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RAMSEY NUMBERS OF A CYCLE

Yusheng Li

Dedicated to Professor Ko-Wei Lih on the occasion of his 60th birthday.

Abstract. We sketch the ideas in the proofs of results on Ramsey numbers of a cycle, particularly in many colors, in which one is due to professor Ko-Wei Lih and the author for the right order of magnitude of Ramsey number $r_k(C_{2m})$ as $k \to \infty$ for $m = 2, 3, 5$.

1. Introduction

Let G be a graph. The *Ramsey Number* $r_k(G)$ is defined to be the smallest positive integer N such that if the edge set of K_N is colored by k colors, then there exists a monochromatic copy of G.

We shall concentrate on Ramsey numbers $r_k(C_n)$ in this article, where C_n is a cycle of length n.

It is trivial to see that $r_1(C_n) = n$. All the exact values of $r_2(C_n)$ are known, which will be given in the next section. For $r_3(C_n)$, only asymptotical formulas are known as $n \to \infty$. For general $k \geq 4$, even the asymptotical formulas are open.

2. Fixed Number of Colors

In Ramsey theory, it is a folklore that $r_2(C_3) = r_2(C_4) = 6$. However, all other exact values of $r_2(C_n)$ have been found. The corect lower bound of $r_2(C_n)$ with $n \geq 5$ is a special case of the following general bound.

Lemma 1. *Let m be a positive integer. Then*

$$r_k(C_{2m+1}) \geq 2^k m + 1,$$

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and

$$r_k(C_{2m}) \geq (k+1)m - k + 1.$$

Proof. It is easy to see that

$$r_k(G) \geq (\chi - 1)(r_{k-1}(G) - 1) + 1,$$

where $\chi = \chi(G)$ is the chromatic number of G, and the first lower bound follows immediately. The second is also easy. Let $N_k = r_k(C_{2m}) - 1$. Then there is an edge-coloring of the complete graph of order N_k by k colors such that there is no monochromatic C_{2m}. Consider such a colored complete graph and a new complete graph of order $m - 1$. Color all the edges of new graph and those between the two complete graphs by a new color. Clearly, there is no monochromatic C_{2m}, thus $N_{k+1} \geq N_k + m - 1$, which together with the fact that $N_1 = r_1(C_{2m}) - 1 = 2m - 1$ implies the second assertion. ■

Rosta [17], and Faudree and Schelp [8] independently obtained the following result, which together with $r_2(C_3)$ and $r_2(C_4)$ gives all the exact values of $r_2(C_n)$.

Theorem 1.

$$r_2(C_n) = \begin{cases} 2n - 1 & \text{for odd } n \geq 5, \\ 3n/2 - 1 & \text{for even } n \geq 6. \end{cases}$$

For three colors, it was shown that the lower bounds in Lemma 1 are asymptotically equal to the exact values as $n \to \infty$.

Theorem 2.

$$r_3(C_n) = \begin{cases} (4 + o(1))n & \textit{for odd } n, \\ (2 + o(1))n & \textit{for even } n, \end{cases}$$

where $o(1)$ is a small term tending to zero as $n \to \infty$.

The result for the odd length case was obtained by Luczak [16], and Gyárfás, Ruszinkó, Sárközy and Szemerédi [11], and the other by Figaj and Luczak [9]. They used the Regularity Lemma of Szemerédi, a powerful tool in modern graph theory, which can be found in some standard textbooks, see, e.g., Bollobás [2]. For four or more colors, we believe that the lower bounds in Lemma 1 are asymptotically sharp.

Problem 1. Prove or disprove the asymptotical equalities

$$r_k(C_{2n+1}) \sim 2^k n \quad \text{and} \quad r_k(C_{2n}) \sim (k+1)n$$

for k fixed and $n \to \infty$.

3. An Odd Cycle in Many Colors

For $n \geq 3$ fixed and $k \to \infty$, it seems to be very hard to estimate $r_k(C_n)$. From the definition, we know that $N = r_k(G) - 1$ is the largest integer for which K_N has a k-edge coloring so that there is no monochromatic G; such an edge coloring of K_N is called a *Ramsey coloring* for $r_k(G)$. In a Ramsey coloring, any graph induced by monochromatic edges is called a *Ramsey graph*. It was shown [1, 6, 7, 18] that

$$c_1 321^{k/5} \leq r_k(C_3) \leq c_2\, k!,$$

where c_1 and c_2 are positive constants with $c_2 < e$. For general odd cycles, Bondy and Erdős [3] obtained

$$r_k(C_{2m+1}) \leq (2m+1)(k+2)!. \tag{1}$$

The upper bound was refined by Graham, Rothschild, and Spencer [10] as

$$r_k(C_{2m+1}) < 2m(k+2)!. \tag{2}$$

Recently we [13] improved the above upper bounds as follows.

Theorem 3. *Let $k \geq 2$ be an integer. Then*

$$r_k(C_5) \leq \sqrt{18^k k!}.$$

Theorem 4. *Let $\epsilon > 0$ be a constant. If each Ramsey graph G of $r_k(C_{2m+1})$ satisfies $\delta(G) \geq \epsilon d(G)$, where $d(G)$ is the average degree of G, then there is a constant $c = c(\epsilon, m) > 0$ such that*

$$r_k(C_{2m+1}) \leq \left(c^k k!\right)^{1/m}.$$

The background for the assumption in Theorem 4 is a widespread belief that the Ramsey graphs for $r_k(G)$ are nearly regular. Various known Ramsey colorings and random graphs can serve as supporting evidence.

The proofs of Theorem 3 and 4, which we shall sketch, are similar. The idea in the proofs of upper bounds (1) and (2) is as follows. Let G_i be a Ramsey graph for $r_k(C_{2m+1})$ in color i, and let v be a vertex of G_i. Then the neighborhood of v contains contains no path of length $2m-1$, thus it contains an independent set of size at least $d_i(v)/(2m-1)+1$, where $d_i(v)$ is the degree of v in G_i. However, the edges of the complete subgraph induced by this independent set are colored by $k-1$ colors other than i, thus its size is at most $r_{k-1}(G)$. Our idea is to get a global independent set of G_i instead of that in a neighborhood. Our proof relies heavily on the following lower bound for the independence number of a graph with a certain forbidden cycle, which is proved by probabilistic method [15].

Lemma 2. *Let $m \geq 2$ be an integer and let $G = (V, E)$ be a graph of order N that contains no C_{2m+1}. Then*

$$\alpha(G) \geq c \left(\sum_{v \in V} d(v)^{1/(m-1)} \right)^{(m-1)/m},$$

where $c = c(m) > 0$ is a constant. In particular, $c(2) = \sqrt{2}/6$. So if G contains no C_5, then $\alpha(G) \geq \sqrt{Nd/18}$, where d is the average degree of G.

Problem 2. Prove or disprove that $r_k(C_{2m+1}) = o((k!)^{1/m})$ for m fixed and $k \to \infty$.

4. An Even Cycle in Many Colors

If G is a bipartite graph, then $r_k(G)$ is closely related to its Turán number $ex(n; G)$, which is the maximum number of edges in an n-vertex graph that does not contain G. It was shown by Bondy and Simonovits [4] that

$$ex(n; C_{2m}) \leq c\, n^{1+1/m}, \tag{3}$$

where here and henceforth c is a constant depending on m only. However the constants may vary in different contexts.

Lemma 3. *Let $m \geq 2$ be an integer. Then*

$$r_k(C_{2m}) \leq c\, k^{m/(m-1)}.$$

Furthermore, if the order of $r_k(C_{2m})$ is $k^{m/(m-1)}$ as $k \to \infty$, then the order of $ex(n; C_{2m})$ is $n^{1+1/m}$ as $n \to \infty$.

Proof. Let $N = r_k(C_{2m}) - 1$. Then there is an edge-coloring of K_N in k colors containing no monochromatic C_{2m}. So each monochromatic subgraph of K_N has at most $ex(N; C_{2m})$ edges. Thus from (3), we have

$$\binom{N}{2} \leq k\, ex(N; C_{2m}) \leq k c_1 N^{1+1/m},$$

yielding $r_k(C_{2m}) = N+1 \leq ck^{m/(m-1)}$ for some constant $c = c(m)$. Similarly, it can be proved that, if $r_k(C_{2m}) \geq c_1 k^{m/(m-1)}$ for some constant c_1, then $ex(n; C_{2m}) \geq c_2 n^{1+1/m}$ for some constant c_2. ∎

From the above result, we know that it is harder to obtain the exact order of magnitude of $r_k(C_{2m})$ than that of $ex(n; C_{2m})$. The asymptotic formula (so the order) of $r_k(C_4)$ is k^2, obtained by Chung and Graham [5], by by Irving [12].

Recently, professor Ko-Wei Lih and the author [14] obtained the right order of $r_k(C_{2m})$ for $m = 2, 3, 5$.

Theorem 5. *Fix* $m = 2, 3,$ *or* 5*. As* $k \to \infty$*, we have*

$$r_k(C_{2m}) \geq ck^{m/(m-1)}.$$

The key step of our proof is an edge coloring of $K_{N,N}$ by k colors such that there is no monochromatic C_{2m}, which is a generalization of Wenger's constructions [19]. In order to show our main result, let us define $br_k(G)$ for a bipartite graph G as the minimum integer N such that, in any edge-coloring of the complete bipartite graph $K_{N,N}$ by k colors, there is a monochromatic G. Using the dichotomy method, we can prove the following result easily.

Lemma 4. *If* $br_k(C_{2m}) \geq c_1 k^{m/(m-1)}$ *as* $k \to \infty$*, then*

$$r_k(C_{2m}) \geq c_2 k^{m/(m-1)},$$

where c_1 *and* c_2 *are positive constants.*

Our edge coloring of $K_{N,N}$ is as follows. Let $m \geq 2$ be an integer and let $q \geq m$ be a prime power. Let $F(q)$ be the Galois field of q elements, and let both X and Y be copies of the Cartesian product $F^m(q)$. Denote by N the number $q^m = |X| = |Y|$. We shall use vectors in $F^{m-1}(q)$ as colors to color the complete bipartite graph $K_{N,N}$ on partite sets X and Y such that there is no monochromatic C_{2m} for $m = 2, 3, 5$. For vertices $A \in X$ and $B \in Y$ with

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

color the edge AB with color $S \in F^{m-1}(q)$ where

$$S = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_{m-1} + b_{m-1} \end{pmatrix} + b_m \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ a_m \end{pmatrix}.$$

Let us denote by $H_S(m, q)$ the subgraph induced by all edges in color S, whose edge sets form a partition of $K_{N,N}$. Our main task is to show that $H_S(m, q)$

contains no C_{2m}. The definition of $H_S(m,q)$ implies that for any vertex x, the last coordinates of neighbors of x are pairwise distinct hence form $F(q)$. In particular, $H_S(m,q)$ is q-regular. Then we can show that if $H_S(m,q)$ contains a cycle $C_{2m} = (A_1, B_1, \ldots, A_m, B_m)$ with $A_i \in X$ and $B_i \in Y$, then for each B_i there exists a B_j, $i \neq j$, such that they have the same mth (last) coordinates. This implies that $H_S(m,q)$ contains no C_{2m} for $m = 2, 3, 5$ immediately.

Problem 3. Prove or disprove that the order of magnitude of $r_k(C_{2m})$ is $k^{m/(m-1)}$ for $m \geq 2$ fixed and $n \to \infty$.

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Yusheng Li
Department of Mathematics,
Tongji University,
Shanghai 200092,
P. R. China
E-mail: li_yusheng@mail.tongji.edu.cn