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A METHOD TO OBTAIN LOWER BOUNDS FOR CIRCULAR CHROMATIC NUMBER

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Dedicated to Professor Ko-Wei Lih on the occasion of his 60th birthday.

Abstract. The circular chromatic number $\chi_c(G)$ of a graph G is a very natural generalization of the concept of chromatic number $\chi(G)$, and has been studied extensively in the past decade. In this paper we present a new method for bounding the circular chromatic number from below. Let ω be an acyclic orientation of a graph G. A sequence of acyclic orientations ω_1, ω_2, $\omega_3, \ldots$ is obtained from ω in such a way that $\omega_1 = \omega$, and ω_i $(i \geq 2)$ is obtained from ω_{i-1} by reversing the orientations of the edges incident to the sinks of w_{i-1}. This sequence is completely determined by ω, and it can be proved that there are positive integers p and M such that $\omega_i = \omega_{i+p}$ for every integer $i \geq M$. The value p at its minimum is denoted by p_ω. To bound $\chi_c(G)$ from below, the methodology we develop in this paper is based on the acyclic orientations $\omega_M, \omega_{M+1}, \cdots, \omega_{M+p_\omega-1}$ of G. Our method demonstrates for the first time the possibility of extracting some information about $\chi_c(G)$ from the period $\omega_M, \omega_{M+1}, \cdots, \omega_{M+p_\omega-1}$ to derive lower bounds for $\chi_c(G)$.

1. Introduction

The purpose of this paper is to explore the possibilities of using dynamic techniques to obtain lower bounds for circular chromatic number. We use Bondy and Murty's book [4] for terminology and notation not defined here and consider only finite, simple and connected graphs. First let us give a definition of the circular chromatic number $\chi_c(G)$ of a graph G. Suppose $k \geq 2d$ are positive integers. A (k,d)-*coloring* of a graph G is a mapping $f : V(G) \to \{0, 1, \ldots, k-1\}$ such that

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for any edge xy of G, $d \leq |f(x) - f(y)| \leq k - d$. If G has a (k, d)-coloring then we say G is (k, d)-*colorable*. The *circular chromatic number* $\chi_c(G)$ of a graph G [6, 7, 9] is defined as

$$\chi_c(G) = \inf\{k/d : G \text{ is } (k, d)-\text{colorable}\}.$$

In fact, to determine the circular chromatic number of a graph G, it suffices to check finitely many k, d whether G is (k, d)-colorable. In [8, 9, 11] we see the following fact

Fact 1. For any graph G with n vertices, we have

$$\chi_c(G) \in \{\tfrac{k}{d} : k \leq n,\ d \leq \alpha(G) \text{ and } \tfrac{n}{\alpha(G)} \leq \tfrac{k}{d} \leq \chi(G)\},$$

where $\alpha(G)$ is the maximum size of an independent set in G and $\chi(G)$ is the chromatic number of G.

A graph G is called k-*colorable* if $V(G)$ can be colored by at most k colors so that adjacent vertices are colored by different colors. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that G is k-colorable. For any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, that is $\chi_c(G)$ is a refinement of $\chi(G)$. The study of circular chromatic number $\chi_c(G)$ has been very active in the past decade [9, 11]. In this paper we present a new method for bounding the circular chromatic number from below.

To explain the main point of our method we introduce a discrete dynamical system on a graph G. Let ω be an acyclic orientation of G. A vertex in ω with zero outdegree (resp., zero indegree) is called a *sink* (resp., *source*) of ω. Let sink(ω) (resp., source(ω)) denote the set of sinks (resp., sources) in ω. One can obtain a sequence of acyclic orientations $\omega_1, \omega_2, \omega_3, \ldots$ from ω in such a way that $\omega_1 = \omega$, and ω_i $(i \geq 2)$ is obtained from ω_{i-1} by reversing the orientations of the edges incident to the sinks of w_{i-1}. This sequence is completely determined by ω, and hence we say that this sequence $\{\omega_i\}_{i=1}^{\infty}$ is *generated by* ω. Obviously the sequence of $\{\omega_i\}_{i=1}^{\infty}$ has the following periodic behavior [1, 2, 3]: There exist positive integers p and M such that $\omega_i = \omega_{i+p}$ for every integer $i \geq M$. The value p at its minimum is denoted by p_ω and is called the *period of* ω. For any $i \geq M$, the sequence $\omega_i, \omega_{i+1}, \cdots, \omega_{i+p_\omega-1}$ is called *a period generated by* ω. For a vertex u of an acyclic digraph ω, let m_ω^u denote the number of times that u becomes a sink in a period generated by ω. It was shown in [1, 2, 3] that $m_\omega^u = m_\omega^v$ for any two vertices u and v of the acyclic digraph ω. So we write m_ω instead of m_ω^u, and m_ω is called the *multiplicity of* ω. In Figure 1 we depict a sequence of acyclic orientations $\{\omega_i\}_{i=1}^{\infty}$ which is generated by ω_1. This sequence has the periodic property that $\omega_i = \omega_{i+5}$ for every $i \geq 1$, moreover, $p_{\omega_1} = 5$ and $m_{\omega_1} = 2$.

Suppose that w is an orientation of G and C is a closed walk of G. Denote by C_w^+ and C_w^- the set of *forward arcs* and the set of *backward arcs* of C in the orientation w, respectively. That is, C_w^+ is the collection of edges of C whose direction in the digraph w agree with the direction of the traversal (clockwise or counter-clockwise) of the closed walk C. From now on, for simplicity of notation, we write $\max_C |C|/|C_\omega^+|$ instead of $\max\{|C|/|C_\omega^+|, |C|/|C_\omega^-| : C$ is a closed walk of $G\}$. In 1989 [3], Barbosa and Gafni showed that if G is a tree with at least one edge then $p_\omega/m_\omega = 2$ for any acyclic orientation ω of G. Furthermore, if G contains at least one closed walk, they proved the following result.

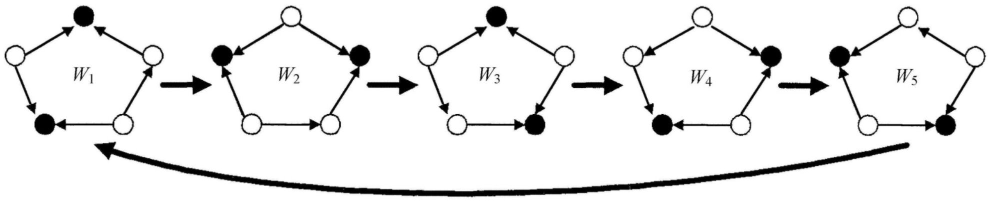

Fig. 1. A sequence of acyclic orientations $\{\omega_i\}_{i=1}^{\infty}$ generated by ω_1.

Theorem 2. ([3]). *Suppose G is not a tree. For any acyclic orientation ω of G we have*

$$\frac{p_\omega}{m_\omega} = \max_C \frac{|C|}{|C_\omega^+|},$$

where the maximum is over all closed walks of G.

In 1998 [5], the following result was proved by Goddyn et al.

Theorem 3. ([5]). *The circular chromatic number $\chi_c(G)$ of a graph G equals*

$$\min_\omega \max_C \frac{|C|}{|C_\omega^+|},$$

where the minimum is over all acyclic orientations of G and the maximum is over all closed walks of G.

It is clear that the following result follows from Theorems 2 and 3 immediately.

Theorem 4. *Suppose G is a connected simple graph. Then*

$$\chi_c(G) = \min_\omega \frac{p_\omega}{m_\omega},$$

where the minimum is over all acyclic orientations of G.

In Section 2, we use Theorem 4 to develop a new method for bounding the circular chromatic number $\chi_c(G)$ from below. The central feature of our method is that, for a period $\omega_i, \omega_{i+1}, \cdots, \omega_{i+p_\omega-1}$ generated by an acyclic orientation ω of a graph G, we are going to derive lower bounds on $\chi_c(G)$ by considering the sets sink(ω_i), sink(ω_{i+1}),$\cdots$, sink($\omega_{i+p_\omega-1}$) of this period. The aim of this paper is to develop a methodological framework for deriving lower bounds on $\chi_c(G)$ by using a period generated by an "optimal" acyclic orientation of G. To demonstrate our methodology, throughout this paper several lower bounds for circular chromatic number are derived in a somewhat unified manner. Some of these bounds are new, and some of these bounds might follow from existing theorems.

2. Lower Bounds for Circular Chromatic Number

In this section, lower bounds on the circular chromatic number $\chi_c(G)$ of a graph G are derived by using the dynamic characterization of $\chi_c(G)$ shown in Theorem 4. To simplify our expressions, throughout this section we assume that if $\omega_1, \omega_2, \cdots, \omega_{p_\omega}$ is a period generated by an acyclic digraph ω then, for any integer $j > p_\omega$, we define ω_j to be the digraph ω_{j-p_ω}. For a vertex u of a graph G, let $N_k(u)$ denote all vertices of distance k from u in G, i.e. $N_k(u) = \{v \in V(G) : d_G(u, v) = k\}$. For a set $S \subseteq V(G)$, we define $N_1(S) = \{v \in V(G) \setminus S : vu \in E(G)$ for some $u \in S\}$. We write $N_1(x, y)$ instead of $N_1(\{x, y\})$ for short. Let $\alpha_k(G)$ (or simply α_k if it cause no confusion) denote the maximum number of vertices in a vertex-induced k-colorable subgraph of G. Notice that $\alpha_1(G) = \alpha(G)$. For a vertex v of a graph G, let α_v denote the maximum size of an independent set of G containing v. For a vertex subset S of G, by abuse of notation, we also use S to denote the subgraph of G induced by S.

The following theorem reveals connection between the circular chromatic number $\chi_c(G)$ of a graph G and the chromatic number of the subgraph induced by a vertex's distance-1 neighborhood $N_1(u)$.

Theorem 5. (a) *For any vertex u of a graph G, $\chi_c(G) \geq \chi(N_1(u)) + 1$.*
(b) *For any graph G we have $\chi_c(G) \geq \sum_{v \in V(G)} 1/\alpha_v$.*

Proof. By Theorem 4, there is an acyclic orientation ω of G such that $p_\omega/m_\omega = \chi_c(G)$. Let $\omega_1, \omega_2, \cdots, \omega_{p_\omega}$ be a period generated by ω. Let I_i denote the indicator function on the set $\mathrm{sink}(\omega_i)$ i.e., $I_i(v) = 1$ if $v \in \mathrm{sink}(\omega_i)$ and 0 otherwise. Note that $\sum_{i=1}^{p_\omega} I_i(v) = m_\omega$ for any vertex v of G.

(a) Let $\xi = \chi(N_1(u))$. Note that if u and v are adjacent in G, and u is a sink of w_i and w_{i+t}, then there must be an index j such that $i < j < i + t$

and v is a sink of w_j. Moreover, since each $\mathrm{sink}(\omega_j)$ is an independent set of G, the subgraph induced by the neighbors of u is $(t-1)$-colorable. Therefore it must be that $t \geq \xi + 1$ and $u \notin \bigcup_{s=1}^{\xi} \mathrm{sink}(\omega_{i+s})$. It follows that $p_\omega \geq \sum_{i=1}^{p_\omega} (\xi+1) I_i(u) = (\xi+1) m_\omega$, and hence $\chi_c(G) = p_\omega / m_\omega \geq \xi + 1$.

(b) This part follows from the fact that

$$p_\omega = \sum_{i=1}^{p_\omega} \sum_{v \in V(G)} I_i(v)/|\mathrm{sink}(\omega_i)| = \sum_{v \in V(G)} \sum_{i=1}^{p_\omega} I_i(v)/|\mathrm{sink}(\omega_i)| \geq \sum_{v \in V(G)} m_\omega/\alpha_v. \blacksquare$$

Note that Theorem 5(a) yields the following well-known result that if H has a universal vertex, i.e., a vertex adjacent to every other vertex, then $\chi_c(H) = \chi(H)$.

From now on, we say that ω is an *optimal acyclic orientation* of G with period $\omega_1, \omega_2, \cdots, \omega_{p_\omega}$ if $\omega_1, \omega_2, \cdots, \omega_{p_\omega}$ is a period generated by ω and $p_\omega / m_\omega = \chi_c(G)$. The following theorem is a special case of Lemma 1 in [10], here we give a different proof based on arguments similar in concept to the proofs of Theorem 5.

Theorem 6. *Let H be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Let G be the graph obtained from $n+1$ disjoint graphs $H, H_1, H_2, \ldots, H_n$ by joining all vertices in $H_1, H_2, \ldots, H_n$ to a new vertex x, and joining all vertices in H_i to v_i, for $i = 1, 2, \ldots, n$. The graph G is represented diagrammatically in Figure 2 left. If $H_1, H_2, \ldots, H_n$ are t-chromatic graphs and $\chi(H) \geq 3$, then $\chi_c(G) \geq t + 2$.*

Proof. Let ω be an optimal acyclic orientation of G with period $\omega_1, \omega_2, \cdots, \omega_{p_\omega}$. Assume that $x \in \mathrm{sink}(\omega_i)$. Let s be the largest integer such that $x \notin \cup_{k=1}^{s} \mathrm{sink}(\omega_{i+k})$. Since x is adjacent to all vertices of H_j $(j = 1, 2, \ldots, n)$ in G, we have $V(H_j) \subseteq \cup_{k=1}^{s} \mathrm{sink}(\omega_{i+k})$ $(j = 1, 2, \ldots, n)$. It is clear that $s \geq t$. Let $r = \chi(H)$. To prove this theorem, we make the following stronger claim.

Claim. Either $s \geq t+1$ holds or $s = t$ and $x \notin \bigcup_{k=2}^{t+r} \mathrm{sink}(\omega_{i+s+k})$.

To prove the claim, it suffices to assume that $s = t$. In this case, for any $j = 1, 2, \ldots, n$ and any $k = 1, 2, \ldots, s$, we have $V(H_j) \cap \mathrm{sink}(\omega_{i+k}) \neq \emptyset$. Let ℓ be the largest integer such that $x \notin \cup_{k=2}^{\ell} \mathrm{sink}(\omega_{i+s+k})$. We should show that $\ell \geq t + r$. Note that $x \in \mathrm{sink}(\omega_i) \cap \mathrm{sink}(\omega_{i+s+1})$. According to the above arguments, in the digraph ω_{i+s+1} we see that $V(H_j) \subseteq N^-(v_j)$ for $j = 1, 2, \ldots, n$ (as depicted in Figure 2 right). Next, since $x \in \mathrm{sink}(\omega_{i+s+1}) \cap \mathrm{sink}(\omega_{i+s+\ell+1})$, we see that each vertex in the graphs $H_1, H_2, \ldots, H_n$ is a sink in one of the digraphs $\omega_{i+s+2}, \omega_{i+s+3}, \ldots, \omega_{i+s+\ell}$. Therefore it must be that $V(H) \subseteq \cup_{k=1}^{\ell} \mathrm{sink}(\omega_{i+s+k})$. Let $\bar{\ell}$ be the smallest integer such that $V(H) \subseteq \cup_{k=1}^{\bar{\ell}} \mathrm{sink}(\omega_{i+s+k})$. Note that $\bar{\ell} \geq \chi(H) = r \geq 3$, since $\mathrm{sink}(\omega_{i+s+1}), \ldots, \mathrm{sink}(\omega_{i+s+\bar{\ell}})$ are independent sets of G. By the choice of $\bar{\ell}$ there is a vertex in H, say v_n, such that $v_n \notin \mathrm{sink}(\omega_{i+s+k})$ for

$k = 1, 2, \ldots, \bar{\ell}-1$ and $v_n \in \text{sink}(\omega_{i+s+\bar{\ell}})$. It follows that $V(H_n) \cap \text{sink}(\omega_{i+s+k}) = \emptyset$ for each $k = 1, 2, \ldots, \bar{\ell}$. However, in the above discussion we have shown that $V(H_n) \subseteq \cup_{k=2}^{\ell} \text{sink}(\omega_{i+s+k})$. Therefore we conclude that $V(H_n) \subseteq \cup_{k=\bar{\ell}+1}^{\ell} \text{sink}(\omega_{i+s+k})$, and hence $\ell - \bar{\ell} \geq \chi(H_n) = t$. That is $\ell \geq t + r$, since $\bar{\ell} \geq r$, and this proves the claim.

Now we are in the position to be able to prove the theorem. We know that there are exactly m_ω integers $1 \leq i_1 < i_2 < \ldots < i_{m_\omega} \leq p_\omega$ such that $x \in \text{sink}(\omega_{i_k})$ for $k = 1, 2, \ldots, m_\omega$. Let $\ell_k = i_{k+1} - i_k$ for $k = 1, 2, \ldots, m_\omega - 1$, and let $\ell_{m_\omega} = p_\omega - (i_{m_\omega} - i_1)$. From what was shown in the first paragraph of this proof, we see that $\ell_k \geq t + 1$ for each $k = 1, 2, \ldots, m_\omega$. Moreover, by the claim we proved above, if $\ell_k = t + 1$ then $\ell_{k+1} \geq t + r \geq t + 3$ (the addition in the subscript of ℓ_{k+1} is taken modulo m_ω). Consequently, we have

$$p_\omega = \sum_{k=1}^{m_\omega} \ell_k \geq \sum_{k=1}^{m_\omega} (t + 2) = m_\omega(t + 2),$$

and therefore $\chi_c(G) = p_\omega / m_\omega \geq t + 2$. ■

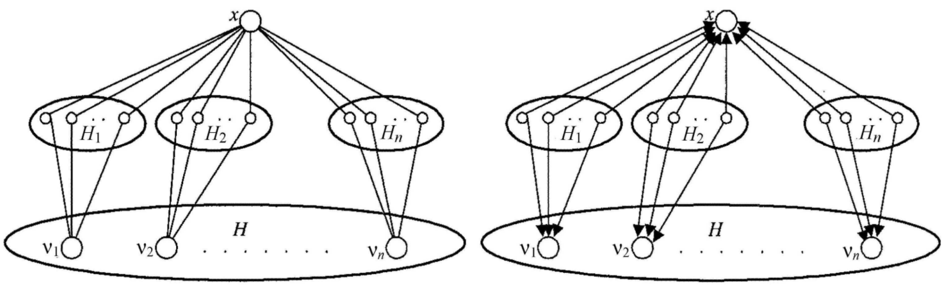

Fig. 2. The graph G (left) and the digraph ω_{i+s+1} (right).

Next, in the following two theorems, lower bounds of the form $|V(G)|/(\alpha_1(G)-\epsilon)$ are established for circular chromatic number $\chi_c(G)$ of a graph G. From now on, if $\omega_1, \omega_2, \cdots, \omega_{p_\omega}$ is a period generated by ω, then for arbitrary positive integers $i \geq 1$ and $\ell \leq p_\omega - 1$ the vector $(|\text{sink}(\omega_i)|, |\text{sink}(\omega_{i+1})|, \cdots, |\text{sink}(\omega_{i+\ell})|)$ is called a *sub-pattern of* ω.

Theorem 7. *If graph G has the following three properties P1:* $\chi(N_1(u, v)) \geq 2$ *for any two nonadjacent vertices u and v in G, P2:* $|V(G)| \leq 3\alpha_1(G) - 3$*, and P3:* $\alpha_2(G) < 2\alpha_1(G)$*, then we have* $\chi_c(G) \geq |V(G)|/(\alpha_1(G) - \frac{2}{3})$.

Proof. Let ω be an optimal acyclic orientation of G with period $\omega_1, \omega_2, \cdots, \omega_{p_\omega}$. Throughout the proof, let I_i denote the independent set $\text{sink}(\omega_i)$ for $i = 1, 2, 3, \cdots$.

Claim A. For any index i, we have $|I_i| + |I_{i+1}| \leq 2\alpha_1(G) - 1$.

Note that, for any index i, the vertex subset $I_i \bigcup I_{i+1}$ induces a bipartite subgraph of G. Since G has property P3, we see that $2\alpha_1(G) > \alpha_2(G) \geq |I_i \bigcup I_{i+1}| = |I_i| + |I_{i+1}|$ which proves the claim.

Claim B. For any index i, we have $|I_i| + |I_{i+1}| + |I_{i+2}| \leq 3\alpha_1(G) - 2$.

To prove this claim by contradiction, let us assume that, for some index i, $|I_i| + |I_{i+1}| + |I_{i+2}| \geq 3\alpha_1(G) - 1$. We must have $(|I_i|, |I_{i+1}|, |I_{i+2}|) = (\alpha_1(G), \alpha_1(G) - 1, \alpha_1(G))$, for otherwise either $|I_i| + |I_{i+1}| = 2\alpha_1(G)$ or $|I_{i+1}| + |I_{i+2}| = 2\alpha_1(G)$ would hold, contrary to Claim A. From property P2 and the fact that $I_i \cap I_{i+1} = \emptyset = I_{i+1} \cap I_{i+2}$, we conclude that there exist two distinct nonadjacent vertices u and v in the set $I_i \cap I_{i+2}$, and hence it must be $N_1(u, v) \subseteq I_{i+1}$. But which is impossible since G has property P1. This completes the proof of Claim B.

We conclude from Claim B that $p_\omega(3\alpha_1(G) - 2) \geq \sum_{i=1}^{p_\omega}(|I_i| + |I_{i+1}| + |I_{i+2}|) = 3m_\omega|V(G)|$, hence that $p_\omega/m_\omega \geq |V(G)|/(\alpha_1(G) - \frac{2}{3})$. This completes the proof. ■

Theorem 8. *Suppose* t *is a positive integer. If a graph* G *has the following three properties P1:* $\chi(N_1(v)) \geq t - 2$ *for any vertex* v *in* G*, P2:* $\chi(N_1(I)) \geq t - 1$ *for any maximum independent set* I *of* G*, and P3: any two different maximum independent sets of* G *intersect in exactly one vertex, then* $\chi_c(G) \geq |V(G)|/(\alpha_1(G) - \frac{t-1}{t})$.

Proof. Let ω be an optimal acyclic orientation of G with period $\omega_1, \omega_2, \cdots, \omega_{p_\omega}$. To shorten notation, let I_i stand for the independent set $\mathrm{sink}(\omega_i)$ for $i = 1, 2, 3, \cdots$.

Claim. For any index i, we have $\sum_{s=0}^{t-1} |I_{i+s}| \leq t(\alpha_1(G) - 1) + 1$.

To prove this claim by contradiction, let us assume that there exists an index i such that $\sum_{s=0}^{t-1} |I_{i+s}| \geq t(\alpha_1(G) - 1) + 2$. Since each independent set I_{i+s} has size at most $\alpha_1(G)$, there exist two maximum independent sets I_{i+a} and I_{i+b} with $0 \leq a < b \leq t - 1$ such that $|I_{i+k}| < \alpha_1(G)$ for each $k \in [a + 1, b - 1]$. If $I_{i+a} = I_{i+b}$ then we must have $N_1(I_{i+a}) \subseteq \cup_{s=a+1}^{b-1} I_{i+s}$ and hence $\chi(N_1(I_{i+a})) \leq (b-1) - (a+1) + 1 \leq t-2$. This contradicts the fact that G has the property P2. If $I_{i+a} \neq I_{i+b}$ then, by property P3, there exists a vertex v such that $I_{i+a} \cap I_{i+b} = \{v\}$, which leads to $N_1(v) \subseteq \cup_{s=a+1}^{b-1} I_{i+s}$, and hence $\chi(N_1(v)) \leq (b-1) - (a+1) + 1 \leq t - 2$. Which follows that $b - a = t - 1$ and hence $a = 0$, $b = t - 1$, since G has property P1 and $0 \leq a < b \leq t - 1$. We see at once that $\sum_{s=0}^{t-1} |I_{i+s}| = \sum_{s=a}^{b} |I_{i+s}| \leq t(\alpha_1(G) - 1) + 1$, since $|I_{i+a}| = |I_{i+b}| = \alpha_1(G)$, $|I_{i+a} \cap I_{i+b}| = 1$, and $|I_{i+k}| < \alpha_1(G)$ for each $k \in [a + 1, b - 1]$. This contradicts our assumption that $\sum_{s=0}^{t-1} |I_{i+s}| \geq t(\alpha_1(G) - 1) + 2$. This proves the claim.

It follows that $p_\omega[t(\alpha_1(G)-1)+1] \geq \sum_{i=1}^{p_\omega}\sum_{s=0}^{t-1}|I_{i+s}| = tm_\omega|V(G)|$, and finally that $p_\omega/m_\omega \geq |V(G)|/(\alpha_1(G)-\frac{t-1}{t})$. This proves the theorem. ■

In the following, we give two examples to show that the lower bounds obtained above are non-trivial, and the methodology we used in this paper throws some interesting light on arguments regarding circular chromatic number of a graph. Let Q be the graph obtained from the Petersen graph by deleting one vertex.

Example 9. $\chi_c(Q) = 3$.

Proof. By Fact 1 we have $\chi_c(Q) \in \{\frac{k}{d} : k \leq 9,\, d \leq 4 \text{ and } \frac{9}{4} \leq \frac{k}{d} \leq 3\}$, it follows that $\chi_c(Q) \in \{\frac{5}{2}, \frac{8}{3}, 3\}$. Since $\alpha_1(Q)=4$, $\alpha_2(Q)<8$ and $\chi(Q)=3$, we can easily check that the graph Q satisfies all the properties stated in the Theorem 7. It follows that $\chi_c(Q) \geq \frac{|V(Q)|}{\alpha_1(Q)-(2/3)} = 27/10 > 8/3$, which completes the proof. ■

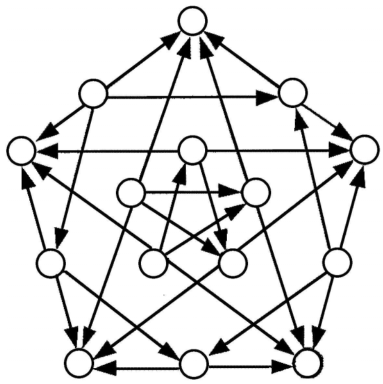

Fig. 3. An acyclic orientation ω on P_L.

Example 10. Suppose P_L is the line graph of the Petersen graph. Then $\chi_c(P_L) = 11/3$.

Proof. The acyclic orientation ω of P_L (depicted in Figure 3) has $p_\omega/m_\omega = 11/3$, and hence $\chi_c(P_L) \leq 11/3$. Since $\alpha_1(P_L) = 5$ and $\chi(P_L) = 4$, similar to the proof of Example 9, by Fact 1 we have $\chi_c(P_L) \in \{3, \frac{13}{4}, \frac{10}{3}, \frac{7}{2}, \frac{11}{3}\}$. Since each vertex of the Petersen graph has degree 3, P_L has the property P1 of Theorem 8 for $t = 4$. Since the subgraph left by deleting a perfect matching from the Petersen graph contains an odd cycle, thus P_L has the property P2 of Theorem 8 for $t = 4$. We also see that any two different maximum matchings of the Petersen graph intersect in exactly one edge, thus P_L has the property P3 of Theorem 8. Therefore

P_L satisfies all the properties stated in Theorem 8 for $t = 4$. We conclude that

$$\chi_c(P_L) \geq \frac{|V(P_L)|}{\alpha_1(P_L) - \frac{3}{4}} = \frac{15}{5 - \frac{3}{4}} = \frac{60}{17} > \frac{7}{2}.$$

Thus it must be $\chi_c(P_L) = 11/3$, since $\chi_c(P_L) \in \{3, \frac{13}{4}, \frac{10}{3}, \frac{7}{2}, \frac{11}{3}\}$. ■

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