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THE PARAMETER SELECTION PROBLEM FOR MANN'S FIXED POINT ALGORITHM

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Abstract. Mann's fixed point algorithm can be written as a line search method that generates a sequence $\{x_n\}$ through the recursive manner $x_{n+1} = x_n - \alpha_n v_n$, where α_n is the stepsize and where v_n is the search direction given by $v_n = x_n - Tx_n$, with T being a nonexpansive mapping. This line search method has widely been used in optimization, variational inequalities and fixed point problems. In this paper, we address the problem of selection of the sequence of parameters, $\{\alpha_n\}$, so as to have optimal convergence of this algorithm.

1. Introduction

A variational inequality problem (VIP) is formulated as finding a point $x^* \in K$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \quad x \in K \tag{1.1}$$

where K is a closed convex subset of a Hilbert space H (finite or infinite-dimensional) with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and $f : K \to H$ is a mapping. It is known that the VIP (1.1) is equivalent to a fixed point problem (FPP) of finding a point $x^* \in K$ such that

$$Tx^* = x^* \tag{1.2}$$

where T is a self-mapping of K given by

$$T = P_K(I - \lambda f) \tag{1.3}$$

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with $\lambda > 0$ being any positive real number, and P_K being the (nearest point) projection from H onto K; that is, for $x \in H$, $P_K x$ is the only point in K with the property

$$\|x - P_K x\| = \inf\{\|x - y\| : y \in K\}. \tag{1.4}$$

It is known that if $f : K \to H$ is strongly monotone (i.e., there exists a constant $\gamma > 0$ such that $\langle f(x) - f(y), x - y\rangle \geq \gamma\|x - y\|^2$ for all $x, y \in K$) and if f is also Lipschitzian (i.e., there exists a constant L such that $\|f(x) - f(y)\| \leq L\|x - y\|$ for all $x, y \in K$), and if $\lambda > 0$ is small enough (precisely, $0 < \lambda < 2\gamma/L^2$), then the mapping T given by (1.3) is a contraction:

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad x, y \in K \tag{1.5}$$

where $\alpha = \sqrt{1 - \lambda(2\gamma - \lambda L^2)} < 1$. Hence, Banach's contraction principle guarantees that T has a unique fixed point, which is also the unique solution of the VIP (1.1), and for each $x \in K$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to this unique solution.

This argument however fails if f is either not strongly monotone or non-Lipschitzian because the mapping T would not be a contraction; instead, T would be nonexpansive; that is,

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in K. \tag{1.6}$$

Thus iterative methods for finding fixed points of nonexpansive mappings are needed, among which is Mann's fixed point algorithm [7] which, starting with an arbitrary $x_0 \in C$, generates a sequence $\{x_n\}$ via the recursive manner:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n = 0, 1, \cdots, \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in the interval $[0, 1]$. This algorithm has extensively been investigated (see [4, 5, 6, 9, 10, 11] and the references therein).

In this paper we will look at Mann's fixed point algorithm (1.7) from another angle. More precisely, we rewrite (1.7) in the form:

$$x_{n+1} = x_n - \alpha_n v_n, \quad v_n = x_n - T x_n. \tag{1.8}$$

Hence, Mann's algorithm (1.7) can indeed be viewed as a line search algorithm. The problem to be addressed in this paper is the optimal parameter selection problem. In other words, we try to select the sequence of parameters, $\{\alpha_n\}$, so as to have optimal convergence of the sequence $\{x_n\}$. This problem remains open. We will however provide with some partial answers.

2. Some Properties and Convergence Results for Mann's Fixed Point Algorithm

Let K be a nonempty closed convex subset of a Hilbert space H and let $T : K \to K$ be a nonexpansive mapping such that the set of fixed points of T, $\text{Fix}(T) = \{x \in K : Tx = x\}$, is nonempty.

Recall that Mann's fixed point algorithm generates a sequence $\{x_n\}$ via the recursive way:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, \cdots, \tag{2.1}$$

where the initial guess $x_0 \in K$ is arbitrary and the sequence of parameters, $\{\alpha_n\}$, is a sequence in the interval $[0, 1]$. We say that a sequence $\{x_n\}$ is a Mann's sequence (defined by the parameter sequence $\{\alpha_n\}$) if it is generated by Mann's algorithm (2.1). Below we discuss some properties of Mann's sequences.

Proposition 2.1. *Let $\{x_n\}$ be a Mann sequence. Then the sequence $\{\|x_n - Tx_n\|\}$ is decreasing. In particular,* $\lim_{n\to\infty} \|x_n - Tx_n\|$ *exists.*

Proof. We have

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &= \|(1 - \alpha_n)(x_n - Tx_{n+1}) + \alpha_n(Tx_n - Tx_{n+1})\| \\
&\leq (1 - \alpha_n)\|x_n - Tx_{n+1}\| + \alpha_n\|Tx_n - Tx_{n+1}\| \\
&\leq (1 - \alpha_n)(\|x_n - Tx_n\| + \|Tx_n - Tx_{n+1}\|) \\
&\quad +\alpha_n\|x_n - x_{n+1}\| \\
&\leq (1 - \alpha_n)\|x_n - Tx_n\| + \|x_n - x_{n+1}\| \\
&= (1 - \alpha_n)\|x_n - Tx_n\| + \alpha_n\|x_n - Tx_n\| \\
&= \|x_n - Tx_n\|.
\end{aligned}$$

Let $\{x_n\}$ be a Mann sequence. For $x^* \in \text{Fix}(T)$ and $n \geq 1$ such that $x_n \neq Tx_n$, define $A_n x^*$ ([8]) by

$$A_n x^* = \frac{\|x_n - x^*\|^2 - \|Tx_n - x^*\|^2}{\|x_n - Tx_n\|^2} + 1 - \alpha_n. \tag{2.1}$$

Note that, since T is nonexpansive, it is always true that $A_n x^* \geq 0$ for all $x^* \in \text{Fix}(T)$ and $n \geq 1$.

Lemma 2.2.

(i) *There holds the identity*

$$\|x_{n+1} - x^*\|^2 = \|x_n - x^*\|^2 - a_n A_n x^* \|x_n - Tx_n\|^2. \tag{2.1}$$

(ii) *The sequence* $\{\|x_n - x^*\|\}$ *is decreasing; in particular,* $\lim_{n\to\infty} \|x_n - x^*\|$ *exists.*

(iii) $\lim_{n\to\infty} a_n A_n x^* \|x_n - Tx_n\|^2 = 0$.

Proof. (i) We compute

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Tx_n - x^*)\|^2 \\ &= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|Tx_n - x^*\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - x^*\|^2 - \alpha_n(\|x_n - x^*\|^2 - \|Tx_n - x^*\|^2 \\ &\quad +(1-\alpha_n)\|x_n - Tx_n\|^2) \\ &= \|x_n - x^*\|^2 - a_n A_n x^* \|x_n - Tx_n\|^2.\end{aligned}$$

(ii) and (iii) follow from (i).

Lemma 2.3. (Demiclosedness Principle [3]). $I - T$ *is demiclosed in the sense that whenever* $\{u_n\}$ *is a sequence in* K *such that* $u_n \to u_\infty$ *weakly and* $u_n - Tu_n \to 0$ *strongly, it follows that* $u_\infty = Tu_\infty$.

One of the fundamental convergence results on Mann's fixed point algorithm is the following.

Theorem 2.4. *A Mann sequence* $\{x_n\}$ *converges at least weakly to a fixed point of* T *provided the condition*

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty \tag{2.4}$$

is satisfied.

Theorem 2.5. *Let* $\{x_n\}$ *be a Mann sequence. Assume the following condition is satisfied*

(A) *For any fixed point* x^* *of* T*, if* $\{\alpha_n A x^*\}$ *converges to zero, then every weak limit point of* $\{x_n\}$ *is a fixed point of* T*.*

Then $\{x_n\}$ *converges weakly to a fixed point of* T*.*

Proof. It suffices to prove that the weak limit point set of the sequence $\{x_n\}$, $\omega_w(x_n) \subset \text{Fix}(T)$ which together with the facts that $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in \text{Fix}(T)$ and the Opial property of a Hilbert space implies the weak convergence of $\{x_n\}$. To see $\omega_w(x_n) \subset \mathrm{Fix}(T)$, let $\delta = \lim_{n\to\infty} \|x_n - Tx_n\|$. Then, if $\delta = 0$, Lemma 2.3 implies that $\omega_w(x_n) \subset \mathrm{Fix}(T)$. If $\delta > 0$, then by Lemma 2.2(iii), $\lim_{n\to\infty} a_n A_n x^* = 0$. Hence, condition (A) implies that $\omega_w(x_n) \subset \mathrm{Fix}(T)$. ■

The following result shows that condition (A) is weaker than condition (2.4).

Theorem 2.6. *Assume that $\{x_n\}$ is a Mann sequence where the sequence of parameters, $\{\alpha_n\}$, satisfies the condition* (2.4). *Then condition (A) is satisfied. Hence $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. It suffices to show that condition (A) implies that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ which in turns implies that $\omega_w(x_n) \subset \mathrm{Fix}(T)$.

Let $\delta = \lim_{n\to\infty} \|x_n - Tx_n\|$. If $\delta > 0$, then $\|x_n - Tx_n\| \geq \delta$ for all n. Hence, by (2.3), we get

$$\sum_{n=0}^{\infty} \alpha_n A x^* < \infty. \tag{2.5}$$

However, $\alpha_n A x^* \geq \alpha_n(1 - \alpha_n)$ for all n. Relation (2.5) contradicts (2.4). ■

3. Potential Optimization Method

Suppose in Mann's algorithm, the nth iterate x_n has been constructed. We then select the stepsize α_n and define the $(n + 1)$th iterate x_{n+1} by

$$x_{n+1} = x_n - \alpha_n(x_n - Tx_n). \tag{3.1}$$

We hope to select such an α_n so that $\lim_{n\to\infty} \alpha_n A_n x^* = 0$ and also condition (A) is satisfied; hence by Theorem 2.5, the Mann sequence $\{x_n\}$ constructed converges at least weakly to a fixed point of T. We select α_n by solving the following one-dimensional optimization problem:

$$\alpha_n = \arg\min_{\alpha\in S} g^{x_n}(\alpha), \tag{3.2}$$

where S is some closed subset of the interval [0,1] and g is some continuous function defined over S.

We use the notation: $x(\alpha) := x - \alpha(x - Tx)$, where $x \in H$ and $\alpha \in [0, 1]$.

Lemma 3.1. *For all $\alpha \in [0, 1]$ and n, we have*

$$\|x_n(\alpha) - Tx_n(\alpha)\| \leq \|x_n - Tx_n\|. \tag{3.3}$$

Proof. Since $x_n(\alpha) := (1-\alpha)x_n + \alpha Tx_n$, we compute

$$\begin{aligned} \|x_n(\alpha) - Tx_n(\alpha)\| &\le (1-\alpha)\|x_n - Tx_n(\alpha)\| + \alpha\|Tx_n - Tx_n(\alpha)\| \\ &\le (1-\alpha)(\|x_n - Tx_n\| + \|Tx_n - Tx_n(\alpha)\|) \\ &\qquad +\alpha\|Tx_n - Tx_n(\alpha)\| \\ &\le (1-\alpha)\|x_n - Tx_n\| + \|x_n - x_n(\alpha)\| \\ &= (1-\alpha)\|x_n - Tx_n\| + \alpha\|x_n - Tx_n\| \\ &= \|x_n - Tx_n\|. \end{aligned}$$

We now consider two potential functions [8] as follows:

$$\begin{aligned} g_1^x(\alpha) &= \|x(\alpha) - Tx(\alpha)\|^2 - \beta\alpha^2\|x - Tx\|^2, \\ g_2^x(\alpha) &= \|x(\alpha) - Tx(\alpha)\|^2 - \beta\alpha(1-\alpha)\|x - Tx\|^2, \end{aligned}$$

where $\beta > 0$ is a parameter.

Lemma 3.2. *Fix $c \in (0,1)$ and choose $\alpha_n \in [0,1]$ such that*

$$g_1^{x_n}(\alpha_n) = \min_{0\le\alpha\le c} g_1^{x_n}(\alpha). \tag{3.4}$$

Then

$$\|x_{n+1} - Tx_{n+1}\|^2 \le [1 - \beta(c^2 - \alpha_n^2)]\|x_n - Tx_n\|^2. \tag{3.5}$$

If $\alpha_n \in [0,1]$ is chosen such that

$$g_2^{x_n}(\alpha_n) = \min_{0\le\alpha\le 1} g_2^{x_n}(\alpha), \tag{3.6}$$

then

$$\|x_{n+1} - Tx_{n+1}\|^2 \le \left[1 - \beta(\frac{1}{2} - \alpha_n)^2\right]\|x_n - Tx_n\|^2. \tag{3.7}$$

Proof. First observe that $x_n(0) = x_n$, $x_n(1) = Tx_n$, and $x_n(\alpha_n) = x_{n+1}$.

(i) The relation $g_1^{x_n}(\alpha_n) \le g_1^{x_n}(c)$ implies that

$$\begin{aligned} &\|x_{n+1} - Tx_{n+1}\|^2 - \beta\alpha_n^2\|x_n - Tx_n\|^2 \\ &\le \|x_n(c) - Tx_n(c)\|^2 - \beta c^2\|x_n - Tx_n\|^2. \end{aligned}$$

This together with Lemma 3.1 implies

$$\|x_{n+1} - Tx_{n+1}\|^2 \le [1 - \beta(c^2 - \alpha_n^2)]\|x_n - Tx_n\|^2.$$

(ii) Noticing the fact that $\max_{0\le\alpha\le1}\alpha(1-\alpha)=\frac{1}{4}$, we have that the relation $g_2^{x_n}(\alpha_n)\le g_2^{x_n}(c)$ implies that, for any $\alpha\in[0,1]$,

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\|^2 &\le \beta\alpha_n(1-\alpha_n)\|x_n-Tx_n\|^2\\ &\qquad +\|x_n(\alpha)-Tx_n(\alpha)\|^2-\beta\alpha(1-\alpha)\|x_n-Tx_n\|^2\\ &\le [(1-\beta(\frac{1}{4}-\alpha_n(1-\alpha_n))^2]\|x_n-Tx_n\|^2\\ &= \left[1-\beta(\frac{1}{2}-\alpha_n)^2\right]\|x_n-Tx_n\|^2.\end{aligned}$$

Theorem 3.3. *Let $\{x_n\}$ be a Mann sequence. Suppose that the sequence of parameters, $\{\alpha_n\}$, is chosen according to either* (3.4) *or* (3.6). *Then $\{x_n\}$ is convergent at least weakly to a fixed point of T.*

Proof. By Theorem 2.5, all we need to prove is that condition (A) is satisfied. That is, if $x^*\in\mathrm{Fix}(T)$ is such that $\alpha_nA_nx^*\to0$, then we must prove that every weak limit point of $\{x_n\}$ is a fixed point of T. In other words, if $x_{n_j}\to z$ weakly, then $Tz=z$. As a matter of fact, since the sequence $\{\|x_n-Tx_n\|\}$ is decreasing, $\lim_{n\to\infty}\|x_n-Tx_n\|=:\delta$ always exists. If $\delta=0$, then we are done. Assume next $\delta>0$. We then distinguish two cases and we will find a contradiction in either case. First observe

$$\alpha_nA_nx^*=\frac{\alpha_n(\|x_n-x^*\|^2-\|Tx_n-x^*\|^2)}{\|x_n-Tx_n\|^2}+\alpha_n(1-\alpha_n).$$

Since both terms on the right side of the above equation are nonnegative, we find that the assumption $\alpha_nA_nx^*\to0$ must imply that

$$\alpha_n(1-\alpha_n)\to0. \tag{3.8}$$

Case 1. By (3.5), we derive that the series

$$\sum_{n=1}^{\infty}(c^2-\alpha_n^2)\|x_n-Tx_n\|^2<\infty.$$

Since $\|x_n-Tx_n\|^2\to\delta^2>0$, we must have that $\alpha_n\to c\in(0,1)$. This contradicts (3.8).

Case 2. By (3.7), we see that

$$\sum_{n=1}^{\infty}(\frac{1}{2}-\alpha_n)^2\|x_n-Tx_n\|^2<\infty.$$

Again since $\|x_n - Tx_n\|^2 \to \delta^2 > 0$, we must have that $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$, which again contradicts (3.8).

Can the choice of $\{\alpha_n\}$ via (3.4) or (3.6) improve the convergence of the Mann sequence $\{x_n\}$? The answer is yes if the space is finite-dimensional, but unknown if the space is infinite-dimensional.

This however can improve the convergence to zero of the sequence $\{\|x_n - Tx_n\|\}$. Indeed, setting $\gamma_n = \beta(c^2 - \alpha_n^2)$ or $\beta(\frac{1}{2} - \alpha_n)^2$, then in either case we have

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\|^2 &\le (1-\gamma_n)\|x_n - Tx_n\|^2 \\ &\le \exp(-\gamma_n)\|x_n - Tx_n\|^2 \\ &\vdots \\ &\le \exp\left(-\sum_{j=0}^{n} \gamma_j\right)\|x_0 - Tx_0\|^2.\end{aligned}$$

4. An Application to VIP

The VIP

$$\langle f(x^*), x - x^* \rangle \ge 0, \quad x \in K, \tag{4.1}$$

where K is a closed convex subset of a Hilbert space H, is equivalent to the FPP

$$Tx^* = x^* \tag{4.2}$$

where T is a self-mapping of K given by

$$T = P_K(I - \lambda f) \tag{4.3}$$

with $\lambda > 0$ a positive real number, and P_K the (nearest point) projection from H onto K.

If f is γ-strongly monotone and L-Lipschitzian, then for $0 < \lambda < 2\gamma/L^2$, the mapping T defined by (4.3) is a contraction. Thus, for any x_0, the sequence $\{T^n x_0\}$ converges strongly to the unique solution of the VIP (4.1).

Fukushima [1] considered the following potential

$$g_3^x(\alpha) = g_3(x(\alpha)) = -\langle f(x(\alpha), Tx(\alpha) - x(\alpha)\rangle - \frac{1}{2}\|Tx(\alpha) - x(\alpha)\|^2. \tag{4.4}$$

The criteria to select the control sequence $\{\alpha_n\}$ is

$$\alpha_n = \arg\min_{\alpha\in[0,1]} g_3^{x_n}(\alpha). \tag{4.5}$$

Define $\{x_n\}$ as a Mann sequence by

$$x_{n+1} = x_n + \alpha_n(Tx_n - x_n).$$

In [1] Fukushima assumed that $f : H \to H$ is continuously differentiable and the gradient $\nabla f(x)$ is positive definite for all $x \in K$. Under these conditions, he was able to show that each direction $Tx_n - x_n$ satisfies the descent condition:

$$\langle \nabla g_3(x_n), Tx_n - x_n \rangle < 0.$$

As a result, the sequence $\{\|Tx_n - x_n\|\}$ is decreasing.

Fukushima then was able to prove that, under the additional condition that the set K is compact convex, the Mann sequence converges to a solution of the VIP (4.1).

We now briefly look at the VIP (4.1) in the case where f is an ν-inversely strongly monotone (ν-ism, for short); this is,

$$\langle f(x) - f(y), x - y \rangle \geq \nu \|f(x) - f(y)\|^2 \tag{4.6}$$

for all $x, y \in K$ and some $\nu > 0$ (f is not necessarily Lipschitzian). This class of monotone operators are introduced due to their applications in transportation networks (see, for instance, [2] for more details).

Again we convert this VIP to its equivalence FPP (4.2) with T given by (4.3). It can be shown that if $0 < \lambda < 2\nu$, then the mapping $T = P_K(I - \lambda f)$ is nonexpansive. In fact, since the projection operator P_K is nonexpansive, we get by (4.6),

$$\begin{aligned}
\|Tx - Ty\|^2 &= \|P_K(I - \lambda f)x - P_K(I - \lambda f)y\|^2 \\
&\leq \|(I - \lambda f)x - (I - \lambda f)y\|^2 \\
&= \|(x - y) - \lambda[f(x) - f(y)]\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, f(x) - f(y)\rangle + \lambda^2\|f(x) - f(y)\|^2 \\
&\leq \|x - y\|^2 - \lambda(2\nu - \lambda)\|f(x) - f(y)\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Thus, our results presented in Section 3 are applicable. In particular, we have the following result.

Theorem 4.1. *Consider the VIP* (4.1) *and its FPP equivalence* (4.2) *with* T *given by* (4.3). *Given* $x_0 \in K$. *Define* $\{x_n\}$ *by Mann's fixed point algorithm*

$$x_{n+1} = x_n - \alpha_n v_n, \quad v_n = x_n - Tx_n, \quad n \geq 0$$

where the sequence of parameters, $\{\alpha_n\}$, *is selected by the potential optimization method* (3.4) *or* (3.6). *Then* $\{x_n\}$ *converges weakly to a solution of the VIP* (4.1).

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