the maximum deviation of adjusting a_{ij}, $W = \begin{bmatrix} 120 & 90 \\ 80 & 100 \end{bmatrix}$ denotes the unit price for adjusting a_{ij}, and $p = (75, 85)$ denotes the unit price for purchasing extra resources.

According to Program 3.3, the optimal adjustment problem can be formulated as the following mathematical programming.

$$\begin{aligned} \min \quad & (\delta_{11}^{-}/4) + (\delta_{12}^{-}/2) + (\delta_{21}^{-}/2) + (\delta_{22}^{-}/3) + (\gamma_1^{+}/200) + (\gamma_2^{+}/240) \\ \text{s.t.} \quad & 20\delta_{11}^{-} + 90\delta_{12}^{-} + \gamma_1^{+} \geq 60, \\ & 20\delta_{21}^{-} + 90\delta_{22}^{-} + \gamma_2^{+} \geq 70, \\ & \delta_{11}^{-} \leq 0.8, \\ & \delta_{12}^{-} \leq 0.8, \\ & \delta_{21}^{-} \leq 0.6, \\ & \delta_{22}^{-} \leq 0.75, \\ & 120\delta_{11}^{-} + 90\delta_{12}^{-} + 80\delta_{21}^{-} + 100\delta_{22}^{-} + 75\gamma_1^{+} + 85\gamma_2^{+} \leq 2000, \\ & \delta_{11}^{-} \geq 0, \delta_{12}^{-} \geq 0, \delta_{21}^{-} \geq 0, \delta_{22}^{-} \geq 0, \gamma_1^{+} \geq 0, \gamma_2^{+} \geq 0. \end{aligned}$$

By using LINGO software, an optimal solution (adjustment) is obtained as follows.

$$D^{-*} = \begin{bmatrix} 0 & 1115/2664 \\ 0 & 3/4 \end{bmatrix}, \gamma^{+*} = (3305/148, 5/2).$$

The total ratio of changes is 0.581344 and the total cost for adjustment is $2,000$. The adjusted LP problem is presented as follows.

$$\begin{aligned} \max \quad & 90x_1 + 70x_2 \\ \text{s.t.} \quad & 4x_1 + 1.58x_2 \leq 222.33, \\ & 2x_1 + 2.25x_2 \leq 242.5, \\ & x_1 \geq 0, x_2 \geq 0, \end{aligned}$$

where the optimal solution $x^* = (20, 90)$. Note that the numerical values of the parameters are rounded off.

The graphical representation of the optimal adjustment of competence set is depicted in Figure 1 as follows.

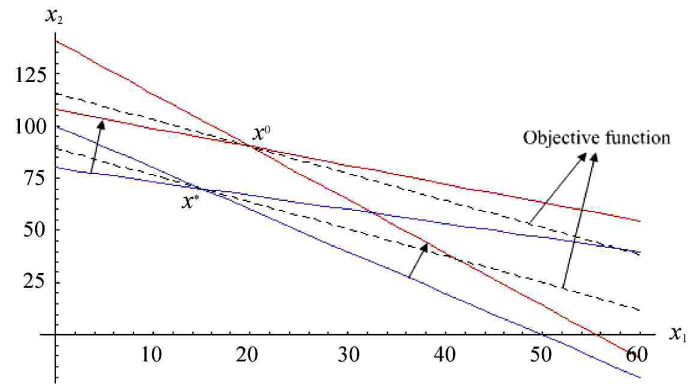

Fig. 1. Graphical representation of the optimal adjustment of competence set in Example 3.1.

4. Bisection Algorithm

We have so far described how CSA models could be used to obtain the optimal adjustment so that x^0 becomes reachable. However, after having solved the CSA model, we may have no solution because of the limitation in the budget level and the bounds of adjustments. This section applies the *bisection algorithm* to find a revised target solution which approximates the original one and obtains its corresponding optimal adjustment.

Utilizing bisection algorithm is motivated in part by the behavior mechanism [9, 10, 11] in *Habitual Domains Theory* (HDs) which characterized two modes of behavior: *active problem solving* or *avoidance justification*. The former attempts to work actively to move the perceived states closer to the ideal states; while the latter tries to rationalize the situations so as to lower the ideal states closer to the perceived states.

Figure 2 graphically illustrates the bisection method. When x^0 is unlikely to be reached, (refer to Figure 2) we bisect the interval $[x^L, x^R]$ and try to obtain the optimal adjustment of competence set with $x^M(1)$ as the target. If $x^M(1)$ is still impossible to be reached, then we bisect the interval $[x^L, x^M(1)]$ and try to obtain the optimal adjustment of competence set with $x^M(3)$ as the target. Otherwise, we bisect the interval $[x^M(1), x^R]$ and try to obtain the optimal adjustment of competence set with $x^M(2)$ as the target. The above procedures continue until the sequence $\{x^M(n)\}$ converges within certain bound.

Based on the two modes of behavior in HDs, we operate the bisection algorithm as Algorithm 1 to find a revised target solution and its corresponding optimal adjustment.

Let x^* be the optimal solution in the original system, and $z(x^*)$ be the objective value at x^* (with respect to original objective function in (1)),

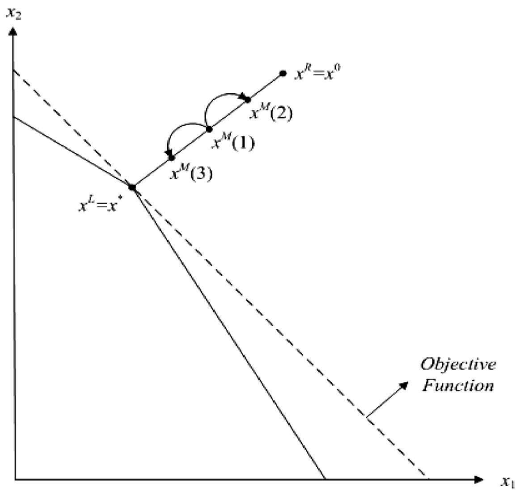

Fig. 2. Graphical representation of the bisection method.

Algorithm 4.1.

Step 1. Set $x^L = x^*$, where x^L denotes the left end point, $x^R = x^0$, where x^R denotes the right end point, and $x^M = x^R$, where x^M denotes the middle point of the interval $[x^L, x^R]$.

Step 2. Choose an $\epsilon > 0$, where ϵ denotes the tolerant discrepancy between $z(x^M)$ and $z(x^L)$.

Step 3. Solve (18) with x^M as the target to obtain D^- and γ^+.

Step 4. If (18) has no solution, set $x^R = x^M$ and go to Step 6; otherwise, go to Step 5.

Step 5. If the deviation of the objective value between x^M and x^L is smaller than ϵ, that is $|z(x^M) - z(x^L)| < \epsilon$, stop and x^M is the desired critical target; otherwise, set $x^L = x^M$ and go to Step 6.

Step 6. Set $x^M = (x^L + x^R)/2$ and go back to Step 3.

The flow chart of the Algorithm 1 is depicted in Figure 3.

Theorem 4.1. *Let $\{x^M(n)\}$ be the sequence of middle points generated by the above algorithm. The sequence converges to a critical point x^c.*

Proof. The proof of this theorem is similar to that of the bisection method. For details, see Mathews and Fink [5].

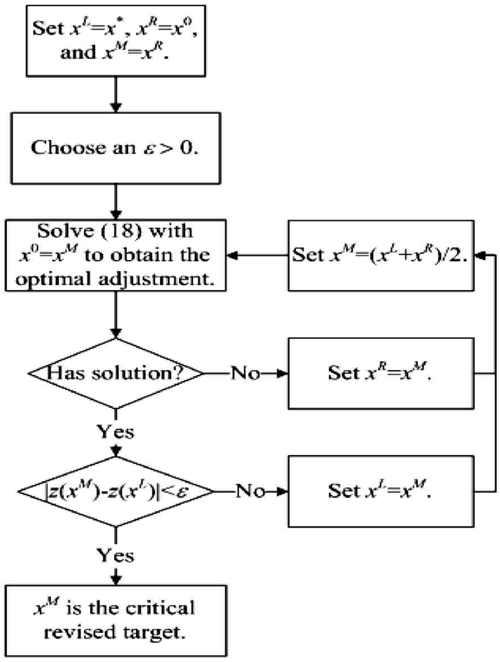

Fig. 3. Flow chart of the Algorithm 4.1.

Remark 4.1. Let $\Delta x = x^0 - x^c$. We may interpret Δx as the quantity of products that needs outsourcing if the target must be reached.

Example 4.1. (Continue Example 3.1.) When the decision maker sets the target to $x^0 = (70, 100)$, we may obtain no solution by applying the CSA model. In this case, we utilize the bisection algorithm. By choosing $\epsilon = 10$, we obtain the sequence of middle points, $x^M(n)$, as follows:

$$x^M(n) = (42.5, 85), (28.75, 77.5), (35.63, 81.25), (39.06, 83.13), (37.34, 82.19),$$
$$(38.2, 82.66), (38.63, 82.89), (38.42, 82.77), (38.53, 82.83), (38.47, 82.8).$$

The revised target solution is $(38.47, 82.80)$. The optimal adjustment corresponding to the revised target can be derived from the CSA model as follows.

$$D^- = \begin{bmatrix} 0.8 & 0.8 \\ 0.6 & 0.75 \end{bmatrix}, \gamma^+ = (22.47, 0.28).$$

The total ratio of changes is 1.26206 and the total cost for adjustment is $2,000$. The adjusted LP problem is listed as follows.

$$\begin{aligned} \max \quad & 90x_1 + 70x_2 \\ \text{s.t.} \quad & 3.2x_1 + 1.2x_2 \le 222.47, \\ & 1.4x_1 + 2.25x_2 \le 240.28, \\ & x_1 \ge 0, x_2 \ge 0, \end{aligned}$$

where the optimal solution $x^* = (38.47, 82.80)$. Note that the numerical values of the parameters are rounded off.

5. Target Revision by the Fuzzy Linear Programming

The use of the *fuzzy linear programming* (FLP) technique in this study is motivated in part by the nature of the optimal adjustment of competence set problems that some constraint coefficients may be adjusted within some tolerant ranges. We may treat these coefficients with the fuzzy sets and then formulate a FLP model with a crisp objective function. In turn, the revised target could be derived by solving the FLP model.

While the bisection method finds a revised target solution which approximates the original optimal solution (a status quo), the FLP techniques allow the decision maker (DM) to interactively select an achievable target. This section demonstrates how the FLP can help the DM to interactively revise the unattainable targets as to get the final target.

FLP problems [2] with a crisp objective function could be represented as follows.

$$\text{(19)} \qquad \begin{aligned} \max \quad & z(x) = cx \\ \text{s.t.} \quad & \tilde{A}x \le \tilde{b}, \\ & x \ge 0, \end{aligned}$$

where $\tilde{A}$ is the $m \times n$ consumption (or productivity) matrix whose elements $\tilde{a}_{ij}$ are fuzzy sets with membership function $\mu_{\tilde{a}_{ij}}$, and $\tilde{b}$ is the $m \times 1$ resource availability vector whose components $\tilde{b}_i$ are fuzzy sets with membership function $\mu_{\tilde{b}_i}$. After having defined the appropriate membership function and α parameter, we could transform (19) into (20) as follows.

$$\text{(20)} \qquad \begin{aligned} \max \quad & z(x) = \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \mu_{\tilde{a}_{ij}}^{-1}(\alpha) x_j \le \mu_{\tilde{b}_i}^{-1}(\alpha), \forall i = 1, 2, \ldots, m, \\ & x_j \ge 0, \forall j = 1, 2, \ldots, n. \end{aligned}$$

In order to apply (20) to solve FLP problems, membership functions have to be defined for the fuzzy sets of the constraint coefficients first. Assume that $a_{ij} \in \left[a_{ij}^0, a_{ij}^0 + d_{ij}\right]$ (interval from a_{ij}^0 to $a_{ij}^0 + d_{ij}$), and $b_i \in \left[b_i^0 - h_i, b_i^0\right]$. Note that d_{ij} and h_i are the maximum tolerable deviation from a_{ij}^0 and b_i^0 respectively. Given $\alpha \in [0, 1]$, let $a_{ij}(\alpha) = a_{ij}^0 + (1-\alpha)d_{ij}$, and $b_i(\alpha) = b_i^0 - (1-\alpha)h_i$. To illustrate the method, assume that the membership functions of the fuzzy sets $\tilde{a}_{ij}$ and $\tilde{b}_i$ are linear as follows.

$$\mu_{\tilde{a}_{ij}}\left(a_{ij}(\alpha)\right) = \begin{cases} 1, & \text{if } a_{ij}(\alpha) < a_{ij}^0, \\ \alpha, & \text{if } a_{ij}^0 \le a_{ij}(\alpha) \le a_{ij}^0 + d_{ij}, \\ 0, & \text{if } a_{ij}(\alpha) > a_{ij}^0 + d_{ij}. \end{cases}$$

$$\mu_{\tilde{b}_i}\left(b_i(\alpha)\right) = \begin{cases} 1, & \text{if } b_i(\alpha) > b_i^0, \\ \alpha, & \text{if } b_i^0 - h_i \le b_i(\alpha) \le b_i^0, \\ 0, & \text{if } b_i(\alpha) < b_i^0 - h_i. \end{cases}$$

The graphical representations of the above two membership functions are depicted in Figure 4.

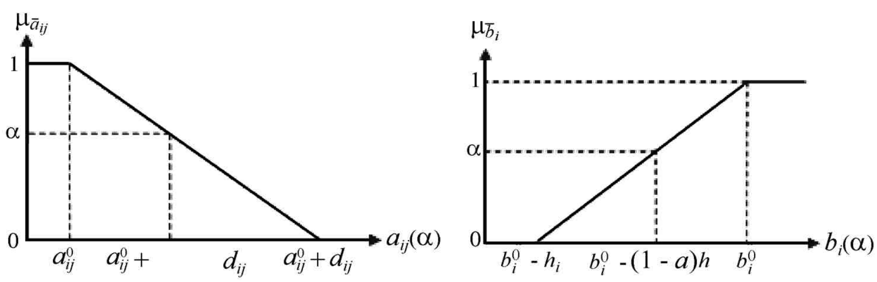

Fig. 4. The membership functions of the fuzzy sets $\tilde{A}$ and $\tilde{b}$.

Given α, confidence or tolerable level, the following linear programming problem can be set to find the desired target.

$$\text{(21)} \quad \begin{aligned} \max \quad & z(x) = \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij}(\alpha) x_j \le b_i(\alpha), \forall i = 1, 2, \ldots, m, \\ & x_j \ge 0, \forall j = 1, 2, \ldots, n. \end{aligned}$$

Therefore, by varying α within 0 to 1 we can derive a set of optimal solutions for the corresponding targets.

Example 5.1. (Continue Example 3.1) When the DM set the target to $x^0 = (70, 100)$, we may obtain no solution by applying the CSA model. In this case, we utilize the fuzzy linear programming techniques as follows.

$$\begin{aligned}
\max\quad & 90x_1 + 70x_2 \\
\text{s.t.}\quad & [3.2^{\alpha=1}, 4^{\alpha=0}]x_1 + [1.2^{\alpha=1}, 2^{\alpha=0}]x_2 \le [200^{\alpha=0}, 226.6^{\alpha=1}], \\
& [1.4^{\alpha=1}, 2^{\alpha=0}]x_1 + [2.25^{\alpha=1}, 3^{\alpha=0}]x_2 \le [240^{\alpha=0}, 263.5^{\alpha=1}], \\
& x_1 \ge 0, x_2 \ge 0.
\end{aligned}$$

Suppose that the value of α has been given by the DM as 0.8, a linear programming problem could be obtained as follows by (21).

$$\begin{aligned}
\max\quad & 90x_1 + 70x_2 \\
\text{s.t.}\quad & 3.36x_1 + 1.36x_2 \le 221.28, \\
& 1.52x_1 + 2.40x_2 \le 258.8, \\
& x_1 \ge 0, x_2 \ge 0,
\end{aligned}$$

where the optimal solution is $x^* = (29.87, 88.92)$. Then, we may solve Program **??** with x^* as the target and obtain the corresponding optimal adjustment of competence set listed as follows.

$$D^- = \begin{bmatrix} 0.8 & 0.8 \\ 0.04 & 0.75 \end{bmatrix}, \gamma^+ = (2.288, 18.614).$$

Table 1 shows the optimal adjustments of competence set by varying the value of α within $\{0, 0.1, 0.2, \ldots, 1\}$, where x^* presents the optimal solution derived from solving (21) with a given α. The optimal adjustment of competence set is obtained by solving Program 3.3 with x^* as the target. The total adjustment ratio corresponding to the optimal adjustment is also listed. Note that by setting $\alpha = 0.9$ and $\alpha = 1$ we obtain the revised target: $x^* = (32.37, 91.99)$ and $x^* = (35.08, 95.28)$ respectively and there is no feasible adjustment. This is due to the constraints imposed on the budget and on the tolerant ranges of adjustment. In this case, the DM may decrease the value of α or apply the aforementioned bisection method.

Table 1. Optimal adjustment of competence set with different α values.

α	x^*	Optimal adjustment of competence set	Total adjustment ratio
0.0	(15,70)	$D^- = 0, \gamma^+ = 0$	0
0.1	(16.47,71,93)	$D^- = 0, \gamma^+ = (9.47, 8.73)$	0.085075
0.2	(18.03,73.97)	$D^- = \begin{bmatrix} 0 & 0 \\ 0 & 0.166781 \end{bmatrix}, \gamma^+ = (9.47, 8.73)$	0.179365
0.3	(19.68,76.12)	$D^- = \begin{bmatrix} 0 & 0.063787 \\ 0 & 0.364162 \end{bmatrix}, \gamma^+ = (26.1046, 0)$	0.283803
0.4	(21.45,78.39)	$D^- = \begin{bmatrix} 0 & 0.214547 \\ 0 & 0.485649 \end{bmatrix}, \gamma^+ = (25.7617, 0)$	0.397965
0.5	(23.35,80.79)	$D^- = \begin{bmatrix} 0 & 0.365915 \\ 0 & 0.607377 \end{bmatrix}, \gamma^+ = (25.4177, 0)$	0.512505
0.6	(25.37,83.33)	$D^- = \begin{bmatrix} 0 & 0.516803 \\ 0 & 0.728789 \end{bmatrix}, \gamma^+ = (25.0748, 0)$	0.626705
0.7	(27.54,86.04)	$D^- = \begin{bmatrix} 0 & 0.78 \\ 0 & 0.75 \end{bmatrix}, \gamma^+ = (14.90, 8.67)$	0.751935
0.8	(29.87,88.92)	$D^- = \begin{bmatrix} 0.8 & 0.8 \\ 0.04 & 0.75 \end{bmatrix}, \gamma^+ = (2.288, 18.614)$	0.959017
0.9	(32.37,91.99)	No feasible adjustment	N/A
1.0	(35.08,95.28)	No feasible adjustment	N/A

6. CONCLUSIONS

In this article, given a specific target, x^0, we have considered a class of *optimal adjustment of competence set problems*. A *competence set adjustment model* (CSA model) has been formulated to provide useful information for the optimal adjustment of the competence set. The *bisection algorithm* (BA) and the *fuzzy linear programming* (FLP) techniques have been utilized to search for a good target, when the original target is not attainable. The former is to find a solution which is as close as possible to the target from a status quo, and the latter is to help the DM to identify an achievable target depending on fuzzy tolerance. The optimal adjustment could then be derived from the aforementioned CSA model with the new target obtained.

A number of research problems remain to be explored. For example: (i) What is the relationship between the optimal adjustment of competence set problem and the ordinary goal programming? (ii) How to effectively determine the optimal adjustment if a set of targets, instead of a single target, is given?

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