TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 6, pp. 1543-1559, September 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

CONVERGENCE RATES FOR ERGODIC THEOREMS OF KIDO-TAKAHASHI TYPE

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. Let $\{T(t); t \geq 0\}$ be a uniformly bounded (C_0)-semigroup of operators on a Banach space X with generator A such that all orbits are relatively weakly compact. Let $\{\phi_\alpha\}$ and $\{\psi_\alpha\}$ be two nets of continuous linear functionals on the space $C_b[0,\infty)$ of all bounded continuous functions on $[0,\infty)$. $\{\phi_\alpha\}$ and $\{\psi_\alpha\}$ determine two nets $\{A_\alpha\}$, $\{B_\alpha\}$ of operators satisfying $\langle A_\alpha x, x^*\rangle = \phi_\alpha(\langle T(\cdot)x, x^*\rangle)$ and $\langle B_\alpha x, x^*\rangle = \psi_\alpha(\langle T(\cdot)x, x^*\rangle)$ for all $x \in X$ and $x^* \in X^*$. Under suitable conditions on $\{\phi_\alpha\}$ and $\{\psi_\alpha\}$, this paper discusses: 1) the convergence of $\{A_\alpha\}$ and $\{B_\alpha\}$ in operator norm; 2) rates of convergence of $\{A_\alpha x\}$ and $\{A_\alpha y\}$ for each $x \in X$ and $y \in R(A)$.

1. Introduction

Throughout this paper we assume that X is a real Banach space with norm $\|\cdot\|$, and denote by X^* its dual space and by $B(X)$ the Banach algebra of all bounded linear operators on X. A semigroup S is called a semitopological semigroup if S is a Hausdorff space and for every $a \in S$, the mappings $s \to sa$ and $s \to as$ of S into itself are continuous. Let $C_b(S)$ (resp. $C_{ub}(S)$) denote the Banach space of all continuous (resp. uniformly continuous) bounded real-valued functions on S with the supremum norm. A linear functional $\mu \in C_b(S)^*$ on $C_b(S)$ is called a *mean or normalized state* on $C_b(S)$ if $\mu(1_S) = \|\mu\| = 1$. It is known that $\mu \in C_b(S)^*$ is a mean on $C_b(S)$ if and only if $\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$ for all $f \in C_b(S)$ (cf. [11, Theorem 1.4.1]). For $a \in S$ let l_a and r_a denote the contractions on $C_b(S)$ defined by $(l_a f)(s) := f(as)$ and $(r_a f)(s) := f(sa)$, respectively. Then

Received March 26, 2008.
2000 *Mathematics Subject Classification*: 47A35, 47A58, 41A25, 47D06, 47D09.
Key words and phrases: Strongly regular net of linear functionals, A-ergodic net, Mean ergodic theorem, (C_0)-semigroup.
Research supported in part by the National Science Council of Taiwan.

$l_a^*, r_a^* \in B(C_b(S)^*)$ and $l_a^*\phi, r_a^*\phi \in C_b(S)^*$ for $\phi \in C_b(S)^*$. Moreover, if μ is a mean on $C_b(S)$, then $l_a^*\mu, r_a^*\mu$ are also means on $C_b(S)$.

Let S be a semitopological semigroup with the identity e and let $\mathcal{S} := \{T(s); s \in S\} \subset B(X)$ be a uniformly bounded semigroup of operators satisfying the following conditions:

$(S1)$ $T(s)T(t) = T(st)$ for all $s, t \in S$ and $T(e) = I$ (the identity operator);

$(S2)$ for every $x \in X$ and $x^* \in X^*$, the function $s \to \langle T(s)x, x^* \rangle$ is continuous;

$(S3)$ for every $x \in X$, the orbit $\mathcal{S}x := \{T(s)x; s \in S\}$ is relatively weakly compact in X.

In particular, condition (S3) always holds for uniformly bounded semigroups on reflexive spaces.

It is known [2, 3] that for a mean μ on $C_b(S)$ there exists a unique operator $A_\mu \in B(X)$ such that $\langle A_\mu x, x^* \rangle = \mu(\langle T(\cdot)x, x^* \rangle)$ for all $x \in X$ and $x^* \in X^*$. In [3, Theorem 2], Kido and Takahashi prove the following mean ergodic theorem for a net $\{A_{\mu_\alpha}\}$ of operators defined by a net $\{\mu_\alpha\}$ of means.

Theorem 1.1. *If $\mathcal{S}$ is a uniformly bounded semigroup satisfying (S1)-(S3), and if $\{\mu_\alpha\}$ is a net of means on $C_b(S)$ such that $w^*\text{-}\lim_\alpha (l_t^*\mu_\alpha - \mu_\alpha) = 0$ and $\lim_\alpha \|r_t^*\mu_\alpha - \mu_\alpha\| = 0$ in $C_b(S)^*$ for all $t \in S$, then the net $\{A_\alpha\}$ $(A_\alpha := A_{\mu_\alpha})$ converges strongly to a linear projection P on X with range $R(P) = F(\mathcal{S}) := \bigcap_{s \in S} N(T(s) - I)$, null space $N(P) = \overline{\sum_{s \in S} R(T(s) - I)}$, and domain $D(P) = X = F(\mathcal{S}) \oplus \overline{\sum_{s \in S} R(T(s) - I)}$.*

It will be seen that under the above conditions on $\{\phi_\alpha\}$ in Theorem 1.1, the net $\{A_\alpha\}$ becomes an $\mathcal{A}$-ergodic net for $\mathcal{A} = \{\mathcal{T} - \mathcal{I}; \mathcal{T} \in \mathcal{S}\}$. We first recall two definitions concerning $\mathcal{A}$-ergodic net.

Definition 1.2. Given a family $\mathcal{A}$ of closed linear operators in X, a net $\{A_\alpha\}$ in $B(X)$ is called an *$\mathcal{A}$-ergodic net* if the following conditions are satisfied:

(a) There is an $M > 0$ such that $\|A_\alpha\| \le M$ for all α;

(b) $\|(A_\alpha - I)x\| \to 0$ for all $x \in \bigcap_{A \in \mathcal{A}} N(A)$, and $R(A_\alpha - I) \subset \overline{\sum_{A \in \mathcal{A}} R(A)}$ eventually;

(c) for every $A \in \mathcal{A}$, $R(A_\alpha) \subset D(A)$ and w-$\lim_\alpha AA_\alpha x = 0$ for all $x \in X$, and $\lim_\alpha \|A_\alpha Ax\| = 0$ for all $x \in D(A)$.

When $\mathcal{A} = \{T - I; T \in \mathcal{S}\}$ for some semigroup $\mathcal{S} \subset B(X)$, $\{A_\alpha\}$ becomes a *right, weakly left $\mathcal{S}$-ergodic net* as defined in [4, p. 75], which was first studied by Eberlein [1]. The special case that $\mathcal{A}$ consists of a single closed operator A and with additional conditions has been studied in [7, 8, 9, 10] to establish general strong ergodic theorem, uniform ergodic theorem, and ergodic theorems with rates.

Definition 1.3. Let $A : D(A) \subset X \to X$ be a closed linear operator, and let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets in $B(X)$ satisfying:

$(C1)$ $\|A_\alpha\| \le M$ for all α;

$(C2)$ $R(B_\alpha) \subset D(A)$ and $B_\alpha A \subset AB_\alpha = I - A_\alpha$ for all α;

$(C3)$ $R(A_\alpha) \subset D(A)$ and $A_\alpha A \subset AA_\alpha$ for all α, and $\|AA_\alpha\| = O(e(\alpha))$;

$(C4)$ $B_\alpha^* x^* = \varphi(\alpha)x^*$ for all $x^* \in R(A)^\perp$, and $|\varphi(\alpha)| \to \infty$;

$(C5)$ $\|A_\alpha x\| = O(f(\alpha))$ $(resp.\ o(f(\alpha)))$ implies $\|B_\alpha x\| = O(\frac{f(\alpha)}{e(\alpha)})$ (resp. $o(\frac{f(\alpha)}{e(\alpha)})$). Here e and f are positive functions satisfying $0 < e(\alpha) \le f(\alpha) \to 0$. They are used as estimators of convergence rates.

Then we call $\{A_\alpha\}$ a *uniform A-ergodic net* and $\{B_\alpha\}$ its *companion net*.

The purpose of this paper is to apply our earlier results on $\mathcal{A}$-ergodic nets to deduce a generalization of Theorem 1.1 for a net $\{\phi_\alpha\} \subset C_b(S)^*$ (Theorem 3.1), and, under suitable stronger conditions on $\{\phi_\alpha\}$, to deduce a convergence theorem (Theorem 3.6) for approximate solutions of $Ax = y$, a uniform ergodic theorem (Theorem 3.7), and a strong ergodic theorem (Theorem 3.8) with rates for C_0-semigroups.

The main results will be given in Section 3. Before that, some related definitions and notations as well as abstract mean ergodic theorems for $\mathcal{A}$-ergodic nets which we need will be recalled in Section 2. Finally, applications to some examples of nets of means will be given in Section 4 for illustration.

2. Preliminaries

We need the following lemma.

Lemma 2.1. *Let $f : S \to X$ be a bounded continuous function such that $f(S)$ is relatively weakly compact in X.*

(i) *For any mean μ on $C_b(S)$, there exists a unique $z_{f,\mu} \in X$ such that $z_{f,\mu} \in \overline{\text{co}} f(S)$, $\langle z_{f,\mu}, x^*\rangle = \mu(\langle f(\cdot), x^*\rangle)$ for all $x^* \in X^*$, and $\|z_{f,\mu}\| \le ||f||_\infty$.*

(ii) *For any $\phi \in C_b(S)^*$, there exists a unique $z_{f,\phi} \in X$ such that $z_{f,\phi} \in \|\phi\|\overline{\text{co}}(f(S) \cup (-f(S)))$, $\langle z_{f,\phi}, x^*\rangle = \phi(\langle f(\cdot), x^*\rangle)$ for all $x^* \in X^*$, and $\|z_{f,\phi}\| \le \|\phi\|||f||_\infty$.*

Proof.

(i) Can be found in [3]. For convenience and completeness, we give a proof here. The linear functional $z_{f,\mu}$ defined on X^* by $z_{f,\mu}(x^*) := \mu(\langle f(\cdot), x^*\rangle), x^* \in X^*$, is continuous, i.e., $z_{f,\mu} \in X^{**}$, and

$$\|z_{f,\mu}\| \leq \|\mu\| \sup\{\|f(s)\|; s \in S\} = \sup\{\|f(s)\|; s \in S\} = \|f\|_\infty.$$

We show that $z_{f,\mu} \in X$. Since $f(S)$ is relatively weakly compact, the strongly and weakly closed set $\overline{\text{co}}\{\text{w-cl}f(S)\}$ is a weakly compact subset of X, and so the strongly and weakly closed subset $\overline{\text{co}}f(S)$ is also a weakly compact subset of X. This subset of X can also be written as $\sigma(X^{**}, X^*)\text{-cl}\,(\text{co}f(S))$ when considered as a subset of X^{**}. It remains to show that $z_{f,\mu} \in \sigma(X^{**}, X^*)\text{-cl}\,(\text{co}f(S))$. If it is not, then by the Hahn-Banach separation theorem and the property of a mean, there would exist an $x^* \in X^*$ such that

$$\begin{aligned} z_{f,\mu}(x^*) &< \inf\{\langle x^{**}, x^*\rangle; x^{**} \in \sigma(X^{**}, X^*)\text{-cl}\,(\text{co}f(S))\} \\ &\leq \inf\{\langle f(s), x^*\rangle; s \in S\} \\ &\leq \mu(\langle f(\cdot), x^*\rangle) = z_{f,\mu}(x^*). \end{aligned}$$

This is a contradiction. Thus such $z_{f,\mu}$ belongs to X. Since $\langle z_{f,\mu}, x^*\rangle = \mu(\langle f(\cdot), x^*\rangle)$ for all $x^* \in X^*$, clearly $z_{f,\mu}$ is uniquely determined by μ and f.

(ii) By part (i), we see that the map $\mu \to z_{f,\mu}$ is linear. Let $\phi \in C_b(S)^*$ be arbitrary. If $\phi = 0$, the result is obvious. So, we assume $\phi \neq 0$. If ϕ is positive, then $||\phi|| = \phi(\mathbf{1})$, so $\mu := \frac{\phi}{\phi(\mathbf{1})}$ is a mean on S and $z_{f,\phi} = \phi(\mathbf{1}) z_{f,\mu} \in ||\phi||\overline{co}(f(S))$.

Now, if ϕ is arbitrary, then $\phi = \phi^+ - \phi^-$, where ϕ^+ and ϕ^- are the positive part and negative part of ϕ, respectively. Since $||\phi|| = ||\phi^+|| + ||\phi^-||$, we have

$$\begin{aligned} z_{f,\phi} &= z_{f,\phi^+} - z_{f,\phi^-} \in ||\phi^+||\overline{co}(f(S)) - ||\phi^-||\overline{co}(f(S)) \\ &= (||\phi^+|| + ||\phi^-||)[\alpha\overline{co}(f(S)) + \beta\overline{co}(-f(S))] \\ &\subset ||\phi||\overline{co}(\overline{co}(f(S)) \bigcup \overline{co}(-f(S)))) \\ &= ||\phi||\overline{co}(f(S) \bigcup (-f(S))), \end{aligned}$$

where $\alpha := \frac{||\phi^+||}{||\phi||}$ and $\beta := \frac{||\phi^-||}{||\phi||}$. This also implies $\|z_{f,\phi}\| \leq \|\phi\| ||f||_\infty$.

Corollary 2.2. *Let $\mathcal{S} := \{T(s); s \in S\} \subset B(X)$ be a uniformly bounded semigroup satisfying (S1)-(S3).*

(i) *For any mean* μ *on* $C_b(S)$*, there exists a unique operator* $A_\mu \in B(X)$ *such that* $A_\mu x \in \overline{\text{co}}(\mathcal{S}x)$*,* $\langle A_\mu x, x^* \rangle = \mu(\langle T(\cdot)x, x^* \rangle)$ *for all* $x \in X$ *and* $x^* \in X^*$*, and* $\|A_\mu\| \leq \sup\{\|T(s)\|; s \in S\}$.

(ii) *For any* $\phi \in C_b(S)^*$*, there exists a unique operator* $A_\phi \in B(X)$ *such that* $A_\phi x \in \|\phi\|\overline{\text{co}}((\mathcal{S}x) \cup (-\mathcal{S}x))$*,* $\langle A_\phi x, x^* \rangle = \phi(\langle T(\cdot)x, x^* \rangle)$ *for all* $x \in X$ *and* $x^* \in X^*$*, and* $\|A_\phi\| \leq \|\phi\| \sup\{\|T(s)\|; s \in S\}$.

(iii) *If* S *is a commutative semigroup, then, for any two linear functionals* $\phi, \psi \in C_b(S)^*$*,* $A_\phi T(\cdot) = T(\cdot)A_\phi$*, and* $A_\phi A_\psi = A_\psi A_\phi$*. Further, when* $\mathcal{S}$ *is a* (C_0)*-semigroup with generator* A*, one has* $A_\phi Ax = AA_\phi x$ *for* $x \in D(A)$.

Proof. Set $A_\phi x := z_{T(\cdot)x,\phi}$ for all $x \in X$. Then (i) and (ii) follow immediately from Lemma 2.1.

(iii) Let $\phi, \psi \in C_b[0,\infty)^*$, $x \in X$, $x^* \in X^*$, and $t > 0$. Then we have

$$\begin{aligned}\langle A_\phi T(t)x, x^* \rangle &= \phi(\langle T(\cdot)T(t)x, x^* \rangle) = \phi(\langle T(t)T(\cdot)x, x^* \rangle)\\ &= \phi(\langle T(\cdot)x, (T(t))^* x^* \rangle) = \langle A_\phi x, (T(t))^* x^* \rangle\\ &= \langle T(t)A_\phi x, x^* \rangle\end{aligned}$$

and so $T(t)$ and A_ϕ commute. Therefore

$$\begin{aligned}\langle A_\phi A_\psi x, x^* \rangle &= \phi(\langle T(\cdot)A_\psi x, x^* \rangle) = \phi(\langle A_\psi T(\cdot)x, x^* \rangle)\\ &= \phi(\langle T(\cdot)x, (A_\psi)^* x^* \rangle) = \langle A_\phi x, (A_\psi)^* x^* \rangle\\ &= \langle A_\psi A_\phi x, x^* \rangle.\end{aligned}$$

This proves that A_ϕ and A_ψ commute.

Remark 2.3. Since every mean on $C_{ub}(S)$ can be extended to a mean on $C_b(S)$, Lemma 2.1 and Corollary 2.2 still hold if f is bounded and uniformly continuous on S and $C_b(S)$ is replaced by $C_{ub}(S)$.

The following mean ergodic theorem is proved in [5, Theorem 1].

Theorem 2.4. *Let* $\{A_\alpha\}$ *be an* $\mathcal{A}$*-ergodic net. Then the operator* P*, defined by*

$$\begin{cases} D(P) := \{x \in X; s - \lim_\alpha A_\alpha x \text{ exists}\}, \\ Px = \text{s-}\lim_\alpha A_\alpha x, \; x \in D(P), \end{cases}$$

is a linear projection with norm $\|P\| \leq M$*, range* $R(P) = \bigcap_{A \in \mathcal{A}} N(A)$*, null space* $N(P) = \overline{\sum_{A \in \mathcal{A}} R(A)}$*, and domain*

$$D(P) = \bigcap_{A \in \mathcal{A}} N(A) \oplus \overline{\sum_{A \in \mathcal{A}} R(A)} = \{x \in X; \{A_\alpha x\} \text{ has a weak cluster point}\}.$$

Here $\sum_{A\in\mathcal{A}} R(A)$ *denotes the linear space spanned by the spaces* $R(A),\ A\in\mathcal{A}$.

Let P and B_1 be the operators defined respectively by

$$\begin{cases} D(P) := \{x\in X; \lim_\alpha A_\alpha x \text{ exists}\}; \\ Px := \lim_\alpha A_\alpha x \text{ for } x\in D(P), \end{cases} \quad \begin{cases} D(B_1) := \{y\in X; \lim_\alpha B_\alpha y \text{ exists}\}; \\ B_1 x := \lim_\alpha B_\alpha y \text{ for } y\in D(B_1). \end{cases}$$

$\{A_\alpha\}$ is said to be strongly (resp. uniformly) ergodic if $D(P)=X$ and $A_\alpha x\to Px$ for all $x\in X$ (resp. $\|A_\alpha - P\|\to 0$).

In [7, Theorem 1.1, Corollary 1.4 and Remark 1.7] we proved the following theorem.

Theorem 2.5. (Strong Ergodic Theorem). *Under conditions* (C1) - (C4) *the following are true.*

(i) *P is a bounded linear projection with range $R(P)=N(A)$, null space $N(P)=\overline{R(A)}$, and domain $D(P)=N(A)\oplus\overline{R(A)}=\{x\in X; \{A_\alpha x\}$ has a weak cluster point$\}$.*

(ii) *B_1 is the inverse operator A_1^{-1} of the restriction $A_1 := A|\overline{R(A)}$ of A to $\overline{R(A)}$; it has range $R(B_1)=D(A_1)=D(A)\cap\overline{R(A)}$ and domain $D(B_1)=R(A_1)=A(D(A)\cap\overline{R(A)})$. Moreover, for each $y\in D(B_1)$, $B_1 y$ is the unique solution of the functional equation $Ax=y$ in $\overline{R(A)}$.*

Theorem 2.6. (Uniform Ergodic Theorem [8]). *Under conditions* (C1) - (C3), *we have: $D(P)=X$ and $\|A_\alpha - P\|\to 0$ if and only if $\|B_\alpha|_{R(A)}\| = O(1)$, if and only if B_1 is bounded and $\|B_\alpha|_{R(A)} - B_1\|\to 0$, if and only if $R(A)$ (or $R(A_1)$) is closed, if and only if $R(A^2)$ (or $R(A_1^2)$) is closed, if and only if $X=N(A)\oplus R(A)$.*

Let X be a Banach space with norm $\|\cdot\|_X$, and Y a submanifold with seminorm $\|\cdot\|_Y$. The *K-functional* is defined by

$$K(t,x) := K(t,x,X,Y,\|\cdot\|_Y) = \inf_{y\in Y}\{\|x-y\|_X + t\|y\|_Y\}.$$

If Y is a Banach space with norm $\|\cdot\|_Y$, the *completion of Y relative to X* is defined as

$$\tilde{Y}_X := \{x\in X : \exists\{x_m\}\subset Y \text{such that} \lim_{m\to\infty}\|x_m - x\|_X = 0 \text{ and } \sup\|x_m\|_Y < \infty\}.$$

$K(t,x)$ is a bounded, continuous, monotone increasing and subadditive function of t for each $x\in X$, and $K(t,x,X,Y,\|\cdot\|_Y) = O(t)$ $(t\to 0^+)$ if and only if $x\in \tilde{Y}_X$.

Let $X_1 := \overline{R(A)}$ and $X_0 := D(P) = N(A) \oplus X_1$. Since the operator $B_1 : D(B_1) \subset X_1 \to X_1$ is closed, its domain $D(B_1)\,(= R(A_1))$ is a Banach space with respect to the norm $\|x\|_{B_1} := \|x\| + \|B_1 x\|$.

Let $B_0 : D(B_0) \subset X_0 \to X_0$ be the operator $B_0 := 0 \oplus B_1$. Then its domain

$$D(B_0)\,(= N(A) \oplus D(B_1) = N(A) \oplus A(D(A) \cap \overline{R(A)}))$$

is a Banach space with norm $\|x\|_{B_0} := \|x\| + \|B_0 x\|$, and $[D(B_0)]^{\sim}_{X_0} = N(A) \oplus [D(B_1)]^{\sim}_{X_1}$.

The following theorem from [9, 10] is concerned with optimal convergence and non-optimal convergence rates of ergodic limits and approximate solutions.

Theorem 2.7. *Under conditions* (C1) - (C5) *the following statements hold.*

(i) *For $x \in X_0 = N(A) \oplus \overline{R(A)}$, one has:*

$$\begin{aligned}\|A_\alpha x - Px\| = O(f(\alpha)) &\Leftrightarrow K(e(\alpha), x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(f(\alpha))\\ &\Leftrightarrow x \in [D(B_0)]^{\sim}_{X_0} \text{ (in case } f = e).\end{aligned}$$

(ii) *For $y \in D(B_1) = R(A_1)$ one has:*

$$\begin{aligned}\|B_\alpha y - B_1 y\| = O(f(\alpha)) &\Leftrightarrow K(e(\alpha), B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(f(\alpha))\\ &\Leftrightarrow \in A(D(A) \cap [D(B_1)]^{\sim}_{X_1}) \text{ (in case } f = e).\end{aligned}$$

3. Main Results

We first deduce from Theorem 2.4 the following generalized version of the Kido-Takahashi ergodic theorem, in which a more general net $\{\phi_\alpha\}$ of linear functionals has replaced the net $\{\mu_\alpha\}$ of means in Theorem 1.1.

Theorem 3.1. *If $\mathcal{S}$ is a uniformly bounded semigroup satisfying* (S1)-(S3)*, and if $\{\phi_\alpha\}$ is a bounded net in $C_b(S)^*$ or $C_{ub}(S)^*$ satisfying $\phi_\alpha(1) = 1$ for all α, w^*-$\lim\limits_{\alpha}(l_t^* \phi_\alpha - \phi_\alpha) = 0$ and $\lim\limits_{\alpha} \|r_t^* \phi_\alpha - \phi_\alpha\| = 0$ in $C_b(S)^*$ for all $t \in S$, then the net $\{A_\alpha\}$ ($A_\alpha := A_{\phi_\alpha}$) converges strongly to a linear projection P on X with range $R(P) = F(\mathcal{S}) := \bigcap_{s \in S} N(T(s) - I)$, null space $N(P) = \overline{\sum_{s \in S} R(T(s) - I)}$, and domain $D(P) = X = F(\mathcal{S}) \oplus \overline{\sum_{s \in S} R(T(s) - I)}$.*

Proof. (a) We prove the case that $\{\phi_\alpha\} \subset C_b(S)^*$; the proof for the case $\{\phi_\alpha\} \subset C_{ub}(S)^*$ is similar. Suppose $\|T(s)\| \le M$ for all $s \in S$. Take $\mathcal{A} = \{T(s) - I; s \in S\}$. Then $\|A_\alpha\| \le M \sup_\alpha \|\phi_\alpha\|$ for all α, by Corollary 2.2(ii). Under the assumptions of the theorem we verify conditions (b) and (c) of Definition 1.2.

(b) If $x \in \bigcap_{A\in\mathcal{A}} N(A) = F(\mathcal{S})$, then $T(s)x = x$ for all $s \in S$, so that $A_\alpha x = x$ for all α. On the other hand, clearly we have

$$(A_\alpha - I)x \in \overline{\text{co}}[\{(T(s)-I)x; s \in S\} \cup \{-(T(s)-I)x; s \in S\}] \subset \overline{\sum_{s\in S} R(T(s) - I)}$$

for all $x \in X$ and α. Hence $R(A_\alpha - I) \subset \overline{\sum_{s\in S} R(T(s) - I)}$ for all α.

(c) The assumption that $w^*\text{-}\lim_\alpha (l_t^*\phi_\alpha - \phi_\alpha) = 0$ in $C_b(S)^*$ for all $t \in S$ implies that

$$\begin{aligned}\langle (T(t) - I)A_\alpha x, x^*\rangle &= \langle A_\alpha x, (T(t) - I)^* x^*\rangle = \phi_\alpha(\langle T(\cdot)x, (T(t) - I)^* x^*\rangle) \\ &= \phi_\alpha(\langle (T(t\cdot) - T(\cdot))x, x^*\rangle) = \phi_\alpha((l_t - I)\langle T(\cdot)x, x^*\rangle) \\ &= (l_t^*\phi_\alpha - \phi_\alpha)(\langle T(\cdot)x, x^*\rangle) \to 0\end{aligned}$$

for all $x \in X$, $x^* \in X^*$, and $t \in S$. Hence $\text{w-}\lim_\alpha (T(t) - I)A_\alpha x = 0$ for all $x \in X$ and $t \in S$.

On the other hand, the assumption that $\lim_\alpha \|r_t^*\phi_\alpha - \phi_\alpha\| = 0$ in $C_b(S)^*$ for all $t \in S$ implies

$$\begin{aligned}|\langle A_\alpha(T(t) - I)x, x^*\rangle| &= |\phi_\alpha(\langle T(\cdot)(T(t) - I)x, x^*\rangle)| \\ &= |\phi_\alpha((r_t - I)\langle T(\cdot)x, x^*\rangle)| = |(r_t^*\phi_\alpha - \phi_\alpha)(\langle T(\cdot)x, x^*\rangle)| \\ &\le \|r_t^*\phi_\alpha - \phi_\alpha\| M \|x\| \|x^*\|\end{aligned}$$

for all $x \in X$, $x^* \in X^*$, and $t \in S$. Hence $\|A_\alpha(T(t) - I)\| \le \|r_t^*\phi_\alpha - \phi_\alpha\| M \to 0$ for all $t \in S$.

Thus $\{A_\alpha\}$ is an $\mathcal{A}$-ergodic net, and it follows from Theorem 2.4 that $\{A_\alpha\}$ converges strongly to a linear projection P on X with range $R(P) = F(\mathcal{S})$, null space $N(P) = \overline{\sum_{s\in S} R(T(s) - I)}$, and domain

$$D(P) = F(\mathcal{S}) \oplus \overline{\sum_{s\in S} R(T(s) - I)} = \{x \in X; \{A_\alpha x\} \text{ has a weak cluster point}\}.$$

Since $A_\alpha x \in \overline{\text{co}}(\mathcal{S}x)$, condition (S3) implies that $\{A_\alpha x\}$ has a weak cluster point for every $x \in X$. Thus $D(P) = X$. This proves Theorem 3.1.

In particular, if $S = [0, \infty)$, then a semigroup $\mathcal{S} = \{T(s); s \ge 0\}$ satisfying (S1)-(S3) has to be strongly continuous, i.e., it is a (C_0)-semigroup. Since this semigroup S is commutative, the assumption in Theorem 2.1 on the net $\{\phi_\alpha\}$ becomes

($*0$) $\lim_\alpha \|r_t^*\phi_\alpha - \phi_\alpha\| = 0$ in $C_b[0, \infty)^*$ for all $t \ge 0$. In this case, $\{\phi_\alpha\}$ is said to be *strongly regular* (cf. [3]).

Let A be the infinitesimal generator of $T(\cdot)$. Using the facts that $x \in N(A)$ if and only if $T(s)x = x$ for all $s \geq 0$, $Ax = \lim_{t\to 0^+} t^{-1}(T(t) - I)x$ for $x \in D(A)$, and $(T(t) - I)x = A\int_0^t T(s)xds$ for all $x \in X$, we can formulate the following corollary.

Corollary 3.2. *Let $\{T(s); s \geq 0\}$ be a uniformly bounded (C_0)-semigroup with generator A such that all orbits are relatively weakly compact in X, and let $\{\phi_\alpha\}$ be a bounded strongly regular net in $C_b[0,\infty)^*$ or $C_{ub}(S)^*$ such that $\phi_\alpha(1) = 1$ for all α. Then the net $\{A_\alpha\}$ converges strongly to a linear projection P on X with range $R(P) = N(A)$, null space $N(P) = \overline{R(A)}$, and domain $D(P) = X = N(A) \oplus \overline{R(A)}$.*

Note that $C_{ub}[0,\infty)$ is invariant under r_t, and the restrictions of $r_t,\ t \geq 0$, to $C_{ub}[0,\infty)$ form a (C_0)-semigroup of operators on $C_{ub}[0,\infty)$; its infinitesimal generator is the differentiation operator $\mathcal{D}$, defined by $\mathcal{D}f(t) = f'(t)$ for differentiable f in $C_{ub}[0,\infty)$. Our uniform ergodic theorem and strong ergodic theorem with rates for C_0-semigroups will be formulated under the following assumptions on a net $\{\phi_\alpha\}$ in $C_{ub}[0,\infty)^*$:

(*1) $\limsup_{t\to 0^+} t^{-1}\|r_t^*\phi_\alpha - \phi_\alpha\| < \infty$ for all α and $e(\alpha) := \limsup_{t\to 0^+} t^{-1}\|r_t^*\phi_\alpha - \phi_\alpha\| \to 0$;

(*2) There exists a companion net $\{\psi_\alpha\} \subset C_{ub}[0,\infty)^*$ such that $\overline{\psi_\alpha \circ \mathcal{D}} = \delta - \phi_\alpha$, where δ is the mean on $C_{ub}[0,\infty)$ defined by $(\delta f) := f(0)$ for all $f \in C_{ub}[0,\infty)$.

Note that condition (*1) is stronger than condition (*0) and under condition (*1) we actually have $\lim_{t\to 0^+} t^{-1}\|r_t^*\phi_\alpha - \phi_\alpha\| = e(\alpha)$ (see Proposition 3.4(iii)).

For convenience of application, we give a condition which is equivalent to (*1) and implies condition (C3) of Definition 1.3. We need the following lemma which was essentially proved in [6, Theorem 3.2.1].

Lemma 3.3. *For $x^* \in X^*$, the following assertions are equivalent:*

(a) $x^* \in D(A^*)$;

(b) $\limsup_{t\to 0^+} t^{-1}\|T^*(t)x^* - x^*\| < \infty$;

(c) $\liminf_{t\to 0^+} t^{-1}\|T^*(t)x^* - x^*\| < \infty$.

Moreover, we have

$$\begin{aligned}\|A^*x^*\| &\leq \liminf_{t\to 0^+} t^{-1}\|T^*(t)x^* - x^*\| \leq \limsup_{t\to 0^+} t^{-1}\|T^*(t)x^* - x^*\| \\ &\leq \limsup_{t\to 0^+} \|T(t)\|\|A^*x^*\|.\end{aligned}$$

Proof. (b) $\Rightarrow$ (c) is obvious.

(a) $\Rightarrow$ (b). If $x^* \in D(A^*)$, then we have for every $x \in X$ and $t > 0$

$$\begin{aligned} t^{-1}|\langle x, T^*(t)x^* - x^*\rangle| &= t^{-1}|\langle (T(t)-I)x, x^*\rangle| = t^{-1}|\langle A(1*T(t))x, x^*\rangle| \\ &\le t^{-1}\|A^*x^*\|\|1*T(t)\|\|x\| \\ &\le \sup\{\|T(s)\|; 0 < s \le t\}\|A^*x^*\|\|x\|. \end{aligned}$$

Hence

$$\limsup_{t\to 0^+} t^{-1}\|T^*(t)x^* - x^*\| \le \limsup_{t\to 0^+} \|T(t)\|\|A^*x^*\|.$$

(c) $\Rightarrow$ (a). Suppose $\liminf_{t\to 0^+} t^{-1}\|T^*(t)x^* - x^*\| < \infty$. Since $t^{-1}(1*T(t))(t) \to I$ strongly as $t \downarrow 0$, we have for every $x \in D(A)$

$$\begin{aligned} |\langle Ax, x^*\rangle| &= \lim_{t\to 0^+} t^{-1}|\langle (1*T(t))Ax, x^*\rangle| = \lim_{t\to 0^+} t^{-1}|\langle T(t)x - x, x^*\rangle| \\ &= \lim_{t\to 0^+} t^{-1}|\langle x, T^*(t)x^* - x^*\rangle| \le \liminf_{t\to 0^+} t^{-1}\|T^*(t)x^* - x^*\|\|x\|. \end{aligned}$$

Therefore $x^* \in D(A^*)$ and $\|A^*x^*\| \le \liminf_{t\to 0^+} t^{-1}\|T^*(t)x^* - x^*\|$.

Proposition 3.4.

(i) *For a linear functional $\phi \in C_{ub}[0,\infty)^*$, $\limsup_{t\to 0^+} t^{-1}\|r_t^*\phi - \phi\| < \infty$ if and only if $\liminf_{t\to 0^+} t^{-1}\|r_t^*\phi - \phi\| < \infty$, if and only if $\overline{\phi \circ \mathcal{D}} \in C_{ub}[0,\infty)^*$ (i.e., $\phi \in D(\mathcal{D}^*)$). In this case, $\limsup_{t\to 0^+} t^{-1}\|r_t^*\phi - \phi\| = \liminf_{t\to 0^+} t^{-1}\|r_t^*\phi - \phi\| = \|\phi \circ \mathcal{D}\|$.*

(ii) *A net $\{\phi_\alpha\} \subset C_{ub}[0,\infty)^*$ satisfies the condition* (*1) *if and only if $\overline{\phi_\alpha \circ \mathcal{D}} \in C_{ub}[0,\infty)^*$ eventually and $\|\phi_\alpha \circ \mathcal{D}\| = e(\alpha)$.*

(iii) *If $\{\phi_\alpha\}$ satisfies the condition* (*1)*, then it satisfies condition* (*0) *and the net $\{A_\alpha\}$ satisfies condition* (C3) *of Definition 1.3.*

Proof. Since $\{r_t; t \ge 0\}$ is a contraction C_0-semigroup on $C_{ub}[0,\infty)^*$,

(i) follows from Lemma 3.3.

(ii) follows from (i).

(iii) Since $\{r_t; t \ge 0\}$ is a (C_0)-semigroup on $C_{ub}[0,\infty)$, by (ii) we have for all $f \in C_{ub}[0,\infty)$

$$\begin{aligned}\|(r_t^*\phi_\alpha - \phi_\alpha)f\|_\infty &= \|\phi_\alpha((r_t - I)f)\|_\infty = \|\phi_\alpha(\mathcal{D}\int_0^t r_s f ds)\|_\infty \\ &\le \|\phi_\alpha \circ \mathcal{D}\| \|\int_0^t r_s f ds)\|_\infty = e(\alpha)\|\int_0^t r_s ds\| \|f\|_\infty.\end{aligned}$$

Hence

$$\|r_t^*\phi_\alpha - \phi_\alpha\| \le e(\alpha)\|\int_0^t r_s ds\| \to 0 \text{ for all } t \ge 0.$$

To verify condition (C3) we see that

$$\begin{aligned}&|\langle A_\alpha t^{-1}(T(t) - I)x, x^*\rangle| = |t^{-1}\phi_\alpha(\langle T(\cdot)(T(t) - I)x, x^*\rangle)| \\ &= |t^{-1}\phi_\alpha((r_t - I)\langle T(\cdot)x, x^*\rangle)| = t^{-1}|(r_t^*\phi_\alpha - \phi_\alpha)(\langle T(\cdot)x, x^*\rangle)| \\ &\le t^{-1}\|r_t^*\phi_\alpha - \phi_\alpha\| M\|x\|\|x^*\|\end{aligned}$$

for all $x \in X$, $x^* \in X^*$, and $t > 0$. Hence $\|A_\alpha t^{-1}(T(t) - I)x\| \le t^{-1}\|r_t^*\phi_\alpha - \phi_\alpha\|M\|x\|$ for all $x \in X$ and $t > 0$. If $x \in D(A)$, then

$$\begin{aligned}\|A_\alpha Ax\| &\le \left\|A_\alpha\left(t^{-1}(T(t) - I)x - Ax\right)\right\| + \|A_\alpha t^{-1}(T(t) - I)x\| \\ &\le M\left\|t^{-1}(T(t) - I)x - Ax\right\| + t^{-1}\|r_t^*\phi_\alpha - \phi_\alpha\|M\|x\|.\end{aligned}$$

This being true for all $t > 0$ and all $x \in D(A)$, it follows that $A_\alpha A$ has a bounded closure $\overline{A_\alpha A}$ on X with norm $\|\overline{A_\alpha A}\| \le \limsup_{t\to 0^+} t^{-1}\|r_t^*\phi_\alpha - \phi_\alpha\|M$.

By Corollary 2.2(iii), we see that, for $x \in D(A)$,

$$\lim_{t\to 0^+} t^{-1}(T(t) - I)A_\alpha x = A_\alpha\left(\lim_{t\to 0^+} t^{-1}(T(t) - I)x\right) = A_\alpha Ax.$$

Hence $A_\alpha x \in D(A)$ and $AA_\alpha x = A_\alpha Ax$ for all $x \in D(A)$. Since $D(A)$ is dense in X, for any $x \in X$ there is a sequence $\{x_n\}$ in $D(A)$ such that $x_n \to x$. Since $A_\alpha x_n \to A_\alpha x$ and $AA_\alpha x_n = A_\alpha Ax_n \to \overline{A_\alpha A}x$ as $n \to \infty$, it follows from the closedness of A that $A_\alpha x \in D(A)$ and $AA_\alpha x = \overline{A_\alpha A}x$. We have shown that $R(A_\alpha) \subset D(A)$ and $A_\alpha A \subset AA_\alpha = \overline{A_\alpha A}$. The assumption (*1) implies that $\|AA_\alpha\| = O(e(\alpha))$. This verifies (C3).

Next, we observe consequence of condition (*2). For this we need the next proposition.

Proposition 3.5. *Let $T(\cdot)$ be a uniformly bounded (C_0)-semigroup on a Banach space with infinitesimal generator A. Then*

(i) For every $x \in X$ and $x \in X^*$ $\langle T(s)x, x^*\rangle$ is bounded and uniformly continuous on $s \ge 0$, i.e., $\langle T(\cdot)x, x^*\rangle \in C_{ub}[0, \infty)$.

(*ii*) *$\langle T(\cdot)x, x^*\rangle \in D(\mathcal{D})$ and $\mathcal{D}\langle T(\cdot)x, x^*\rangle = \langle T(\cdot)Ax, x^*\rangle$ for all $x \in D(A)$ and $x^* \in X^*$.*

(*iii*) *Let $\phi, \psi \in C_{ub}[0, \infty)^*$ be such that $\overline{\psi \circ \mathcal{D}} = \delta - \phi$. Then $R(A_\psi) \subset D(A)$ and $A_\psi A \subset AA_\psi = I - A_\phi$.*

Proof.

(i) Let $x \in X$ and $x \in X^*$ be arbitrary. It is clear that $|\langle T(s)x, x^*\rangle| \leq \sup_{s\geq 0} ||T(s)|| \cdot ||x|| \cdot ||x^*|| < \infty$ for all $s \geq 0$. For every $t, s \geq 0$, we have

$$\begin{aligned} &|\langle T(t)x, x^*\rangle - \langle T(s)x, x^*\rangle| = |\langle T(t)x - T(s)x, x^*\rangle| \\ &\leq \sup_{r\geq 0} ||T(r)|| \cdot ||T(|t-s|)x|| \cdot ||x^*|| \to 0 \text{ as } |t-s| \to 0. \end{aligned}$$

This proves $\langle T(s)x, x^*\rangle$ is uniformly continuous on $s \geq 0$ and so (i) holds.

(ii) holds because for $x \in D(A)$ and $x^* \in X^*$

$$\begin{aligned} &|t^{-1}(r_t - I)\langle T(\cdot)x, x^*\rangle - \langle T(\cdot)Ax, x^*\rangle| \\ &= |t^{-1}(\langle T(\cdot + t)x, x^*\rangle - \langle T(\cdot)x, x^*\rangle) - \langle T(\cdot)Ax, x^*\rangle| \\ &= |\langle T(\cdot)[t^{-1}(T(t)x - x) - Ax], x^*\rangle| \\ &\leq \sup_{s\geq 0} ||T(s)|| \cdot ||t^{-1}(T(t)x - x) - Ax|| \cdot ||x^*|| \to 0 \text{ as } t \downarrow 0. \end{aligned}$$

(iii) Let $x \in D(A)$. By (ii) and Corollary 2.2(ii), we have $\langle T(\cdot)x, x^*\rangle \in D(\mathcal{D})$ and

$$\psi(\mathcal{D}\langle T(\cdot)x, x^*\rangle) = \psi(\langle T(\cdot)Ax, x^*\rangle) = \langle A_\psi Ax, x^*\rangle$$

for every $x^* \in X^*$. On the other hand, the assumption implies

$$\psi\mathcal{D}(\langle T(\cdot)x, x^*\rangle) = (\delta - \phi)(\langle T(\cdot)x, x^*\rangle) = \langle x - A_\phi x, x^*\rangle.$$

Therefore we have $A_\psi Ax = (I - A_\phi)x$ for all $x \in D(A)$. Clearly, it follows from Corollary 2.2(iii) and the closedness of A that $A_\psi Ax = AA_\psi x$ for all $x \in D(A)$. Hence $AA_\psi x = A_\psi Ax = (I - A_\phi)x$ for all $x \in D(A)$. Again by the closedness of A and the fact that $D(A)$ is dense in X we obtain that $R(A_\phi) \subset D(A)$ and $A_\psi A \subset AA_\psi = I - A_\phi$.

It follows from (iii) of Proposition 3.5 that (*2) implies $R(B_\alpha) \subset D(A)$ and $B_\alpha A \subset AB_\alpha = I - A_\alpha$. We have shown that conditions (*1) and (*2) yield conditions (C3) and (C2), respectively. Therefore we can immediately deduce the following Theorems 3.6, 3.7, and 3.8 from Theorems 2.6(ii), 2.7, and 2.8, respectively.

Let $A_\alpha := A_{\phi_\alpha}$ and $B_\alpha := A_{\psi_\alpha}$. The following theorem is concerned with the convergence of approximate solutions $B_\alpha y$ of $Ax = y$.

Theorem 3.6. *Let $\{T(s); s \geq 0\}$ be a uniformly bounded (C_0)-semigroup with generator A such that all orbits are relatively weakly compact in X. Let $\{\phi_\alpha\}$ be a bounded net in $C_{ub}[0,\infty)^*$ which satisfies conditions* (*1) *and* (*2). *Further, suppose $B_\alpha^* x^* = \varphi(\alpha)x^*$ for all $x^* \in R(A)^\perp$, with $|\varphi(\alpha)| \to \infty$, and that $\|A_\alpha x\| = O(f(\alpha))$ (resp. $o(f(\alpha))$) implies $\|B_\alpha x\| = O(\frac{f(\alpha)}{e(\alpha)})$ (resp. $o(\frac{f(\alpha)}{e(\alpha)})$). Then the operator B_1, defined by $B_1 y := \lim_\alpha B_\alpha y$ with the natural domain $D(B_1)$, is the inverse operator A_1^{-1} of the restriction $A_1 := A|\overline{R(A)}$ of A to $\overline{R(A)}$; it has range $R(B_1) = D(A_1) = D(A)$ and domain $D(B_1) = R(A_1) = R(A)$. Thus, for each $y \in R(A)$, $B_1 y$ is the unique solution of the functional equation $Ax = y$ in $\overline{R(A)}$.*

The following is a uniform ergodic theorem.

Theorem 3.7. *Let $\{T(s); s \geq 0\}$ be a uniformly bounded (C_0)-semigroup with generator A such that all orbits are relatively weakly compact in X. Let $\{\phi_\alpha\}$ be a bounded net in $C_{ub}[0,\infty)^*$ which satisfies conditions* (*1) *and* (*2). *Then $\|A_\alpha - P\| \to 0$ if and only if B_1 is bounded and $\|B_\alpha|_{R(A)} - B_1\| \to 0$, if and only if $R(A)$ is closed.* The following theorem is about the convergence rates of $A_\alpha x$ and $B_\alpha y$.

Theorem 3.8. *Let $\{T(s); s \geq 0\}$ be a uniformly bounded (C_0)-semigroup with generator A such that all orbits are relatively weakly compact in X. Let $\{\mu_\alpha\}$ be a bounded net in $C_{ub}[0,\infty)^*$ which satisfies conditions* (*1) *and* (*2). *Further, suppose $B_\alpha^* x^* = \varphi(\alpha)x^*$ for all $x^* \in R(A)^\perp$, with $|\varphi(\alpha)| \to \infty$, and that $\|A_\alpha x\| = O(f(\alpha))$ (resp. $o(f(\alpha))$) implies $\|B_\alpha x\| = O(\frac{f(\alpha)}{e(\alpha)})$ (resp. $o(\frac{f(\alpha)}{e(\alpha)})$). Then the following statements hold.*

(*i*) *For $x \in X$,*

$$\|A_\alpha x - Px\| = O(f(\alpha)) \Leftrightarrow K(e(\alpha), x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(f(\alpha))$$
$$\Leftrightarrow x \in [D(B_0)]\tilde{}_{X_0} \text{ (in case } f = e).$$

(*ii*) *For $y \in D(B_1) = R(A)$,*

$$\|B_\alpha y - B_1 y\| = O(f(\alpha)) \Leftrightarrow K(e(\alpha), B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(f(\alpha))$$
$$\Leftrightarrow y \in A(D(A) \cap [D(B_1)]\tilde{}_{X_1}) \text{ (in case } f = e).$$

4. Examples

Example 1. Let $S = [0,\infty)$, and for a fixed $\beta > 0$ and each $t > 0$ let $\mu_t \in C_b[0,\infty)^*$ be a mean defined by $\mu_t(f) := (j_\beta * f)(t)/j_{\beta+1}(t)$, and let $\psi_t \in C_b[0,\infty)^*$ be a linear functional defined by $\psi_t(f) = -(j_{\beta+1} * f)(t)/j_{\beta+1}(t)$, $f \in C_b[0,\infty)$, where $j_\beta(t) = t^\beta/\Gamma(\beta+1)$. We consider the net $\{\mu_t\}_{t\to\infty}$. Since

$$\begin{aligned}|(\mu_t \circ \mathcal{D})f| = |\mu_t(f')| &= \frac{|(j_\beta * f')(t)|}{j_{\beta+1}(t)} = \frac{|(j_{\beta-1} * (f - f(0))(t)|}{j_{\beta+1}(t)} \\ &\le 2\|f\|_\infty \frac{j_\beta(t)}{j_{\beta+1}(t)} = 2\|f\|_\infty \frac{\beta+1}{t}\end{aligned}$$

for all differentiable $f \in C_{ub}[0,\infty)$, we have $\|\mu_t \circ \mathcal{D}\| = O(t^{-1})$ $(t \to \infty)$ and hence it follows from Proposition 3.4 that the net $\{\mu_t\}_{t\to\infty}$ satisfies condition (*1) with $e(t) = t^{-1}$. Also (*2) is satisfied:

$$(\psi_t \circ \mathcal{D})f = -\frac{(j_{\beta+1} * f')(t)}{j_{\beta+1}(t)} = -\frac{(j_\beta * (f - f(0))(t)}{j_{\beta+1}(t)} = f(0) - \mu_t f = (\delta - \mu_t)f.$$

Since

$$(j_\beta * \langle T(\cdot)x, x^*\rangle)(t)/j_{\beta+1}(t) = \left\langle \frac{(j_\beta * T(\cdot)x)(t)}{j_{\beta+1}(t)}, x^* \right\rangle$$

and

$$(j_{\beta+1} * \langle T(\cdot)x, x^*\rangle)(t)/j_{\beta+1}(t) = \left\langle \frac{(j_{\beta+1} * T(\cdot)x)(t)}{j_{\beta+1}(t)}, x^* \right\rangle$$

for all $x^* \in X^*$. Thus the operator A_t corresponding to the mean μ_t is the Cesàro mean $C_t^{\beta+1}$ of order $\beta+1$ as defined by

$$C_t^{\beta+1}x = \frac{(j_\beta * T(\cdot)x)(t)}{j_{\beta+1}(t)} = \frac{\beta+1}{t^{\beta+1}} \int_0^t (t-s)^\beta T(s)x\,ds,\ x \in X,$$

and $B_t = -(j_{\beta+1}(t))^{-1}(j_{\beta+1} * T(\cdot))(t) = -\frac{t}{\beta+2} C_t^{\beta+1}$.

If $x^* \in R(A)^\perp$, then for all $x \in X$ and $t \ge 0$ we have

$$\langle x, T^*(t)x^* - x^*\rangle = \langle T(t)x - x, x^*\rangle = \langle A(j_0 * T(\cdot))(t)x, x^*\rangle = 0$$

so that

$$\begin{aligned}\langle x, B_t^* x^*\rangle &= \left\langle (j_{\beta+1}(t))^{-1}(j_{\beta+1} * T(\cdot))(t)x, x^* \right\rangle \\ &= (j_{\beta+1}(t))^{-1}(j_{\beta+1} * \langle x, T^*(\cdot))x^*\rangle)(t) \\ &= \left\langle x, (j_{\beta+1}(t))^{-1}(j_{\beta+1} * 1)(t)x^* \right\rangle = \left\langle x, \frac{t}{\beta+2}x^* \right\rangle.\end{aligned}$$

Hence $B_t^* x^* = \frac{t}{\beta+2} x^*$ for all $x^* \in R(A)^\perp$ with $\frac{t}{\beta+2} \to \infty$ as $t \to \infty$. It can also been shown that if $\|C_t^{\beta+1} x\| = O(t^{-\theta})$ (resp. $o(t^{-\theta})$) with $0 \le \theta \le 1$, then $\|B_t x\| = O(t^{-\theta}/t^{-1})$ (resp. $o(t^{-\theta}/t^{-1})$) (cf. [9, 10]). Hence conditions (C4) and (C5) are satisfied.

Example 2. Let $\mu_\lambda(f) := \lambda L_\lambda(f) := \lambda \int_0^\infty e^{-\lambda t} f(t)dt$ and $\psi_\lambda(f) := -L_\lambda(f) = -\int_0^\infty e^{-\lambda t} f(t)dt$ for $f \in C_b[0,\infty)$ and $\lambda > 0$. Since

$$\begin{aligned}|(\mu_\lambda \circ \mathcal{D})f| = |\mu_\lambda(f')| &= \left|\lambda \int_0^\infty e^{-\lambda t} f'(t)dt\right| \\ &= \left|-\lambda f(0) + \lambda^2 \int_0^\infty e^{-\lambda t} f(t)dt\right| \le 2\lambda \|f\|_\infty\end{aligned}$$

for all differentiable $f \in C_{ub}[0,\infty)$, it follows from Proposition 3.4 that the net $\{\mu_\lambda\}_{\lambda \to 0^+}$ satisfies condition (*1) with $e(\lambda) = \lambda$. Also (*2) is satisfied:

$$(\psi_\lambda \circ \mathcal{D})f = -L_\lambda f' = f(0) - \lambda L_\lambda f = (I - \mu_\lambda)f.$$

It is easy to see that the operator A_λ corresponding to the mean μ_λ is $A_\lambda = \lambda L_\lambda(T(\cdot)) = \lambda(\lambda - A)^{-1}$ and the operator B_λ corresponding to ψ_λ is $B_\lambda = -(\lambda - A)^{-1}$. We also know that conditions (C4) and (C5) are satisfied (cf. [9, 10]).

As applications of Corollary 3.2, and Theorems 3.6 - 3.8 to the above two examples of nets of means, the following known theorems cf. [8, 9, 10]) can be formulated.

Theorem 4.1. *Let $\{T(s); s \ge 0\}$ be a uniformly bounded (C_0)-semigroup with generator A such that all orbits are relatively weakly compact in X.*

(*i*) $\lim_{t\to\infty} C_t^\beta x = \lim_{\lambda \to 0^+} \lambda(\lambda - A)^{-1} x = Px$ *for all* $x \in X = D(P) = N(A) \oplus \overline{R(A)}$ *and for all* $0 \le \theta \le 1$

$$\begin{aligned}&\|C_t^\beta x - Px\| = O(t^{-\theta})\ (t \to \infty) \\ \Leftrightarrow\ &\|\lambda(\lambda - A)^{-1}x - Px\| = O(\lambda^\theta)\ (\lambda \to 0^+) \\ \Leftrightarrow\ &K(t^{-1}, x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(t^{-\theta})\ (t \to \infty) \\ \Leftrightarrow\ &x \in [D(B_0)]\widetilde{\ }_{X_0}\ \textit{(in case } \theta = 1);\end{aligned}$$

(*ii*) *For* $y \in D(B_1) = R(A_1) = A(D(A) \cap \overline{R(A)})$, *we have*

$$-\lim_{t\to\infty} \frac{t}{\beta+2} C_t^{\beta+1} y = -\lim_{\lambda\to 0^+} (\lambda - A)^{-1} y = B_1 y$$

and for all $0 \le \theta \le 1$

$$\begin{aligned} & \|\frac{t}{\beta+2}C_t^{\beta+1}y + B_1 y\| = O(t^{-\theta})\ (t \to \infty) \\ \Leftrightarrow\ & \|(\lambda - A)^{-1}y + B_1 y\| = O(\lambda^{\theta})\ (\lambda \to 0^+) \\ \Leftrightarrow\ & K(t^{-1}, B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(t^{-\theta})\ (t \to \infty) \\ \Leftrightarrow\ & y \in A(D(A) \cap [D(B_1)]\tilde{}_{X_1})\ \textit{(in case } \theta = 1). \end{aligned}$$

(*iii*) $\|C_t^{\beta} - P\| \to 0$ *as* $t \to \infty$ *if and only if* $\|\lambda(\lambda - A)^{-1} - P\| \to 0$ *as* $\lambda \to 0^+$, *if and only if* $R(A)$ *is closed.*

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