

## EPIDERIVATIVES WITH RESPECT TO HALF-SPACES

Elvira Hernández, Luis Rodríguez-Marín and Miguel Sama

**Abstract.** In this paper we extend some results given in [L. Rodríguez-Marín, M. Sama,  $\tau^w$ -contingent epiderivatives in reflexive spaces, *Nonlinear Analysis*, **68** (2008), 3780-3788]. Following the same approach, we associate a set-valued optimization problem with a family of simpler problems by using a decoupling of the ordering cone into half-spaces. In this context we give necessary and sufficient optimality conditions in terms of epiderivatives with respect to half-spaces. Moreover we obtain computation formulas for these conditions in term of derivatives of scalar set-valued maps.

### 1. INTRODUCTION

Let  $X$ ,  $Y$  and  $Z$  be normed spaces where  $C \subset Z$  is a closed convex cone, and let  $F : Y \rightarrow 2^Z$ ,  $g : S \subset X \rightarrow Y$  be a set-valued map and a single-valued map respectively such that  $g(S) \subset \text{dom}(F)$ . In this paper we study the following constrained set-valued optimization problem

$$(1.1) \quad (P) \begin{cases} \text{minimize } F(y) \\ y = g(x), x \in S. \end{cases}$$

**Definition 1.1.**  $(\bar{x}, \bar{z}) \in \text{graph}(F \circ g)$  is said to be a strong minimizer of  $(P)$  if  $\bar{z}$  is a strong minimizer of  $\bigcup_{x \in S} F(g(x))$  with respect to  $C$ , i.e.  $\bigcup_{x \in S} F(g(x)) \subset \bar{z} + C$ .

$(P)$  is an optimization problem of a set-valued map whose effective domain is parameterized by a single valued map, for example this situation occurs if we optimize a set-valued map defined on a manifold. When  $X = Y$ ,  $g = \text{id}$  we recover

---

Received September 29, 2007, accepted January 30, 2008.

2000 *Mathematics Subject Classification*: 26E25, 46G05, 54C60, 58C20.

*Key words and phrases*: Set-valued maps, Family of epiderivatives, Contingent derivative, Calculus rules, Existence conditions, Optimization.

For E. Hernández and M. Sama, this work is partially supported by MEC (Spain), proj. MTM2006-02629. and (i-MATH) CSD2006-00032 (Consolider-Ingenio 2010). E. Hernández is also supported by Junta de Castilla y León (Spain), proj. VA027B06.

the classical unconstrained set-valued optimization problem (see [17, 12]), in this case necessary and sufficient conditions have been given in terms of contingent epiderivatives successfully (see [12] and references therein). In the same context, Jahn and Kahn give several results for the general problem  $(P)$  in [11] by using a chain rule for contingent epiderivatives. Main drawback of contingent epiderivatives is that strong conditions must be imposed in order to assure their existence (see [10, 18, 20]). Taking in account these facts, in [21] in order to save this problems of existence, and also with the aim of obtaining computation formulas, a new approach was proposed to deal with problem  $(P)$ . Mainly this approach is based in the following two facts:

- When the ordering cone is not necessarily pointed, a notion of family of contingent epiderivative is introduced in [18]. This let us to consider epiderivatives with respect to half-spaces that have less restrictive existence conditions. For example, as we shall see, stable set-valued maps are epidifferentiable under general assumptions. Moreover a notion of  $\tau^w$ -contingent epiderivative, that coincide with the classical one when the image space is finite dimensional, was introduced in [21].
- A well known result in functional analysis states that a closed convex subset, in particular a closed convex cone, is determined by a family of closed half-spaces.

Main idea of this approach is to replace contingent epiderivatives with respect to the ordering cone by epiderivatives with respect to half-spaces associated with the ordering cone. Loosely speaking, if  $Z$  is a reflexive Banach space and  $g$  is Hadamard directionally differentiable, in [21] we relate strong minimizers of problem  $(P)$  to strong minimizers of a family of problems  $(P_{\lambda_i})_{i \in I}$  associated with a family of half-spaces  $\{H_{\lambda_i}\}_{i \in I}$  verifying  $C = \bigcap_{i \in I} H_{\lambda_i}$ . This allows us to establish optimality conditions in terms of the family of epiderivatives with respect to each half-space  $H_{\lambda_i}$  [21, Theorem 5.4]. As a consequence of this result, a necessary and sufficient condition, under convexity assumptions, for  $(\bar{x}, \bar{z})$  being a strong minimizer of  $(P)$  is given by

$$(1.2) \quad 0 \leq D_{\uparrow}(\lambda_i \circ F)(\bar{x}, \lambda_i(\bar{z}))(g'_H(\bar{x}, u)) \text{ for any } u \in L_{\lambda_i}, i \in I,$$

where  $L_{\lambda_i} = \text{dom}(D_{\uparrow}(\lambda_i \circ F)(\bar{x}, \lambda_i(\bar{z}))(g'_H(\bar{x}, \cdot)))$  and  $g'_H(\bar{x}, \cdot)$  denoting the Hadamard directional derivative of  $g$  at  $\bar{x}$  [21, Theorem 5.5].

In this paper we relax the condition on  $g$  assuming that  $g$  is stable at  $\bar{x}$ , and therefore not necessarily Hadamard directionally differentiable. As we show in Example 1, under this hypothesis of stability, results of [21] can not be applied even in the finite dimensional case. Consequently, by simplicity, we are going to limit our research to the finite dimensional case.

The contents of this paper are as follows. In Section 2 we give some preliminaries and notations. In Section 3 we establish a chain rule for  $F \circ g$  when  $g$  is stable and the image spaces are finite dimensional extending the one given in [21]. As a consequence, in Section 4 we also extend the optimality conditions given by (1.2) by decoupling problem  $(P)$  into a family of simpler problems. In this case, the necessary and the sufficient conditions differ and they are expressed in terms of a set-relation that compares subsets instead of the natural order associated with the ordering cone, moreover we prove that these conditions collapse in (1.2) when  $g$  is Hadamard directional differentiable recovering the results given in [21].

## 2. PRELIMINARIES AND NOTATIONS

Throughout this work we assume that  $Y = \mathbb{R}^m$ ,  $Z = \mathbb{R}^n$  and  $\text{int}(C) \neq \emptyset$ . As we have mentioned in the previous section by  $C$  we denote a closed convex cone not necessarily pointed. By  $y \leq x$  (or equivalently  $x \geq y$ ) we denote  $x - y \in C$  and by  $B(0, t)$  we represent the closed ball centered at origin of radius  $t \in \mathbb{R}_+$ . Let  $A$  be a subset of  $Z$ , by  $\text{int}(A)$  we denote the topological interior of  $A$  and by  $\text{IMin}(A, C)$  (resp.  $\text{IMax}(A, C)$ ) we denote the set of ideal minimal points with respect to  $C$ , i.e.  $\text{IMin}(A, C) = \{a \in A : A \subset a + C\}$  (resp.  $\text{IMax}(A, C) = \{a \in A : A \subset a - C\}$ ). Nonemptiness of  $C$  is equivalent to the existence of  $C$ -lower bounds of the unit ball  $B(0, 1)$ . We are going to assume the existence of the infimum of  $B(0, 1)$  with respect to  $C$  and we denote it by  $\Phi_B$ , i.e

$$\Phi_B = \text{IMax}\{z \in Z : B(0, 1) \subset z + C\}.$$

Although it does not exist in every space ordered space, this assumption is not very restrictive. For example it does exist for every strongly minihedral ordering cone  $C$ , in particular for  $Z = \mathbb{R}^n$ ,  $C = \mathbb{R}_+^n$  (see [15]). However we remark that in most of the results given in this paper this element can be replaced by any  $C$ -lower bound of  $B(0, 1)$ . In this work the following relation on  $2^Z$  plays a fundamental role. Given  $A, B \in 2^Z \setminus \{\emptyset\}$

$$A \leq^l B \text{ if } B \subset A + C$$

Intuitively with  $\leq^l$  the  $C$ -lower part of both subsets are compared. This and other similar relations arise in many areas of mathematics (see [4, 5, 12, 3, 16]) and have been lately studied in set-valued analysis in connection with a new criterion of set-valued optimization (see [13, 6, 7, 8, 9, 2]).

Let  $Z'$  be the topological dual of  $Z$  and  $C^+$  the positive dual cone of  $C$ , i.e.  $C^+ = \{\lambda \in Z' : \lambda(c) \geq 0 \text{ for any } c \in C\}$ . Given  $\lambda \in C^+$ , by  $H_\lambda$  we denote its associated half-space, i.e.  $H_\lambda = \{z \in Z : \lambda(z) \geq 0\}$ . We recall that the effective

domain, the graph, the epigraph of a set-valued map  $F : Y \rightarrow 2^Z$  are defined by

$$\begin{aligned}\text{dom}(F) &= \{y \in Y : F(y) \neq \emptyset\}, \\ \text{graph}(F) &= \{(y, z) \in Y \times Z : z \in F(y)\}, \\ \text{epi}(F) &= \{(y, z) \in Y \times Z : y \in \text{dom}(F), z \in F(y) + C\}.\end{aligned}$$

Given  $G : X \rightarrow 2^Y$ ,  $F : Y \rightarrow 2^Z$  by  $F \circ G : X \rightarrow 2^Z$  we denote the set-valued defined by  $(F \circ G)(x) = \bigcup_{y \in G(x)} F(y)$  (with convention  $F(\emptyset) = \emptyset$ ).

**Definition 2.1.** Let  $M > 0$ .  $F$  is said to be  $M$ -stable at  $(\bar{y}, \bar{z}) \in \text{graph}(F)$  if there exists a neighborhood  $U$  of  $\bar{y}$  such that  $F(y) \subset \{\bar{z}\} + M\|y - \bar{y}\|B(0, 1)$  for any  $y \in U \setminus \{\bar{y}\}$ .

**Definition 2.2.**  $F$  is said to be locally lipschitz at  $\bar{y} \in \text{dom}(F)$  if there exists a neighborhood  $U$  of  $\bar{y}$  and a real constant  $M > 0$  such that  $F(y) \subset F(y') + M\|y - y'\|B(0, 1)$  for any  $y, y' \in U$ .

By  $T(A, z)$  we denote the contingent cone to  $A$  at  $z \in A$ . We recall that the contingent derivative  $D_c F(\bar{y}, \bar{z})$  of  $F$  at  $(\bar{y}, \bar{z}) \in \text{graph}(F)$  is the set-valued map from  $Y$  to  $Z$  defined by  $\text{graph}(D_c F(\bar{y}, \bar{z})) = T(\text{graph}(F), (\bar{y}, \bar{z}))$ , see [1].

**Definition 2.3.** Let  $S \subset X$ . A map  $f : S \rightarrow Y$  is Hadamard directionally differentiable at  $\bar{x} \in S$  in a direction  $u \in T(S, \bar{x})$  if there exists the following limit (with respect to the norm topology)

$$f'_H(\bar{x}, u) = \lim_{u_n \rightarrow u, h_n \rightarrow 0^+} \frac{f(\bar{x} + h_n u_n) - f(\bar{x})}{h_n}.$$

$f$  is said to be Hadamard directionally differentiable at  $\bar{x}$  if  $f$  is Hadamard directionally differentiable at  $\bar{x} \in S$  in every direction  $u \in T(S, \bar{x})$ .

**Definition 2.4.** Let  $(\bar{y}, \bar{z}) \in \text{graph}(F)$  and  $L = \text{dom}(D_c(F + C)(\bar{y}, \bar{z}))$ . A single-valued map  $\varphi : L \rightarrow Y$  whose epigraph coincides with the contingent cone to the epigraph of  $F$  at  $(\bar{y}, \bar{z})$ , i.e.

$$\text{epi}(\varphi) = T(\text{epi}(F), (\bar{y}, \bar{z})),$$

is called a contingent epiderivative of  $F$  at  $(\bar{y}, \bar{z})$  with respect to  $C$ . The set of all these elements is called the family of contingent epiderivative of  $F$  at  $(\bar{y}, \bar{z})$  with respect to  $C$  and it is denoted by  $\Gamma(F, (\bar{y}, \bar{z}), C)$ .

When  $C$  is pointed by  $DF(\bar{y}, \bar{z})$  we denote the unique element of  $\Gamma(F, (\bar{y}, \bar{z}), C)$ . Furthermore if  $Y = \mathbb{R}$ ,  $C = \mathbb{R}_+$  following the notation given in [1] by  $D_\uparrow F(\bar{y}, \bar{z})$

we denote the corresponding epiderivative. In the same context we will denote  $\leq^l$  by  $\leq_{\mathbb{R}_+}^l$ .

As we have mentioned before; in general, epiderivatives have strong conditions in order to exist, contrary to this when the ordering cone is given by a half-space this situation is improved. Epiderivatives with respect to half-spaces have less restrictive existence conditions, indeed stable set-valued maps are in general epidifferentiable with respect to half-spaces; moreover a certain computation formula can be given for these kind of epiderivatives. Let us see this in the following finite dimensional version of [21, Theorem 4.4].

**Theorem 2.5.** *Let  $\lambda \in C^+$  and let  $\bar{z} \in \text{IMin}(F(\bar{y}), H_\lambda)$ . If  $F$  is stable at  $(\bar{y}, \bar{z})$ , then  $\Gamma(F, (\bar{y}, \bar{z}), H_\lambda) \neq \emptyset$ . Moreover, for every  $\varphi \in \Gamma(F, (\bar{y}, \bar{z}), H_\lambda)$  one has*

$$\lambda(\varphi(u)) = D_\uparrow(\lambda \circ F)(\bar{y}, \lambda(\bar{z}))(u) \text{ for any } u \in \text{dom}(D_c(F + H_\lambda)(\bar{y}, \bar{z})).$$

### 3. CHAIN RULE

In [21, Theorem 2.4], if  $g$  is Hadamard directionally differentiable at  $\bar{x} \in \text{int}(S)$  and  $F$  is epidifferentiable and locally lipschitz at  $(g(\bar{x}), \bar{z})$ , then  $F \circ g$  is epidifferentiable at  $(g(\bar{x}), \bar{z})$  and moreover the following chain rule is verified.

$$(3.1) \quad \begin{aligned} \Gamma(F \circ g, (\bar{x}, \bar{z}), C) &= \{\varphi : L^* \rightarrow Z : \varphi(u) \\ &= \varphi_1(g'_H(\bar{x}, u)), \varphi_1 \in \Gamma(F, (g(\bar{x}), \bar{z}), C)\}. \end{aligned}$$

where  $L^* = \text{dom}(D_c(F \circ g)(\bar{x}, \bar{z}))$ ,

In general if  $g$  is not necessarily Hadamard directionally differentiable, so  $D_c g(\bar{x}, g(\bar{x}))$  is not necessarily single-valued,  $F \circ g$  may be not epidifferentiable and of course the chain rule given in (3.1) has not sense. Even if we consider a order structure on  $Y$  given by closed convex cone  $K \subset Y$  and we assume the existence of the corresponding contingent epiderivative  $Dg(\bar{x}, g(\bar{x}))$  we can not assure the epidifferentiability of  $F \circ g$ .

**Example 1.** Let  $X = \mathbb{R}$ ,  $S = \mathbb{R}_-$ ,  $Y = Z = \mathbb{R}^2$ ,  $C = K = \mathbb{R}_+^2$  and let us consider the maps  $g : \mathbb{R}_- \rightarrow \mathbb{R}^2$ ,  $F : \mathbb{R}^2 \rightarrow 2^{\mathbb{R}^2}$  defined by

$$g(r) = \begin{cases} (0, -r) & \text{if } r \in \{-\frac{1}{n}, n \in \mathbb{N}\}, \\ (r, 0) & \text{if } r \notin \{-\frac{1}{n}, n \in \mathbb{N}\}, \end{cases}$$

$$F(x, y) = \{(-x, y)\},$$

Both epiderivatives,  $Dg(0, (0, 0))$  and  $DF((0, 0), (0, 0))$ , exist and they are given by

$$Dg(0, (0, 0))(u) = (u, 0) \text{ for every } u \in \mathbb{R}_-,$$

$$DF((0, 0), (0, 0))(u_1, u_2) = (-u_1, u_2) \text{ for every } u = (u_1, u_2) \in \mathbb{R}^2,$$

but it is easy to check that  $D(F \circ g)((0, 0), (0, 0))$  does not exist. However, in this section we give a chain rule for  $F \circ g$  not in terms of an equality but in terms of the set relation  $\leq^l$ . Therefore there is a loss of exactness in the formulation of the chain rule that is measured by a family of parameters depending on the size of the image sets of  $D_c g(\bar{x}, g(\bar{x}))$ . As a consequence we can recover the results of [21] when  $g$  is Hadamard directionally differentiable. Firstly we need the following technical result. We recall we are denoting  $L^* = \text{dom}(D_c(F \circ g)(\bar{x}, \bar{z}))$ .

**Proposition 3.1.** *Let  $g$  be  $M$ -stable at  $\bar{x} \in \text{int}(S)$  and let  $F$  be locally lipschitz at  $g(\bar{x})$ , then*

$$\begin{aligned} D_c(F \circ g + C)(\bar{x}, \bar{z})(u) &\subset (D_c(F + C)(g(\bar{x}), \bar{z}) \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset \\ &\subset D_c(F \circ g + C)(\bar{x}, \bar{z})(u) + B(0, R_u) \text{ for any } u \in L^*, \end{aligned}$$

where  $R_u = \sup\{\|v - v'\| : v, v' \in D_c g(\bar{x}, g(\bar{x}))(u)\}$ .

*Proof.* Let us prove the first content, let  $(u, v) \in T(\text{epi}(F \circ g), (\bar{x}, \bar{z}))$ , there exist  $(t_n) \subset \mathbb{R}_+$ ,  $(x_n, z_n) \subset \text{graph}(F \circ g)$ ,  $(c_n) \subset C$  such that  $(x_n, z_n + c_n) \rightarrow (\bar{x}, \bar{z})$  and

$$t_n(x_n - \bar{x}, z_n + c_n - \bar{z}) \rightarrow (u, v).$$

Since  $g$  is stable at  $\bar{x}$ , for  $n$  sufficiently large we have that  $t_n(g(x_n) - g(\bar{x})) \in B(0, \|t_n(x_n - \bar{x})\|)$ . Therefore we can assume, by taking subsequences if necessary, that  $t_n(g(x_n) - g(\bar{x}))$  converges to an element  $w$ . Thus

$$t_n(g(x_n) - g(\bar{x}), z_n + c_n - \bar{z}) \rightarrow (w, v),$$

and consequently  $w \in \text{dom}(D_c(F + C)(g(\bar{x}), \bar{z})) \cap D_c g(\bar{x}, g(\bar{x}))(u)$  and

$$v \in D_c(F + C)(g(\bar{x}), \bar{z})(w).$$

Now let us prove the second content. Let  $v \in D_c(F + C)(g(\bar{x}), \bar{z})(w)$  with  $w \in D_c g(\bar{x}, g(\bar{x}))(u)$ , there exist  $(t_n) \subset \mathbb{R}_+$ ,  $(y_n, z_n) \subset \text{graph}(F)$ ,  $(c_n) \subset C$  such that  $(y_n, z_n + c_n) \rightarrow (g(\bar{x}), \bar{z})$  and

$$t_n(y_n - g(\bar{x}), z_n + c_n - \bar{z}) \rightarrow (w, v).$$

As  $g$  is  $M$ -stable there exists  $u_n \rightarrow u$  such that  $x_n := \bar{x} + \frac{1}{t_n}u_n \rightarrow \bar{x}$  and  $\hat{w} \in D_c g(\bar{x}, g(\bar{x}))(u)$  verifying  $\frac{g(\bar{x} + (1/t_n)u_n) - g(\bar{x})}{(1/t_n)} \rightarrow \hat{w}$ .

Since  $F$  is locally lipschitz at  $g(\bar{x})$  and  $y_n - g(x_n) \rightarrow 0$ , for  $n$  big enough we have

$$F(y_n) \subset F(g(x_n)) + M \|y_n - g(x_n)\| B(0, 1)$$

and there exists  $z_n^* \in F(g(x_n))$  such that

$$(3.2) \quad z_n \in z_n^* + M \|y_n - g(x_n)\| B(0, 1).$$

It is easily seen that  $z_n - z_n^* \rightarrow 0$ , hence  $z_n^* + c_n - \bar{z} \rightarrow 0$ . Furthermore

$$(3.3) \quad t_n(y_n - g(x_n)) = t_n(y_n - g(\bar{x})) - t_n(g(x_n) - g(\bar{x})) \rightarrow w - \hat{w}.$$

From (3.2), as  $Y$  is finite dimensional, there exists  $(e_n) \subset B(0, 1)$  such that  $t_n(z_n - z_n^*) = M \|t_n(y_n - g(x_n))\| e_n$ , therefore from (3.3) there exists  $e \in B(0, 1)$  such that, without loss of generality,  $t_n(z_n - z_n^*) \rightarrow \|w - \hat{w}\| e$ . Then

$$\begin{aligned} t_n(z_n^* + c_n - \bar{z}) &= t_n(z_n^* - z_n + z_n + c_n - \bar{z}) \\ &= t_n(z_n^* - z_n) + t_n(z_n + c_n - \bar{z}) \rightarrow \|w - \hat{w}\| (-e) + v. \end{aligned}$$

Hence  $\hat{v} := \|w - \hat{w}\| (-e) + v \in D_c(F \circ g + C)(\bar{x}, \bar{z})(u)$  and consequently

$$D_c(F + C)(g(\bar{x}), \bar{z})(w) \subset D_c(F \circ g + C)(\bar{x}, \bar{z})(u) + \|w - \hat{w}\| e.$$

**Theorem 3.2.** *Let  $g$  be  $M$ -stable at  $\bar{x} \in \text{int}(S)$  and let  $F$  be locally lipschitz at  $g(\bar{x})$ . If  $\Gamma(F, (g(\bar{x}), \bar{z}), C) \neq \emptyset$ ,  $\Gamma(F \circ g, (\bar{x}, \bar{z}), C) \neq \emptyset$  then for every  $\varphi \in \Gamma(F, (g(\bar{x}), \bar{z}), C)$ ,  $\Psi \in \Gamma(F \circ g, (\bar{x}, \bar{z}), C)$  we have*

$$\Psi(u) + R_u \Phi_B \leq^l (\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \leq^l \Psi(u)$$

for any  $u \in L^*$ , where  $R_u = \sup\{\|v - v'\| : v, v' \in D_c g(\bar{x}, g(\bar{x}))(u)\}$ .

*Proof.* Let  $\varphi \in \Gamma(F, (g(\bar{x}), \bar{z}), C)$ ,  $\Psi \in \Gamma(F \circ g, (\bar{x}, \bar{z}), C)$ . By [18, Theorem 3.1]  $\varphi(\cdot) \in \text{IMin}(D_c(F + C)(g(\bar{x}), \bar{z}))(\cdot, C)$  and  $\Psi(\cdot) \in \text{IMin}(D_c(F \circ g + C)(\bar{x}, \bar{z}))(\cdot, C)$

Let  $u \in L^*$  and let us prove that

$$\Psi(u) + R_u \Phi_B \leq^l (\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u).$$

From Proposition 3.1

$$(3.4) \quad \begin{aligned} & (D_c(F + C)(g(\bar{x}), \bar{z}) \circ D_c g(\bar{x}, g(\bar{x}))) (u) \\ & \subset D_c(F \circ g + C)(\bar{x}, \bar{z})(u) + B(0, R_u) \\ & \subset \Psi(u) + R_u \Phi_B + C, \end{aligned}$$

where we have applied that  $B(0, R_u) = R_u B(0, 1) \subset R_u \Phi_B + C$ . Moreover, noticing that

$$\begin{aligned} (\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) + C &= \left( \bigcup_{v \in D_c g(\bar{x}, g(\bar{x}))(u)} \varphi(v) \right) + C \\ &\subset \bigcup_{v \in D_c g(\bar{x}, g(\bar{x}))(u)} D_c(F + C)(g(\bar{x}), \bar{z})(v) \\ &= (D_c(F + C)(g(\bar{x}), \bar{z}) \circ D_c g(\bar{x}, g(\bar{x}))) (u), \end{aligned}$$

from (3.4) we have that

$$(\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset \Psi(u) + R_u \Phi_B + C.$$

Conversely, applying Proposition 3.1

$$\Psi(u) \in D_c(F \circ g + C)(\bar{x}, \bar{z})(u) \subset (D_c(F + C)(g(\bar{x}), \bar{z}) \circ D_c g(\bar{x}, g(\bar{x}))) (u)$$

By an analogous reasoning as before we get

$$\begin{aligned} \Psi(u) \in D_c(F \circ g + C)(\bar{x}, \bar{z})(u) &\subset (D_c(F + C)(g(\bar{x}), \bar{z}) \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset \\ &\subset (\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) + C. \end{aligned}$$

When  $C$  is pointed, and uniqueness holds for contingent epiderivatives, we have the following corollary.

**Corollary 3.3.** *Under the same hypotheses as in Theorem 3.2 and in addition  $C$  pointed, then*

$$D(F \circ g)(\bar{x}, \bar{z})(u) + R_u \Phi_B \leq^l (DF(g(\bar{x}), \bar{z}) \circ D_c g(\bar{x}, g(\bar{x}))) (u) \leq^l D(F \circ g)(\bar{x}, \bar{z})(u)$$

for any  $u \in L^*$ , where  $R_u = \sup\{\|v - v'\| : v, v' \in D_c g(\bar{x}, g(\bar{x}))(u)\}$ .

If  $g$  is Hadamard directionally differentiable at  $\bar{x}$ , as a direct consequence of Theorem 3.2 we assure the epidifferentiability of  $F \circ g$  and, furthermore, the chain rule becomes an equality. That is, we recover a finite dimensional version of Theorem 3.2 of [21].

**Corollary 3.4.** *Let  $g$  be Hadamard directionally differentiable at  $\bar{x} \in \text{int}(S)$  and let  $F$  be locally lipschitz at  $g(\bar{x})$ . If  $\Gamma(F, (g(\bar{x}), \bar{z}), C) \neq \emptyset$ , then:*



- (i)  $\Gamma(F \circ g, (\bar{x}, \bar{z}), C) \neq \emptyset$ .
- (ii)  $\Gamma(F \circ g, (\bar{x}, \bar{z}), C) = \{\varphi : L^* \rightarrow Z : \varphi(u) = \varphi_1(g'_H(\bar{x}, u)), \varphi_1 \in \Gamma(F, (g(\bar{x}), \bar{z}), C)\}$ .

*Proof.* From [21, Proposition 2.4]

$$D_c g(\bar{x}, g(\bar{x}))(u) = \{g'_H(\bar{x}, u)\} \text{ for every } u \in X,$$

therefore  $R_u = 0$  for every  $u \in X$ . Hence from Proposition 3.1

$$D_c(F + C)(g(\bar{x}), \bar{z})(g'_H(\bar{x}, u)) = D_c(F \circ g + C)(\bar{x}, \bar{z})(u) \text{ for every } u \in L^*$$

Then, proof follows by following a similar reasoning as in [21, Theorem 3.2]. ■

#### 4. OPTIMALITY CONDITIONS FOR $(P)$

A well-known result in functional analysis states that a closed convex cone is given by the intersection of half-spaces that contain it. In this section, we exploit this property in order to decouple a set-valued optimization problem into a family of simpler problems. Optimality conditions are given by using families of contingent epiderivatives with respect to half-spaces and the chain rule given in the previous section. First we need the following definition of representation of a cone.

**Definition 4.1.** Let  $\{\lambda_i\}_{i \in I} \subset C^+$ .  $\{H_{\lambda_i}\}_{i \in I}$  is said to be a representation of  $C$  if  $C = \bigcap_{i \in I} H_{\lambda_i}$ .

Let  $\{H_{\lambda_i}\}_{i \in I}$  be any representation of  $C$ , associated with it we have the following family of set-valued optimization problems

$$(P_i)_{i \in I} \equiv (P_{H_{\lambda_i}})_{i \in I} \begin{cases} \text{minimize } F(y) \\ y = g(x), x \in S. \end{cases}$$

**Definition 4.2.** Let  $i \in I$ .  $(\bar{x}, \bar{z}) \in \text{graph}(F \circ g)$  is said to be a strong minimizer of  $(P_i)$  if  $\bar{z} \in \text{IMin}(\bigcup_{x \in S} F(g(x)), H_{\lambda_i})$ .

In [21] strong minimizers  $(P)$  are related to strong minimizers of  $(P_i)_{i \in I}$ , indeed by Proposition 5.3 of that work the following equivalence is easily deduced:

$$(4.1) \quad \begin{aligned} & \text{“}(\bar{x}, \bar{z}) \text{ is a strong minimizer of } (P) \text{ if and only if } (\bar{x}, \bar{z}) \\ & \text{is a strong minimizer of } (P_i) \text{ for every } i \in I \text{”} \end{aligned}$$

As a consequence we have the following result.

**Theorem 4.3.** Assume  $\Gamma(F \circ g, (\bar{x}, \bar{z}), H_{\lambda_i}) \neq \emptyset$  for every  $i \in I$ .

- (a) If  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ , then for every  $i \in I$ ,  $\varphi \in \Gamma(F \circ g, (\bar{x}, \bar{z}), H_{\lambda_i})$ ,  $u \in L_{\lambda_i}$ , we have  $\varphi(u) \in H_{\lambda_i}$ .
- (b) If  $F \circ g$  is  $C$ -convex, and for every  $i \in I$  there exists  $\varphi \in \Gamma(F \circ g, (\bar{x}, \bar{z}), H_{\lambda_i})$  such that  $\varphi(u) \in H_{\lambda_i}$  for every  $u \in L_{\lambda_i}$ , then  $(\bar{x}, \bar{y})$  is a strong minimizer of  $(P)$ .

*Proof.* Results follows from (4.1) and [21, Theorem 5.2]. ■

By applying the chain rule of previous section, we can give a version of previous theorem in terms of contingent epiderivatives of  $F$  with respect to each  $H_{\lambda_i}$ ,  $i \in I$ . We recall that we are denoting

$$R_u = \sup\{\|v - v'\| : v, v' \in D_c g(\bar{x}, g(\bar{x}))(u)\}.$$

**Theorem 4.4.** Let  $\bar{z} \in \text{IMin}(F(g(\bar{x})), C)$ . If  $g$  is  $M$ -stable at  $\bar{x} \in \text{int}(S)$  and  $F$  is locally lipschitz at  $g(\bar{x})$  and stable at  $(g(\bar{x}), \bar{z})$ , the following conditions hold:

- (i) If  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ , then for every  $i \in I$ ,  $\varphi \in \Gamma(F, (g(\bar{x}), \bar{z}), H_{\lambda_i})$ ,  $u \in L_{\lambda_i}$  we have

$$(4.2) \quad (\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset R_u \Phi_B + H_{\lambda_i}.$$

- (ii) If  $F \circ g$  is  $C$ -convex and for every  $i \in I$  there exists  $\varphi \in \Gamma(F, (g(\bar{x}), \bar{z}), H_{\lambda_i})$  such that

$$(4.3) \quad (\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset H_{\lambda_i} \text{ for every } u \in L_{\lambda_i},$$

then  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ .

*Proof.* As  $F$  is stable at  $(g(\bar{x}), \bar{z})$ , clearly  $F \circ g$  is stable at  $(\bar{x}, \bar{z})$ , therefore by Theorem 2.5 (taking in account that  $\bar{z} \in \text{IMin}(F(g(\bar{x})), C) \subset \text{IMin}(F(g(\bar{x})), H_{\lambda_i})$  for every  $i \in I$ )  $\Gamma(F \circ g, (\bar{x}, \bar{z}), H_{\lambda_i}) \neq \emptyset$  and  $\Gamma(F, (g(\bar{x}), \bar{z}), H_{\lambda_i}) \neq \emptyset$  for every  $i \in I$ .

Let any  $i \in I$ ,  $\Psi \in \Gamma(F \circ g, (\bar{x}, \bar{z}), H_{\lambda_i})$ ,  $\varphi \in \Gamma(F, (g(\bar{x}), \bar{z}), H_{\lambda_i})$ ,  $u \in L_{\lambda_i}$ .

- (i) If  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ , then by Theorem 4.3  $\Psi(u) \subset H_{\lambda_i}$ , therefore by Theorem 2.2 (with  $C = H_{\lambda_i}$ ) we have

$$(\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset \Psi(u) + R_u \Phi_B + H_{\lambda_i} \subset R_u \Phi_B + H_{\lambda_i}.$$

- (ii) Conversely, if (4.3) is verified then by Theorem 2.2 the following content is verified

$$\Psi(u) \in (\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset H_{\lambda_i}.$$

From this and Theorem 4.3 we deduce that  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ . ■

**Theorem 4.5.** *Let  $\bar{z} \in \text{IMin}(F(g(\bar{x})), C)$ . If  $g$  is  $M$ -stable at  $\bar{x} \in \text{int}(S)$ , and  $F$  is locally lipschitz at  $g(\bar{x})$  and stable at  $(g(\bar{x}), \bar{y})$ , the following conditions hold:*

(i) *If  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ , then for every  $i \in I$ ,  $u \in L_{\lambda_i}$*

$$(4.4) \quad R_u \lambda_i(\Phi_B) \leq_{\mathbb{R}_+}^l (D_{\uparrow}(\lambda_i \circ F)(g(\bar{x}), \lambda_i(\bar{z})) \circ D_c g(\bar{x}, g(\bar{x}))) (u).$$

(ii) *If  $F \circ g$  is  $C$ -convex and for every  $i \in I$ ,  $u \in L_{\lambda_i}$*

$$(4.5) \quad 0 \leq_{\mathbb{R}_+}^l (D_{\uparrow}(\lambda_i \circ F)(g(\bar{x}), \lambda_i(\bar{z})) \circ D_c g(\bar{x}, g(\bar{x}))) (u),$$

*then  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ .*

*Proof.* Let any  $i \in I$ ,  $\Psi \in \Gamma(F \circ g, (\bar{x}, \bar{z}), H_{\lambda_i})$ ,  $\varphi \in \Gamma(F, (g(\bar{x}), \bar{z}), H_{\lambda_i})$ ,  $u \in L_{\lambda_i}$ .

(i) If  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ , from Theorem 4.4 we have

$$(\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset R_u \Phi_B + H_{\lambda_i},$$

therefore it is straightforward

$$(4.6) \quad (\lambda_i \circ \varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset \lambda_i(R_u \Phi_B) + \lambda_i(H_{\lambda_i}) = \lambda_i(R_u \Phi_B) + \mathbb{R}_+.$$

On the other hand, by Theorem 2.5  $\lambda_i(\varphi(v)) = D_{\uparrow}(\lambda_i \circ F)(g(\bar{x}), \lambda_i(\bar{z}))(v)$ , thus

$$(4.7) \quad (\lambda_i \circ \varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) = (D_{\uparrow}(\lambda_i \circ F)(g(\bar{x}), \lambda_i(\bar{z})) \circ D_c g(\bar{x}, g(\bar{x}))) (u)$$

Consequently by (4.6) and (4.7) we conclude that

$$(D_{\uparrow}(\lambda_i \circ F)(g(\bar{x}), \lambda_i(\bar{z})) \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset \lambda_i(R_u \Phi_B) + \mathbb{R}_+.$$

(ii) Following an analogous reasoning, condition

$$0 \leq_{\mathbb{R}_+}^l (D_{\uparrow}(\lambda_i \circ F)(g(\bar{x}), \lambda_i(\bar{z})) \circ D_c g(\bar{x}, g(\bar{x}))) (u),$$

is equivalent to

$$(\lambda_i \circ \varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset 0 + \mathbb{R}_+,$$

So

$$(\lambda_i \circ \varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset 0 + \mathbb{R}_+ \text{ for every } i \in I$$

and since  $\{H_{\lambda_i}\}_{i \in I}$  is a representation of  $C$ , it yields

$$(\varphi \circ D_c g(\bar{x}, g(\bar{x}))) (u) \subset 0 + C = C.$$

Therefore, by Theorem 4.4,  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ . ■

Following the same reasoning as in the proof of Corollary 3.4, if  $g$  is Hadamard directionally differentiable at  $\bar{x}$ , the optimality conditions (4.4) and (4.5) collapse in condition (1.2), and therefore from the previous theorem it is straightforward to recover [21, Theorem 5.5].

**Theorem 4.6.** *Suppose that  $\bar{z} \in \text{IMin}(F(g(\bar{x})), C)$ . Let  $g$  be Hadamard directionally differentiable at  $\bar{x} \in \text{int}(S)$ , and let  $F$  be locally lipschitz at  $g(\bar{x})$  and stable at  $(g(\bar{x}), \bar{z})$ . If  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$  with respect to  $C$ , then for every  $i \in I$ ,  $u \in L_{\lambda_i}$*

$$(4.8) \quad 0 \leq D_{\uparrow}(\lambda_i \circ F)(\bar{x}, \lambda_i(\bar{z}))(g'_H(\bar{x}, u)).$$

Reciprocally if  $F \circ g$  is  $C$ -convex and (4.8) holds, then  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ . Conditions (4.4) and (4.5) can be explicitly calculated for particular cases, for example in the multiobjective optimization context, i.e. when  $Z = \mathbb{R}^n$ ,  $C = \mathbb{R}_+^n$ . In this case we have the following result.

By  $F_i$  we denote the set-valued map from  $X$  to  $\mathbb{R}$  defined by

$$F_i(x) = \{t \in \mathbb{R} : \exists (z_j)_{j \in \{1, \dots, n\} \setminus \{i\}} \subset \mathbb{R} \text{ such that } (z_1, \dots, t, \dots, z_n) \in F(x)\},$$

furthermore we denote  $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$ .

**Corollary 4.7.** *Let  $\|\cdot\|$  be the euclidean norm (on  $Y$ ). Under the same hypotheses as in Theorem 4.5 and in addition  $C = \mathbb{R}_+^n$ , the following conditions hold:*

(i) *If  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ , then for every  $i \in \{1, \dots, n\}$ ,  $u \in L_{\lambda_i}$*

$$-2M \|u\| \leq_{\mathbb{R}_+}^l (D_{\uparrow} F_i(g(\bar{x}), \bar{z}_i) \circ D_c g(\bar{x}, g(\bar{x}))) (u).$$

(ii) *If  $F \circ g$  is  $C$ -convex and for every  $i \in \{1, \dots, n\}$ ,  $u \in L_{\lambda_i}$*

$$0 \leq_{\mathbb{R}_+}^l (D_{\uparrow} F_i(g(\bar{x}), \bar{z}_i) \circ D_c g(\bar{x}, g(\bar{x}))) (u),$$

*then  $(\bar{x}, \bar{z})$  is a strong minimizer of  $(P)$ .*

*Proof.* By  $\{e_i\}_{i=1}^n$  we denote the canonical basis of  $\mathbb{R}^n$ , i.e.  $e_i = (0, \dots, \underset{i}{1}, \dots, 0)$  and let us denote by  $\langle \cdot, e_i \rangle$  the associated linear map associated with  $e_i$  for any

$i \in \{1, \dots, n\}$ . It is straightforward that  $\{H_{\langle \cdot, e_i \rangle}\}_{i=1, \dots, n}$  is a natural representation of  $C = \mathbb{R}_+^n$

$$C = \bigcap_{i=1, \dots, n} H_{\langle \cdot, e_i \rangle},$$

and furthermore we have  $F_i = \langle \cdot, e_i \rangle \circ F$ ,  $i = 1, \dots, n$ .

On the other hand, since  $g$  is  $M$ -stable at  $\bar{x}$  and  $Y$  is finite dimensional, by [19, Lemma 4.6]

$$(4.9) \quad D_c g(\bar{x}, g(\bar{x}))(u) \subset B(0, M \|u\|),$$

hence from (4.9) it is easily deduced that

$$(4.10) \quad 2M \|u\| \leq R_u \text{ for every } u \in X.$$

Moreover, as we are considering the euclidean norm on  $Y$ , the element  $\Phi_B$  is given by  $(-1, \dots, -1, \dots, -1)$  and clearly

$$(4.11) \quad \langle \cdot, e_i \rangle \circ \Phi_B = -1$$

Taking in account (4.10) and (4.11), finally proof follows from Theorem 4.5. ■

#### REFERENCES

1. J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, 1990.
2. F. Flores-Bazán, E. Hernández and V. Novo, *Characterizing efficiency without linear structure: a unified approach*, J. Global Optim., 2007. Published online. DOI 10.1007/s10898-007-9165-x.
3. T. X. D. Ha, D. Kuroiwa and T. Tanaka, On cone convexity of set-valued maps, *Nonlinear Anal.*, **30** (1997), 1487-1496.
4. T. X. D. Ha, Some variants of the Ekeland variational principle for a set-valued map. *J. Optim. Theory Appl.*, **124** (2005), 187-206.
5. S. Heikkilä, Inclusion problems in ordered topological vector spaces and applications. *J. Math. Anal. Appl.*, **298** (2004), 94-105.
6. E. Hernández and L. Rodríguez-Marín, Nonconvex scalarization in set-optimization with set-valued maps, *J. Math. Anal. Appl.*, **325** (2007), 1-18.
7. E. Hernández and L. Rodríguez-Marín, Existence theorems for set optimization problems, *Nonlinear Analysis*, **67** (2007), 1726-1736.
8. E. Hernández and L. Rodríguez-Marín, Duality in set optimization with set-valued maps, *Pac. J. Optim.*, **3** (2007), 245-255.
9. E. Hernández and L. Rodríguez-Marín, Lagrangian duality in set-valued optimization, *J. Optim. Theory Appl.*, **134(1)** (2007), 119-134.

10. J. Jahn and R. Rauh, Contingent epiderivatives and set valued optimization, *Math. Meth. Oper. Res.*, **46** (1997) 193-211.
11. J. Jahn and A. A. Khan, Some calculus rules for contingent epiderivatives, *Optimization*, **52** (2003), 113-125.
12. J. Jahn, *Vector optimization. Theory, applications, and extensions*. Springer-Verlag, 2004.
13. D. Kuroiwa, On set-valued optimization. Proceedings of the Third World Congress of Nonlinear Analyst. *Nonlinear Anal.*, **47** (2001), 1395-1400.
14. D. Kuroiwa, Existence theorems of set optimization with set-valued maps, *J. Inf. Optim. Sci.*, **24(1)** (2003), 73-84.
15. M. A. Krasnoselskii, J. A. Lifshits and A. V. Sobolev, *Positive linear systems*, Heldermann Verlag, Berlin, 1989.
16. A. Löhne, Optimization with set relations: conjugate duality, *Optimization*, **54(3)** (2005), 265-282.
17. D. T. Luc, *Theory of vector optimization*, Springer-Verlag, 1989.
18. L. Rodríguez-Marín and M. Sama, About contingent epiderivatives, *J. Math. Anal. Appl.*, **327** (2007), 745-762.
19. L. Rodríguez-Marín and M. Sama,  $(\Lambda, C)$ -contingent derivatives of set-valued maps, *J. Math. Anal. Appl.*, **335** (2007), 974-989.
20. L. Rodríguez-Marín and M. Sama, Variational characterization of the contingent epiderivative, *J. Math. Anal. Appl.*, **335** (2007), 1374-1382.
21. L. Rodríguez-Marín and M. Sama,  $\tau^w$ -contingent epiderivatives in reflexive spaces, *Nonlinear Analysis*, **68** (2008), 3780-3788.

Elvira Hernández, Luis Rodríguez-Marín and Miguel Sama  
Departamento de Matematica Aplicada,  
E.T.S.I.I. Universidad Nacional de Educacion a Distancia,  
Calle Juan del Rosal 12,  
28040 Madrid,  
Spain  
E-mail: msama@ind.uned.es