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OPTIMIZATION THEORY FOR SET FUNCTIONS IN NONDIFFERENTIABLE FRACTIONAL PROGRAMMING WITH MIXED TYPE DUALITY

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Abstract. We revisit optimization theory involving set functions which are defined on a family of measurable subsets in a measure space. In this paper, we focus on a minimax fractional programming problem with subdifferentiable set functions. Using nonparametric necessary optimality conditions, we introduce generalized $(\mathcal{F}, \rho, \theta)$-convexity to establish several sufficient optimality conditions for a minimax programming problem, and construct a new dual model to unify the Wolfe type dual and the Mond-Weir type dual as special cases of this dual programming problem. Finally we establish a weak, strong, and strict converse duality theorem.

1. INTRODUCTION

In optimization theory, various types of mappings are considered. It may be "point to point", "point to set", "set to point", or "set to set". In this article, we will confine ourselves to optimization for set to point mappings, that is set functions defined on a σ-algebra of measurable subsets in a measure space. In 1979, Morris [17] was the first to develop the general theory for optimizing set functions. Several authors have shown interesting in optimization problems with set functions that arise in many situations dealing with an optimal selection of measurable subsets. These problems have been encountered in fluid flow, electrical insulator design, optimal plasma confinement, and regional design problems. For example, see [1, 2, 5, 17]. The analysis of optimization problems involving set functions has been developed by many researchers. For details, one can consult [4, 6-16, 18-20].

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In this paper, we mainly study the duality structure for a minimax fractional programming problem of set functions. Usually a dual programming problem depends on necessary optimality conditions by some additional assumptions to establish the sufficiency of an optimal solution for the primary problem. Then we constitute the dual model relative to the original programming problem. Usually, these duality forms are difficult to understand the motivation in the dual models. It is often given by a more general dual constitution in the literature, but only by the requirement for mathematical analysis.

We do here a mixed type dual to unify two known dual types, the Wolfe type dual and the Mond-Weir type dual, as special cases. We are also established a weak, strong and strict converse duality theorem for the dual problem in this framework.

2. Preliminary

For convenience, we revisit some elementary concepts of set functions. Let (X, Γ, μ) be an atomless finite measure space with $L^1(X, \Gamma, \mu)$ separable. For any $f \in L^1$ and $\Omega \in \Gamma$, we denote by $\int_\Omega f \; d\mu = \langle f, \mathcal{X}_\Omega \rangle$, where $\mathcal{X}_\Omega$ stands for the characteristic function of Ω. Since $\mu(X) < \infty$, each $\Omega \in \Gamma$ corresponds to $\mathcal{X}_\Omega \in L^\infty \subset L^1$. By atomlessness of the measure space, any $\Omega \in \Gamma$ with $\mu(\Omega) > 0$ implies that there exists $\Lambda \subset \Omega$ with $\mu(\Lambda) > 0$. As L^1 is separable, there exists a countable set $\{\mathcal{X}_\Omega\}_{\Omega \in \Gamma} \subset L^\infty$ dense in L^1. It follows that the theory of set functions defined on Γ can be taken over the sequence of measurable subsets in X. Hence even Γ is not a linear space, we can discuss the convexity, differentiability and continuity of set functions defined on Γ with respect to the $weak^*$ topology induced from L^1. Accordingly, for any $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0, 1]$ one can associate a sequence $V_n = \Omega_n \cup \wedge_n \cup (\Omega \cap \wedge)$ in Γ with $\Omega_n \subset \Omega \backslash \wedge$ and $\wedge_n \subset \wedge \backslash \Omega$ such that

$$\mathcal{X}_{\Omega_n} \overset{w^*}{\to} \lambda \mathcal{X}_{\Omega \backslash \Lambda} \text{ and } \mathcal{X}_{\Lambda_n} \overset{w^*}{\to} (1-\lambda)\mathcal{X}_{\Lambda \backslash \Omega}$$

implies

$$\mathcal{X}_{\Omega_n \cup \wedge_n \cup (\Omega \cap \wedge)} \overset{w^*}{\to} \lambda \mathcal{X}_\Omega + (1-\lambda)\mathcal{X}_\Lambda. \tag{2.1}$$

where $\overset{w^*}{\to}$ denotes the $weak^*$ convergence of elements in $L^\infty(X, \Gamma, \mu)$. We call such sequence $\{V_n\}$ a Morris sequence. Recall that a subfamily $\mathcal{S} \subset \Gamma$ is called **convex** if for any $(\Omega, \Lambda, \lambda) \in \mathcal{S} \times \mathcal{S} \times [0, 1]$, there exists a Morris sequence V_n satisfying (2.1).

A set function $F : \mathcal{S} \longrightarrow \mathbb{R}$ is **convex** if for any $(\Omega, \Lambda, \lambda) \in \mathcal{S} \times \mathcal{S} \times [0, 1]$, there exists a Morris sequence $\{V_n\}$ such that

$$\limsup_{n \longrightarrow \infty} F(V_n) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda). \tag{2.2}$$

A vector $f \in L^1(X, \Gamma, \mu)$ is called a **subgradient** of a set function $F : \Gamma \longrightarrow \mathbb{R}$ at Ω_0 if it satisfies the inequality

$$F(\Omega) \geq F(\Omega_0) + \langle \mathcal{X}_\Omega - \mathcal{X}_{\Omega_0}, f \rangle \text{ for all } \Omega \in \Gamma. \tag{2.3}$$

The set of all subgradients f of a set function $F : \Gamma \longrightarrow \mathbb{R}$ is denoted by $\partial F(\Omega_0)$, namely the subdifferential of F at Ω_0. If $\partial F(\Omega_0) \neq \phi$, the set function F is called **subdifferentiable** at Ω_0. Of course

$$\partial F(\Omega) \subset L_1(X, \Gamma, \mu) \text{ for all } \Omega \in \mathcal{S}.$$

A set function $F : \Gamma \longrightarrow \mathbb{R}$ is w^***-lower(upper) semicontinuous** at $\Omega \in \mathcal{S} =$ dom $F \equiv \{\Omega \in \Gamma; F(\Omega) \text{is finite}\}$ if for any sequence Ω_n in $\mathcal{S}$ with $\mathcal{X}_{\Omega_n} \overset{w^*}{\to} \mathcal{X}_\Omega$, we have

$$-\infty < F(\Omega) \leq \liminf_{n \to \infty} F(\Omega_n)$$
$$(\limsup_{n \longrightarrow \infty} F(\Omega_n) \leq F(\Omega) < \infty.)$$

F is w^***-continuous** at Ω if

$$\lim_{n \longrightarrow \infty} F(\Omega_n) = F(\Omega)$$

for any sequence $\{\Omega_n\} \subset \mathcal{S}$ with $\mathcal{X}_{\Omega_n} \overset{w^*}{\to} \mathcal{X}_\Omega$.

3. Programming Problem of Set Functions

Consider the following minimax fractional programming problem with subdifferentiable set functions as the form:

$$\text{(P)} \qquad \begin{aligned} &\min_{\Omega} \left(\max_{1 \leq i \leq p} \frac{F_i(\Omega)}{G_i(\Omega)} \right) \\ &\text{subject to } \Omega \in \mathcal{S} \text{ and } H_j(\Omega) \leq 0, j \in M = \{1, 2, \cdots\} \equiv \underline{m} \end{aligned}$$

where $\mathcal{S}$ is a subfamily of measurable subsets, and the set functions $F_i, G_i, i \in \underline{p}$ and $H_j, j \in \underline{m}$ are subdifferentiable on $\mathcal{S}$. Without loss of generality, we assume that $G_i(\Omega) > 0$ and $F_i(\Omega) \geq 0$ for any $\Omega \in \mathcal{S}$, and all functions are proper.

Let $q = \max_{i \in \underline{p}} \{F_i(\Omega)/G_i(\Omega)\}$. Then $F_i(\Omega) - qG_i(\Omega) \leq 0, i \in \underline{p}$.

The problem $(\bar{P})$ is equivalent to the following parametric programming problem

$$(\widetilde{P}) \quad \text{Minimize} \quad q$$
$$\text{subject to} \quad F_i(\Omega) - qG_i(\Omega) \leq 0, \; i \in \underline{p} \tag{3.1}$$
$$H_j(\Omega) \leq 0, \text{ for all } \Omega \in \mathcal{S}, \quad j \in \underline{m}. \tag{3.2}$$

Writing the objective function of (P) as

$$\varphi(\Omega) \equiv \max_{i \in \underline{p}} F_i(\Omega)/G_i(\Omega) \tag{3.3}$$

one can easily get (cf. Lai and Liu [13, Section 2])

$$\varphi(\Omega) = \max_{y \in I} \langle y, F(\Omega)\rangle / \langle y, G(\Omega)\rangle, \tag{3.4}$$

where $I = \{y \in \mathbb{R}^p_+ | \sum_{i=1}^p y_i = 1\}$.

We will use the inner product

$$\langle y, F(\Omega)\rangle = y^\top F(\Omega) = \sum_{i=1}^p y_i F_i(\Omega) \text{ in } \mathbb{R}^p$$

It can be derived the necessary optimality conditions of (P) from the parametric optimality conditions of $(\tilde{P})$. (cf. Lai and Liu [13, section 2]).

Actually the solution of $(\tilde{P})$ is equivalent to finding the minimax solution $(\Omega^*; y^*)$ for the Lagrangian:

$$L(\Omega, y; q, z) = \langle y, F(\Omega)\rangle - q\langle y, G(\Omega)\rangle + \langle z, H(\Omega)\rangle$$
with multipliers $q^* \in \mathbb{R}_+$ and $z^* \in \mathbb{R}^m_+$.

The minimax solution (Ω^*, y^*) is deduced to the Kuhn-Tucker type condition for subdifferentiable set functions. Hence we get the necessary optimality conditions for problem (P) as follows.

Theorem 3.1. (Necessary Conditions) *Let $\Omega^* \in \mathcal{S}$ be an optimal solution of problem (P) with optimal value $q^* \in \mathbb{R}_+$. If the constraint qualification holds for $(\widetilde{P})$, the same as in (P), and all set functions $F_i, G_i, 1 \le i \le p$ and $H_j, 1 \le j \le m$ are subdifferentiable on $\mathcal{S}$, then there exist*

$$y^* \in I = \{y \in \mathbb{R}^p_+ | \sum_{i=1}^p y_i = 1\} \subset \mathbb{R}^p_+ \textit{ and } z^* \in \mathbb{R}^m_+$$

such that $(\Omega^*, q^*, y^*, z^*)$ satisfies the following conditions:

$$0 \in \partial(y^{*\top} F)(\Omega^*) + q^* \partial(-y^{*\top} G)(\Omega^*) + \partial(z^{*\top} H)(\Omega^*) + N_{\mathcal{S}}(\Omega^*) \tag{3.5}$$

$$y^{*\top} F(\Omega^*) - q^* y^{*\top} G(\Omega^*) = 0 \tag{3.6}$$

$$z^{*\top} H(\Omega^*) = 0 \tag{3.7}$$

where $z_j^* = 0$ if $H_j(\Omega^*) < 0$ and $z_k^* > 0$ if $H_k(\Omega^*) = 0$ and

$$N_{\mathcal{S}}(\Omega^*) = \{f \in L^1(X, \Gamma, \mu) \mid \langle \mathcal{X}_\Omega - \mathcal{X}^*_\Omega, f\rangle \le 0 \text{ for all } \Omega \in \mathcal{S}\} \tag{3.8}$$

Remark 3.1.

(1) In Theorem 3.1, the set $\Omega^* \in \mathcal{S}$ is called a **regular** solution of (P) if there exist no $h \in \partial(z^{*\top} H)$ and $\eta \in \mathcal{N}_{\mathcal{S}}(\Omega^*)$ such that $h + \eta = 0$

(2) From (3.6), $q^* = \varphi(\Omega^*) = \langle y^*, F(\Omega^*)\rangle / \langle y^*, G(\Omega^*)\rangle$ substituting q^* and the equality (3.7) into (3.5), we then get

$$\begin{aligned} 0 \in\ & y^{*\top} G(\Omega^*)[\partial(y^{*\top} F)(\Omega^*) + \partial(z^{*\top} H)(\Omega^*)] \\ & -[(y^{*\top} F)(\Omega^*) + (z^{*\top} H)(\Omega^*)]\ \partial(y^{*\top} G)(\Omega^*) + \mathcal{N}_{\mathcal{S}}(\Omega^*) \end{aligned} \tag{3.9}$$

4. Generalized $(\mathcal{F}, \rho, \theta)$-Convexity

The existence of an optimal solution for problem (P) could be established from the inverse of the necessary conditions with some extra assumptions. Many authors effort to search such conditions, and constitutes the duality programming problem. In this paper we will use the concept of generalized $(\mathcal{F}, \rho, \theta)$-convexity (see Lai and Liu [12]) to treat with these tasks. For convenience, we recall $(\mathcal{F}, \rho, \theta)$-convexity as the following. This $(\mathcal{F}, \rho, \theta)$-convexity is an extension of generalized $(\mathcal{F}, \rho)$-convexity defined in Preda [19] for nondifferentiable set functions.

We consider a sublinear functional with respect to the third variable by:

$$\mathcal{F} : \Gamma \times \Gamma \times L^1(X, \Gamma, \mu) \longrightarrow \mathbb{R}$$

Let $\rho \in \mathbb{R}$ and $\theta : \Gamma \times \Gamma \longrightarrow \mathbb{R}_+ = [0, \infty)$ such that $\theta(\Omega, \Omega_0) \neq 0$ if $\Omega \neq \Omega_0$. Then for a subdifferentiable set function $F : \Gamma \longrightarrow \mathbb{R}$, we give the following definitions:

(1) F is said to be $(\mathcal{F}, \rho, \theta)$-**convex** at Ω_0 if for each $\Omega \in \Gamma$ and $f \in \partial F(\Omega_0)$, we have
$$F(\Omega) - F(\Omega_0) \geq \mathcal{F}(\Omega, \Omega_0; f) + \rho\theta(\Omega, \Omega_0)$$

(2) F is said to be $(\mathcal{F}, \rho, \theta)$-**quasiconvex** $\Big($**prestrict** $(\mathcal{F}, \rho, \theta)$-**quasiconvex**$\Big)$ at Ω_0 if for each $\Omega \in \Gamma$ and $f \in \partial F(\Omega_0)$, we have that
$$F(\Omega) \leq F(\Omega_0)\ \Big(F(\Omega) < F(\Omega_0)\Big) \Longrightarrow \mathcal{F}(\Omega, \Omega_0; f) \leq -\rho\theta(\Omega, \Omega_0).$$

(3) F is said to be $(\mathcal{F}, \rho, \theta)$-**pseudoconvex** $\Big($**strict** $(\mathcal{F}, \rho, \theta)$-**pseudoconvex**$\Big)$ at Ω_0 if for each $\Omega \in \Gamma$ and $f \in \partial F(\Omega_0)$, we have that
$$\mathcal{F}(\Omega, \Omega_0; f) \geq -\rho\theta(\Omega, \Omega_0) \quad \Rightarrow \quad F(\Omega) \geq F(\Omega_0) \quad \Big(F(\Omega) > F(\Omega_0)\Big)$$

Remark 4.1. If the functional $\mathcal{F} : \Gamma \times \Gamma \times L^1(X, \Gamma, \mu) \longrightarrow \mathbb{R}$ is taken by the special case

$$\mathcal{F}(\Omega, \Omega_0; f) = \langle \mathcal{X}_\Omega - \mathcal{X}_{\Omega_0}, f \rangle$$

then F is reduced to be a convex set function at Ω_0 (cf. [12]).

5. Duality Programming Problems

According to Theorem 3.1, the Wolfe type (WD) and Mond-Weier type (MWD) dual models related to problem (P) involving set functions can be formulated as the following forms:

$$\begin{aligned}
\text{(WD)} \quad & \text{Maximize} \quad (y^\top F(U) + z^\top H(U))/y^\top G(U) \\
& \text{subject to} \quad y \in I \equiv \left\{ y \in \mathbb{R}^p_+ \;\middle|\; \sum_{i=1}^{p} y_i = 1 \right\}, \; z \in \mathbb{R}^m_+ \text{ and} \\
& \qquad 0 \in y^\top G(U)[\partial(y^\top F)(U) + \partial(z^\top H)(U)] \\
& \qquad\quad -\partial(y^\top G)(U)[y^\top F(U) + z^\top H(U)] + N_{\mathcal{S}}(U).
\end{aligned}$$

$$\begin{aligned}
\text{(MWD)} \quad & \text{Maximize} \quad y^\top F(U)/y^\top G(U) \\
& \text{subject to} \quad y \in I, \; z \in \mathbb{R}^m_+ \text{ and} \\
& \qquad 0 \in y^\top G(U)[\partial(y^\top F)(U) + \partial(z^\top H)(U)] \\
& \qquad\quad -\partial(y^\top G)(U) y^\top F(U) + N_{\mathcal{S}}(U), \\
& \qquad z^\top H(U) \geq 0.
\end{aligned}$$

where $N_{\mathcal{S}}(U_0) \equiv \{ f \in L_1(X, \Gamma, \mu) \mid \langle \mathcal{X}_U - \mathcal{X}_{U_0}, f \rangle \leq 0 \text{ for all } \; U \in \mathcal{S} \}$.

In this paper, we are mainly concerned to construct a new type dual problem(MD) which is called a mixed type problem containing (WD) and (MWD) as its special cases. It is constituted by dividing the constrained inequalities to be several families of inequalities, and one adds one part of constraints in the numerator of the objective of problem (P) as a new objective fractional function. Such a mixed type dual is then constructed as the form

$$\begin{aligned}
\text{(MD)} \quad & \text{Maximize}(y^\top F(U) + z^\top_{M_0} H(U))/y^\top G(U) \\
& \text{subject to} y \in I, \; z \in R^m_+, \; U \in \mathcal{S} \subset \Gamma \text{and} \\
& \qquad 0 \in y^\top G(U)[\partial(y^\top F)(U) + \sum_{\alpha=0}^{k} \partial(z^\top_{M_\alpha} H)(U)] \\
& \qquad\quad -\partial(y^\top G)(U)[y^\top F(U) + z^\top_{M_0} H(U)] + N_{\mathcal{S}}(U).
\end{aligned} \tag{5.1}$$

$$
\text{(5.2)}\quad
\begin{aligned}
& z_{M_\alpha}^\top H(U) \geq 0,\ \alpha = 0, 1, 2, \cdots, k \\
& z = (z_{M_0}, z_{M_1}, \cdots, z_{M_k})^\top \in \mathbb{R}_+^m \\
& \text{where } M_\alpha \subseteq M,\ \alpha = 0, 1, \cdots, k \text{ with } M_\alpha \cap M_\beta = \varnothing \text{ if } \alpha \neq \beta \\
& \text{and } \textstyle\bigcup_{\alpha=0}^k M_\alpha = M, z_{M_\alpha}^\top H(U) = \sum_{j \in M_\alpha} z_j H_j(U).
\end{aligned}
$$

It is remarkable that if $M_0 = M$ (the others $M_\alpha = \varnothing\ \forall \alpha = 1, \cdots, k$), then (MD) coincides with the Wolfe type dual. If $M_0 = \varnothing$ and $M_1 = M$ (that is all $M_\alpha = M,\ \alpha = 1, \cdots, k$), then (MD)=(MWD). In this dual problem (MD), we assume throughout that $y^\top F(U) + z_{M_0}^\top H(U) \geq 0$ and $y^\top G(U) > 0$.

Now, by the preparations before, we will establish the weak, strong, and strict converse duality theorem in this section.

Theorem 5.1. (Weak Duality). *Let $\Omega \in \mathcal{F}_P$ and $(U, y, z) \in \mathcal{F}_{MD}$ be any feasible solutions of (P) and (MD), respectively, and denoted by*

$$A(\cdot) = y^\top G(U)[y^\top F(\cdot) + z_{M_0}^\top H(\cdot)] - y^\top G(\cdot)[y^\top F(U) + z_{M_0}^\top H(U)].$$

Suppose that $\mathcal{F}(\Omega, U; -\eta) \geq 0$ for each $\eta \in N_{\mathcal{S}}(U)$. Further assume that any one of the following six conditions holds:

(a) *$y^\top F$ is $(\mathcal{F}, \rho_1, \theta)$-convex, $-y^\top G$ is $(\mathcal{F}, \rho_2, \theta)$-convex, $z_{M_\alpha}^\top H$, $\alpha = 0, 1, \cdots, k$ are $(\mathcal{F}, \rho_{3\alpha}, \theta)$-convex and*

$$y^\top G(U)\rho_1 + [y^\top F(U) + z_{M_0}^\top H(U)]\rho_2 + y^\top G(U) \sum_{\alpha=0}^k \rho_{3\alpha} \geq 0;$$

(b) *A is $(\mathcal{F}, \rho_1, \theta)$-pseudoconvex, $z_{M_\alpha}^\top H,\ \alpha = 1, 2, \cdots, k$ are $(\mathcal{F}, \rho_{2\alpha}, \theta)$-quasiconvex, and $\rho_1 + y^\top G(U) \sum_{\alpha=1}^k \rho_{2\alpha} \geq 0$;*

(c) *A is $(\mathcal{F}, \rho_1, \theta)$-quasiconvex, $z_{M_\alpha}^\top H,\ \alpha = 1, 2, \cdots, k$ are strictly $(\mathcal{F}, \rho_{2\alpha}, \theta)$-pseudoconvex, and $\rho_1 + y^\top G(U) \sum_{\alpha=1}^k \rho_{2\alpha} \geq 0$;*

(d) *A is prestrictly $(\mathcal{F}, \rho_1, \theta)$-quasiconvex, $z_{M_\alpha}^\top H,\ \alpha = 1, 2, \cdots, k$ are $(\mathcal{F}, \rho_{2\alpha}, \theta)$-quasiconvex, and*

$$\rho_1 + y^\top G(U) \sum_{\alpha=1}^k \rho_{2\alpha} > 0;$$

(e) *$A + y^\top G(U) \sum_{\alpha=1}^k z_{M_\alpha}^\top H$ is $(\mathcal{F}, \rho, \theta)$-pseudoconvex, and $\rho \geq 0$;*

(f) *$A + y^\top G(U) \sum_{\alpha=1}^k z_{M_\alpha}^\top H$ is prestrictly $(\mathcal{F}, \rho, \theta)$-quasiconvex, and $\rho > 0$.*

Thenfor each $\Omega \in \mathcal{F}_p$,

$$\varphi(\Omega) \geq (y^\top F(U) + z_{M_0}^\top H(U))/y^\top G(U). \tag{5.3}$$

Proof. Suppose on the contrary that the inequality (5.3) were not true, we then have

$$\varphi(\Omega) < (y^\top F(U) + z_{M_0}^\top H(U))/y^\top G(U). \tag{5.4}$$

By (3.4),

$$\begin{aligned} \varphi(\Omega) &= \max_{y \in I} y^\top F(\Omega)/y^\top G(\Omega) \\ &\geq y^\top F(\Omega)/y^\top G(\Omega) \text{ for any } y \in I. \end{aligned}$$

Then (5.4) would be

$$y^\top F(\Omega)/y^\top G(\Omega) < (y^\top F(U) + z_{M_0}^\top H(U))/y^\top G(U),$$

or

$$y^\top G(U) y^\top F(\Omega) - y^\top G(\Omega)[y^\top F(U) + z_{M_0}^\top H(U)] < 0. \tag{5.5}$$

By adding $y^\top G(U) z_{M_0}^\top H(\Omega)$ to both sides of (5.5), we get

$$\begin{aligned} &y^\top G(U)[y^\top F(\Omega) + z_{M_0}^\top H(\Omega)] - y^\top G(\Omega)[y^\top F(U) + z_{M_0}^\top H(U)] \\ &< y^\top G(U) z_{M_0}^\top H(\Omega) \leq 0 \end{aligned} \tag{5.6}$$

since $\Omega \in \mathcal{F}_P$, $z_{M_0}^\top H(\Omega) \leq 0$ as $z \in R_+^m$.
Hence (5.6) implies

$$A(\Omega) < 0 = A(U). \tag{5.7}$$

Since $\Omega \in \mathcal{F}_P$ and (5.2), we have

$$z_{M_\alpha}^\top H(\Omega) \leq 0 \leq z_{M_\alpha}^\top H(U), \alpha = 0, 1, \cdots, k \tag{5.8}$$

From equation (5.1), there exist

$$\begin{aligned} &f \in \partial(y^\top F)(U),\ h_\alpha \in \partial(z_{M_\alpha}^\top H)(U), \alpha = 0, 1, \cdots, k \\ &g \in \partial(-y^\top G)(U) \ \text{ and } \ \eta \in N_{\mathcal{S}}(U) \end{aligned}$$

such that

$$y^\top G(U)[f+\sum_{\alpha=0}^{k} h_\alpha]+[y^\top F(U)+z_{M_0}^\top H(U)]g+\eta=0.$$

By the sublinearity of $\mathcal{F}$, we have

$$\mathcal{F}(\Omega,U;y^\top G(U)[f+\sum_{\alpha=0}^{k} h_\alpha]+[y^\top F(U)+z_{M_0}^\top H(U)]g)\geq \mathcal{F}(\Omega,U;-\eta). \tag{5.9}$$

We have to evaluate the following expressions

$$\begin{aligned}
&y^\top G(U)[y^\top F(\Omega)-y^\top F(U)]\\
&\quad -[y^\top F(U)+z_{M_0}^\top H(U)][y^\top G(\Omega)-y^\top G(U)]\\
&\quad +y^\top G(U)[z^\top H(\Omega)-\sum_{\alpha=0}^{k} z_{M_\alpha}^\top H(U)]\\
&\qquad \Big(\text{ where } z^\top H(\Omega)\equiv\sum_{\alpha=0}^{k} z_{M_\alpha}^\top H(\Omega)=\sum_{j=1}^{m} z_j H_j(\Omega)\Big)\\
&=y^\top F(\Omega)y^\top G(U)-y^\top G(\Omega)[y^\top F(U)+z_{M_0}^\top H(U)]\\
&\quad -y^\top F(U)\cdot y^\top G(U)+[y^\top F(U)+z_{M_0}^\top H(U)]y^\top G(U)\\
&\quad +[z^\top H(\Omega)-\sum_{\alpha=0}^{k} z_{M_\alpha}^\top H(U)]y^\top G(U) \qquad (\bigstar)\\
&<0+y^\top G(U)\cdot[z^\top H(\Omega)+z_{M_0}^\top H(U)-\sum_{\alpha=0}^{k} z_{M_\alpha}^\top H(U)]\Big(\text{by } (5.5)<0\Big)\\
&\text{(since } y^\top G(U)>0 \text{ and } z^\top H(\Omega)\leq 0 \text{ if } \Omega\in\mathcal{F}_{\mathcal{P}}, y^\top G(U)z^\top H(\Omega)\leq 0.)\\
&\leq 0+0+y^\top G(U)[z_{M_0}^\top H(U)-\sum_{\alpha=0}^{k} z_{M_\alpha}^\top H(U)]\\
&=-y^\top G(U)\sum_{\alpha=1}^{k} z_\alpha^\top H(U)\\
&\leq 0 \text{ (by } (5.2), z_{M_\alpha}^\top H(U)\geq 0, \alpha=1,2,\cdots,k).
\end{aligned}$$

Consequently the expression ($\bigstar$) is strictly less than 0.

If hypothesis (a) holds, then we have

$$y^\top F(\Omega)-y^\top F(U)\geq \mathcal{F}(\Omega,U;f)+\rho_1\theta(\Omega,U) \tag{5.10}$$

$$(5.11) \qquad -[y^\top G(\Omega) - y^\top G(U)] \geq \mathcal{F}(\Omega, U; g) + \rho_2\theta(\Omega, U)$$

$$(5.12) \quad z_{M_\alpha}^\top H(\Omega) - z_{M_\alpha}^\top H(U) \geq \mathcal{F}(\Omega, U; h_\alpha) + \rho_{3\alpha}\theta(\Omega, U)\alpha = 0, 1, 2, \cdots, k.$$

We multiply (5.10) by $y^\top G(U)$, (5,11) by $y^\top F(U) + z_{M_0}^\top H(U)$ and (5,12) by $y^\top G(U)$, and adding the resultant inequalities , then using the inequalities (★) and (5.9), and again by sublinearity of $\mathcal{F}$, eventually, we get

$$0 > \Big(y^\top G(U)\rho_1 + [y^\top F(U) + z_{M_0}^\top H(U)]\rho_2 + y^\top G(U)\sum_{\alpha=0}^{k} \rho_{3\alpha}\Big)\theta(\Omega, U).$$

This contradicts the fact of condition (a). Hence the inequality (5.4) is not true. This proves that (5.3) holds.

If hypothesis (b) holds, A is $(\mathcal{F}, \rho_1, \theta)$-pseudoconvex and then we have

$$(5.13) \quad \mathcal{F}\Big(\Omega, U; y^\top G(U)(f + h_0) + [y^\top F(U) + z_{M_0}^\top H(U)]g\Big) < -\rho_1\theta(\Omega, U)$$

Since $z_{M_\alpha}^\top H(U)$, $\alpha = 1, 2, \cdots, k$ are $(\mathcal{F}, \rho_{2\alpha}, \theta)$-quasiconvex, it follows from (5.8) that, for $h_\alpha \in \partial(z_{M_\alpha}^\top H)(U)$,

$$(5.14) \qquad \mathcal{F}(\Omega, U; h_\alpha) \leq -\rho_{2\alpha}\theta(\Omega, U), \alpha = 1, 2, \cdots, k$$

Multiply (5.14) by $y^\top G(U)$ (> 0) and adding to (5.13), then from (5.9) and the sublinearity of $\mathcal{F}$, we get

$$\Big(\rho_1 + y^T G(U)\sum_{\alpha=1}^{k} \rho_{2\alpha}\Big)\theta(\Omega, U) < 0.$$

This is a contradiction, so that (5.4) is not true. Hence (5.3) holds.

The proof of the theorem under the hypotheses (c) and (d) can be carried out along with the same lines of (b).

If hypothesis (e) holds, by the inequalities (5.7) and (5.8), we have

$$(5.15) \qquad A(\Omega) + y^\top G(U)\sum_{\alpha=1}^{k} z_{M_\alpha}^\top H(\Omega) < A(U) + y^\top G(U)\sum_{\alpha=1}^{k} z_{M_\alpha}^\top H(U)$$

Using the $(\mathcal{F}, \rho, \theta)$-pseudoconvexity of $A + y^\top G(U)\sum_{\alpha=1}^{k} z_{M_\alpha}^\top H$ and the inequality (5.15), we have

$$(5.16) \qquad \mathcal{F}\Big(\Omega, U; y^\top G(U)[f + \sum_{\alpha=0}^{k} h_\alpha] + [y^\top F(U) + z_{M_0}^\top H(U)]g\Big)$$

$$< -\rho\theta(\Omega, U).$$

Consequently, the inequalities (5.16) and (5.9) yield

$$\rho\theta(\Omega, U) < 0.$$

This contradicts the fact $\rho \geq 0$. Hence (5.4) is not true, and then (5.3) holds.

Hypothesis (f) follows along with the same line as (e). Therefore the proof of theorem is complete. ■

Next, if $\Omega^* \in \mathcal{S}$ is an optimal solution of (P), then by Theorem 3.1, there exist $y^* \in I \subset \mathbb{R}^p_+$ and $z^* \in \mathbb{R}^m_+$ such that $(\Omega^*, y^*, z^*) \in \mathcal{F}_{MD}$ is a feasible solution of the dual problem (MD). Furthermore if we assume that the conditions of Theorem 5.1 are fulfilled, then one can get

$$\begin{aligned}\min_{\Omega\in\mathcal{S}} \varphi(\Omega) &= \varphi(\Omega^*)\\ &\geq \underset{(\Omega^*, y^*, z^*) \in \mathcal{F}_{MD}}{\text{maximize}} \big((y^{*\top})F(\Omega^*) + z^{*\top}_{M_\circ} H(\Omega^*)\big)/y^{*\top}G(\Omega^*)\\ &\geq \max_{y^*\in I} y^{*\top}F(\Omega^*)/y^{*\top}G(\Omega^*) = \varphi(\Omega^*).\end{aligned}$$

It follows that (Ω^*, y^*, z^*) is an optimal solution of (MD). Hence (P) and (MD) have the same optimal value, and we have the following theorem.

Theorem 5.2. (Strong Duality). *Let $\Omega^* \in \mathcal{S}$ be an optimal solution of (P). Then there exist $y^* \in I \subset \mathbb{R}^p_+$ and $z^* \in \mathbb{R}^m_+$ such that $(\Omega^*, y^*, z^*)\cdot \in \mathcal{F}_{MD}$. is a feasible solution of (MD). Assume further that the conditions of Theorem 5.1 are fulfilled. Then (Ω^*, y^*, z^*) is an optimal solution of (MD), and their optimal values are equal, that is, $min(P) = max(MD)$.* ■

Finally we assume that Ω^* and $(\Omega^\circ, y^\circ, z^\circ)$ are respectively the optimal solutions of (P) and (MD). Then questions arise whether $\Omega^* = \Omega^\circ$ and how their optimal values are related.

The following theorem will answer these questions if we could provide some additional assumptions as in the following theorem.

Theorem 5.3 (Strict Converse Duality). *Let Ω^* and $(\Omega^\circ, y^\circ, z^\circ)$ be the optimal solutions of the problem (P) and the dual problem (MD), respectively. Suppose that the assumptions in Theorem 5.2 are fulfilled. Furthermore, suppose the function $A(\cdot)$ defined in Theorem 5.1 is given by*

$$A(\cdot) = y^{\circ\top}G(\Omega^\circ)[y^{\circ\top}F(\cdot) + z^{\circ\top}_{M_\circ}H(\cdot)] - y^{\circ\top}G(\cdot)[y^{\circ\top}F(\Omega^\circ) + z^{\circ\top}_{M_\circ}H(\Omega^\circ)],$$

and any one of the following two conditions holds.

(i) *$A(\cdot)$ is strictly $(\mathcal{F}, \rho_1, \theta)$-pseudoconvex, $z_{M_\alpha}^{\circ\top} H(\cdot)$, $\alpha = 1, 2, \cdots, k$ are $(\mathcal{F}, \rho_{2\alpha}, \theta)$-quasiconvex, and $\rho_1 + y^{\circ\top} G(\Omega^\circ) \sum_{\alpha=1}^{k} \rho_{2\alpha} \geq 0$;*

(ii) *$A(\cdot) + y^{\circ\top} G(\Omega^\circ) \sum_{\alpha=1}^{k} z_{M_\alpha}^{\circ\top} H(\cdot)$ is strictly $(\mathcal{F}, \rho, \theta)$-pseudoconvex, and $\rho \geq 0$.*

Then $\Omega^\circ = \Omega^*$ is an optimal solution of (P), as well as the optimal values of problem (P) and the dual problem (MD) are equal, that is

$$\min(P) = \max(MD).$$

Proof. Suppose on the contrary that $\Omega^\circ \neq \Omega^*$. By Theorem 5.2, if Ω^* is (P)-optimal, then there exist $y^* \in I$ and $z^* \in \mathbb{R}_+^m$ such that (Ω^*, y^*, z^*) is an optimal solution of problem (MD), and have equal optimal values:

$$\varphi(\Omega^*) = \big(y^{*\top} F(\Omega^*) + z_{M_\circ}^{*\top} H(\Omega^*)\big) / y^{*\top} G(\Omega^*).$$

Using the process given in the proof of Theorem 5.1 replacing Ω by Ω^* and (U, y, z) by $(\Omega^\circ, y^\circ, z^\circ)$, we have the strictly inequality

$$\varphi(\Omega^*) > \big(y^{\circ\top} F(\Omega^\circ) + z_{M_\circ}^{\circ\top} H(\Omega^\circ)\big) / y^{\circ\top} G(\Omega^\circ).$$

This contradicts the fact that

$$\begin{aligned} \varphi(\Omega^*) &= \big(y^{*\top} F(\Omega^*) + z_{M_\circ}^{*\top} H(\Omega^*) / y^{*\top} G(\Omega^*)\big) \\ &= \big(y^{\circ\top} F(\Omega^\circ) + z_{M_\circ}^{\circ\top} H(\Omega^\circ)\big) / y^{\circ\top} G(\Omega^\circ). \end{aligned}$$

Therefore we conclude that the final result follows from Theorem 5.1 that $\Omega^\circ = \Omega^*$, and have $\min(P) = \max(MD)$. ■

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