

ADJACENT VERTEX DISTINGUISHING TOTAL COLORING OF GRAPHS WITH LOWER AVERAGE DEGREE

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Dedicated to Professor Ko-Wei Lih on the occasion of his 60th birthday.

Abstract. An adjacent vertex distinguishing total coloring of a graph G is a proper total coloring of G such that any pair of adjacent vertices are incident to distinct sets of colors. The minimum number of colors required for an adjacent vertex distinguishing total coloring of G is denoted by $\chi''_a(G)$. Let $\text{mad}(G)$ and $\Delta(G)$ denote the maximum average degree and the maximum degree of a graph G , respectively.

In this paper, we prove the following results: (1) If G is a graph with $\text{mad}(G) < 3$ and $\Delta(G) \geq 5$, then $\Delta(G) + 1 \leq \chi''_a(G) \leq \Delta(G) + 2$, and $\chi''_a(G) = \Delta(G) + 2$ if and only if G contains two adjacent vertices of maximum degree; (2) If G is a graph with $\text{mad}(G) < 3$ and $\Delta(G) \leq 4$, then $\chi''_a(G) \leq 6$; (3) If G is a graph with $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) \leq 3$, then $\chi''_a(G) \leq 5$.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we only consider simple graphs, i.e. graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *proper total k -coloring* is a mapping $\phi : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ such that any two adjacent or incident elements in $V(G) \cup E(G)$ have different colors. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a total k -coloring. Let $C_\phi(v) = \{\phi(v)\} \cup \{\phi(xv) \mid xv \in E(G)\}$ denote the set of colors assigned to a vertex v and those edges incident to v . A proper total k -coloring ϕ of G is *adjacent vertex distinguishing*, or a *total- k -avd-coloring*, if $C_\phi(u) \neq C_\phi(v)$ whenever $uv \in E(G)$. The *adjacent vertex distinguishing total chromatic number* $\chi''_a(G)$ is the smallest integer k such that G has a total- k -avd-coloring.

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Let $\Delta(G)$ denote the maximum degree of a graph G . By definition, it is evident that $\chi''_a(G) \geq \chi''(G) \geq \Delta(G) + 1$ for any graph G . Zhang et al. [6] first investigated the adjacent vertex distinguishing total coloring of graphs by determining completely the adjacent vertex distinguishing total chromatic numbers for paths, cycles, fans, wheels, trees, complete graphs, and complete bipartite graphs. The well-known Total Coloring Conjecture, made by Behzad [1] and independently by Vizing [4], says that every simple graph G has $\chi''(G) \leq \Delta(G) + 2$. This conjecture still remains open. Contrastively, Zhang et al. [6] put forward the following conjecture:

Conjecture 1. If G is a graph with at least two vertices, then $\chi''_a(G) \leq \Delta(G) + 3$.

If Conjecture 1 were true, then the upper bound $\Delta(G) + 3$ for $\chi''_a(G)$ is tight. For instance, $\chi''_a(K_{2n+1}) = \Delta(K_{2n+1}) + 3 = 2n + 3$ for any $n \geq 1$. Chen [2] further constructed a class of graphs, i.e. the joint graph $sP_3 \vee K_t$, attaining the upper bound $\Delta + 3$.

More recently, Wang [5] and Chen [2] independently confirmed Conjecture 1 for graphs G with $\Delta(G) \leq 3$.

Let $\chi(G)$ and $\chi'(G)$ denote the (vertex) chromatic number and the edge chromatic number of a graph G , respectively. The following result follows immediately from the definitions of parameters under consideration:

Proposition 1. For any graph G , $\chi''_a(G) \leq \chi(G) + \chi'(G)$.

Proposition 1 implies that Conjecture 1 holds for all bipartite graphs and for Class 1 graphs G with $\chi(G) \leq 3$. We say that a graph G is of *Class 1* if $\chi'(G) = \Delta(G)$.

Another easy observation was given in [6] as follows:

Proposition 2. If G is a graph with two adjacent vertices of maximum degree, then $\chi''_a(G) \geq \Delta(G) + 2$.

Since every simple bipartite graph is of Class 1, Proposition 2 implies that every simple bipartite graph G with a pair of adjacent vertices of maximum degree has $\chi''_a(G) = \Delta(G) + 2$. In particular, this is true for all regular bipartite graphs with at least one edge.

We recall that the *girth* $g(G)$ of a graph G is the length of a shortest cycle in G . The *maximum average degree* $\text{mad}(G)$ of G is defined by

$$\text{mad}(G) = \max_{H \subseteq G} \{2|E(H)|/|V(H)|\}.$$

Proposition 3. ([3]). Let G be a planar graph. Then

$$\text{mad}(G) < \frac{2g(G)}{g(G) - 2}.$$

In this paper, we shall prove the following results:

Theorem A. *Let G be a graph.*

- (1) *If $\text{mad}(G) < 3$ and $\Delta(G) \geq 5$, then $\Delta(G) + 1 \leq \chi''_a(G) \leq \Delta(G) + 2$; and $\chi''_a(G) = \Delta(G) + 2$ if and only if G contains adjacent vertices of maximum degree.*
- (2) *If $\text{mad}(G) < 3$ and $\Delta(G) = 4$, then $\chi''_a(G) \leq 6$.*
- (3) *If $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) = 3$, then $\chi''_a(G) \leq 5$.*

We see from Proposition 3 that if G is a planar graph with $g(G) \geq 6$ then $\text{mad}(G) < 3$, and if $g(G) \geq 8$ then $\text{mad}(G) < \frac{8}{3}$. This fact together with Theorem A gives the following corollary:

Corollary B. *Let G be a planar graph.*

- (1) *If $g(G) \geq 6$ and $\Delta(G) \geq 5$, then $\Delta(G) + 1 \leq \chi''_a(G) \leq \Delta(G) + 2$; and $\chi''_a(G) = \Delta(G) + 2$ if and only if G contains adjacent vertices of maximum degree.*
- (2) *If $g(G) \geq 6$ and $\Delta(G) = 4$, then $\chi''_a(G) \leq 6$.*
- (3) *If $g(G) \geq 8$ and $\Delta(G) = 3$, then $\chi''_a(G) \leq 5$.*

The proof of Theorem A is established in Sections 2 and 3. We need to consider two cases, depending on the value of $\text{mad}(G)$.

2. CASE $\text{Mad}(G) < 3$

Let G be a graph. The *degree* of a vertex v in G , denoted $d_G(v)$, is the number of vertices in G that are adjacent to v . Those vertices are also called the neighbors of v . A k -*vertex* is a vertex of degree k . A 1-vertex is also said to be a *leaf*. Let $|T(G)| = |V(G)| + |E(G)|$.

Lemma 4. *Let G be a graph.*

- (1) *If v is a leaf of G , then $\text{mad}(G - v) \leq \text{mad}(G)$;*
- (2) *If e is an edge of G , then $\text{mad}(G - e) \leq \text{mad}(G)$.*

Proof. By definition, (2) holds obviously. The proof of (1) appeared in [3]. ■

Suppose that ϕ is a total- k -avd-coloring of a graph G with the color set $C = \{1, 2, \dots, k\}$, where $k \geq 5$. Assume that $v \in V(G)$ with $d_G(v) \leq 2$ is not adjacent

to any vertex of the same degree as itself. Since v has at most two adjacent vertices and two incident edges and $|C| \geq 5$, we may first erase the color of v and finally recolor it after arguing. In other words, we omit the coloring for such 1-vertices and 2-vertices in the following discussion.

Theorem 5. *If G is a graph with $\text{mad}(G) < 3$ and $K(G) = \max\{\Delta(G)+2, 6\}$, then $\chi''_a(G) \leq K(G)$.*

Proof. Our proof proceeds by *reductio ad absurdum*. Assume that G is a counterexample to the theorem such that $|T(G)|$ is as small as possible. Since $\chi''_a(G) = \max\{\chi''_a(G_i)\}$ and $\Delta(G) = \max\{\Delta(G_i)\}$, both maxima being taken over all components G_i of G , we know that G is a connected graph with $\text{mad}(G) < 3$ and $\chi''_a(G) > K(G)$. Any proper subgraph H of G with $\text{mad}(H) < 3$ has $\chi''_a(H) \leq K(H) \leq K(G)$ by the minimality of $T(G)$.

We are going to analyze the structure of G with a sequence of auxiliary claims. Then we will derive a contradiction using the discharging method.

In the subsequent proofs, we routinely construct appropriate proper total colorings without verifying in detail that they are adjacent vertex distinguishing because that usually can be supplied in a straightforward manner.

Claim 1. No vertex of degree at most 3 is adjacent to a leaf.

Proof. Assume to the contrary that G contains a vertex v with $d_G(v) \leq 3$ adjacent to a leaf. Without loss of generality, we may assume that $d_G(v) = 3$ and u_1, u_2, u_3 are neighbors of v with $d_G(u_1) = 1$. Let $H = G - u_1$. Then H is a connected graph with $\text{mad}(H) \leq \text{mad}(G) < 3$ by Lemma 4(1). By the minimality of $|T(G)|$, there is a total- $K(G)$ -avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, K(G)\}$. We note that $|C| = K(G) \geq 6$. Suppose that $\phi(v) = 1$, $\phi(vu_2) = 2$, and $\phi(vu_3) = 3$.

If $|\{4, 5, 6\} \cap C_\phi(u_i)| \geq 2$ for all $i = 2, 3$, we color vu_1 with 4. If $|\{4, 5, 6\} \cap C_\phi(u_i)| \leq 1$ for all $i = 2, 3$, we color vu_1 with a color in $\{4, 5, 6\} \setminus (C_\phi(u_2) \cup C_\phi(u_3))$. If $|\{4, 5, 6\} \cap C_\phi(u_2)| \geq 2$ and $|\{4, 5, 6\} \cap C_\phi(u_3)| \leq 1$, say, we color vu_1 with a color in $\{4, 5, 6\} \setminus C_\phi(u_3)$. It is easy to see that ϕ is extended to the whole graph G in every possible case. ■

Claim 2. There does not exist a path $P = x_1x_2 \cdots x_n$ with $d_G(x_1), d_G(x_n) \geq 3$ and $d_G(x_i) = 2$ for all $i = 2, 3, \dots, n-1$, where $n \geq 4$.

Proof. Assume to the contrary that G contains such a path P . Let $H = G - x_2x_3$. Then H is a graph with $\text{mad}(H) \leq \text{mad}(G) < 3$ by Lemma 4(2). By the minimality of $|T(G)|$, there is a total- $K(G)$ -avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, K(G)\}$.

If $n = 4$, we recolor x_2 with a color $a \in C \setminus \{\phi(x_1), \phi(x_3), \phi(x_1x_2), \phi(x_3x_4)\}$, and color x_2x_3 with a color in $C \setminus \{a, \phi(x_3), \phi(x_1x_2), \phi(x_3x_4)\}$.

If $n \geq 5$, we recolor x_3x_4 with $a \in C \setminus \{\phi(x_2), \phi(x_4), \phi(x_5), \phi(x_4x_5)\}$, x_3 with $b \in C \setminus \{a, \phi(x_2), \phi(x_4), \phi(x_4x_5)\}$, and color x_2x_3 with a color in $C \setminus \{a, b, \phi(x_2), \phi(x_1x_2)\}$. ■

Claim 3. There does not exist a k -vertex v , $k \geq 4$, with neighbors v_1, v_2, \dots, v_k such that $d_G(v_1) = 1$, $d_G(v_i) \leq 2$ for $2 \leq i \leq k - 2$.

Proof. Assume to the contrary that G contains such vertex v . For $2 \leq i \leq k - 2$, if v_i is a 2-vertex, we denote by $u_i \neq v$ the second neighbor of v_i . Note that u_i is of degree at least 3 by Claim 2 if it exists. Let $H = G - v_1$. By the minimality of $|T(G)|$, there is a total- $K(G)$ -avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, K(G)\}$. Without loss of generality, we assume that $\phi(v) = 1$, $\phi(vv_i) = i$ for $i = 2, 3, \dots, k$. Since $\Delta(G) \geq d_G(v) = k$, $|C| \geq \Delta(G) + 2 \geq k + 2$. Thus, $k + 1, k + 2 \in C$.

If $k + 1 \in C_\phi(v_{k-1}) \cap C_\phi(v_k)$, we color vv_1 with $k + 2$. If $k + 1 \notin C_\phi(v_{k-1}) \cup C_\phi(v_k)$, we color vv_1 with $k + 1$. The similar argument works for the color $k + 2$. If $\{k + 1, k + 2\} \subseteq C_\phi(v_{k-1}) \setminus C_\phi(v_k)$ or $\{k + 1, k + 2\} \subseteq C_\phi(v_k) \setminus C_\phi(v_{k-1})$, we color vv_1 with $k + 1$.

Now suppose that $k + 1 \in C_\phi(v_{k-1}) \setminus C_\phi(v_k)$ and $k + 2 \in C_\phi(v_k) \setminus C_\phi(v_{k-1})$, say. If $d_G(v_2) = 1$, we recolor (or color) vv_2 with $k + 1$ and vv_1 with $k + 2$. If $d_G(v_2) = 2$, we recolor (or color) vv_2 with a color $a \in \{k + 1, k + 2\} \setminus \{\phi(v_2u_2)\}$, vv_1 with a color in $\{k + 1, k + 2\} \setminus \{a\}$, and v_2 with a color different from $1, a, \phi(u_2), \phi(u_2v_2)$. ■

Claim 4. There does not exist a 2-vertex v adjacent to a 3-vertex u .

Proof. Assume to the contrary that G contains a 2-vertex v adjacent to a 3-vertex u and another vertex w . Let $u_1, u_2 \neq v$ be the other neighbors of u . By Claims 1 and 2, $d_G(w) \geq 3$. Let $H = G - uv$. By the minimality of $|T(G)|$, there is a total- $K(G)$ -avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, K(G)\}$. Without loss of generality, we assume that $\phi(u) = 1$, $\phi(uu_1) = 2$, and $\phi(uu_2) = 3$. Note that at least two colors in $\{4, 5, 6\}$ differ from $\phi(vw)$, say $\phi(vw) \neq 4, 5$.

If $4 \in C_\phi(u_1) \cap C_\phi(u_2)$, we color uv with 5. If $4 \notin C_\phi(u_1) \cup C_\phi(u_2)$, we color uv with 4. The similar argument works for the color 5. If $\{4, 5\} \subseteq C_\phi(u_1) \setminus C_\phi(u_2)$ or $\{4, 5\} \subseteq C_\phi(u_2) \setminus C_\phi(u_1)$, we color uv with 4. Now suppose that $4 \in C_\phi(u_1) \setminus C_\phi(u_2)$ and $5 \in C_\phi(u_2) \setminus C_\phi(u_1)$, say. If $\phi(vw) \neq 6$, we color uv with 6. If $\phi(vw) = 6$, we need to consider some subcases. When $6 \in C_\phi(u_1)$, we color uv with 4. When $6 \in C_\phi(u_2)$, we color uv with 5. When $6 \notin C_\phi(u_1) \cup C_\phi(u_2)$, we recolor u with 6 and color uv with 1. Finally, we recolor v with a color different from $\phi(u), \phi(w), \phi(uv), \phi(vw)$. ■

Claim 5. There does not exist a 4-vertex v adjacent to three 2-vertices.

Proof. Assume to the contrary that G contains a 4-vertex v with neighbors v_1, v_2, v_3, v_4 such that $d_G(v_1) = d_G(v_2) = d_G(v_3) = 2$. Let $u_1 \neq v$ be the second neighbor of v_1 . By Claims 1, 2 and 4, $d_G(u_1) \geq 4$. Let $H = G - vv_1$. By the minimality of $|T(G)|$, there is a total- $K(G)$ -avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, K(G)\}$. Without loss of generality, we assume that $\phi(v) = 1$, $\phi(vv_i) = i$ for $i = 2, 3, 4$. If $5, 6 \in C_\phi(v_4)$, or $5, 6 \notin C_\phi(v_4)$, we color vv_1 with a color in $\{5, 6\} \setminus \{\phi(v_1u_1)\}$. Otherwise, we may assume, without loss of generality, that $5 \in C_\phi(v_4)$ and $6 \notin C_\phi(v_4)$. If $\phi(v_1u_1) \neq 6$, then we color vv_1 with 6. If $\phi(v_1u_1) = 6$, we recolor v with 6 and color vv_1 with 1. ■

Claim 6. There does not exist a 5-vertex v adjacent to five 2-vertices.

Proof. Assume to the contrary that G contains a 5-vertex v adjacent to five 2-vertices v_1, v_2, \dots, v_5 . For $1 \leq i \leq 5$, let $u_i \neq v$ be the second neighbor of v_i . We note that $d_G(u_i) \geq 3$ by Claims 1 and 2. By the minimality, $G - vv_1$ has a total- $K(G)$ -avd-coloring ϕ with the color set $C = \{1, 2, \dots, K(G)\}$. We color vv_1 with a color in $C \setminus \{\phi(vv_2), \phi(vv_3), \phi(vv_4), \phi(vv_5), \phi(v_1u_1)\}$ and recolor v with a color in $C \setminus \{\phi(vv_1), \phi(vv_2), \phi(vv_3), \phi(vv_4), \phi(vv_5)\}$. Finally, we recolor v_1, v_2, \dots, v_5 (if needed). ■

Now we continue the proof of Theorem 5.

Let H be the graph obtained by removing all leaves of G . Then $\text{mad}(H) \leq \text{mad}(G) < 3$ by Lemma 4. The other properties of the graph H are collected in the following Claim 7:

Claim 7.

- (1) There are no vertices of degree less than 2;
- (2) If $v \in V(G)$ with $2 \leq d_G(v) \leq 3$, then $v \in V(H)$ and $d_H(v) = d_G(v)$;
- (3) If $v \in V(H)$ with $d_H(v) = 2$, then $d_G(v) = 2$;
- (4) If $v \in V(G)$ with $d_G(v) \geq 4$, then $d_H(v) \geq 3$.

Proof. (1) Suppose that H contains a vertex v with $d_H(v) \leq 1$, then $d_G(v) \geq 2$ by the definition of H and v is adjacent to at least $d_G(v) - 1$ leaves in G . This contradicts Claims 1 and 3.

The statements (2) to (4) follow immediately from Claim 1. ■

Claim 7 asserts that H can not contain a 2-vertex adjacent to a 2-vertex or a 3-vertex.

In order to complete the proof, we make use of discharging method. First, we define an initial charge function $w(v) = d_H(v)$ for every $v \in V(H)$. Next, we design a discharging rule and redistribute weights accordingly. Once the discharging is finished, a new charge function w' is produced. However, the sum of all charges is kept fixed when the discharging is in progress. Nevertheless, we can show that $w'(v) \geq 3$ for all $v \in V(H)$. This leads to the following obvious contradiction:

$$3 = \frac{3|V(H)|}{|V(H)|} \leq \frac{\sum_{v \in V(H)} w'(v)}{|V(H)|} = \frac{\sum_{v \in V(H)} w(v)}{|V(H)|} = \frac{2|E(H)|}{|V(H)|} \leq \text{mad}(H) < 3.$$

The discharging rule is defined as follows:

(R). Every vertex v of degree at least 4 gives $\frac{1}{2}$ to each adjacent 2-vertex.

Let $v \in V(H)$. Then $d_H(v) \geq 2$ by Claim 7(1). If $d_H(v) = 2$, then v is adjacent to two vertices of degree at least 4 by Claim 4, each of which sends $\frac{1}{2}$ to v by (R). Thus, $w'(v) \geq d_H(v) + 2 \times \frac{1}{2} = 2 + 1 = 3$. If $d_H(v) = 3$, then $w'(v) = w(v) = 3$. If $d_H(v) = 4$, then v is adjacent to at most two 2-vertices by Claim 5. Thus, $w'(v) \geq 4 - 2 \times \frac{1}{2} = 3$. If $d_H(v) = 5$, then v is adjacent to at most four 2-vertices by Claim 6. Thus, $w'(v) \geq 5 - 4 \times \frac{1}{2} = 3$. If $d_H(v) \geq 6$, then v is adjacent to at most $d_H(v)$ 2-vertices and hence $w'(v) \geq d_H(v) - \frac{1}{2}d_H(v) = \frac{1}{2}d_H(v) \geq 3$ by (R). ■

Theorem 6. Let G be a graph with $\text{mad}(G) < 3$ and without adjacent vertices of maximum degree. Let $K'(G) = \max\{\Delta(G) + 1, 6\}$. Then $\chi''_a(G) \leq K'(G)$.

Proof. The proof is proceeded by contradiction. Assume that G is a counterexample to the theorem such that $|T(G)|$ is as small as possible. With the same argument, we can prove that G satisfies Claims 1, 2, 4, 5 and 6.

If G does not satisfy Claim 3, we suppose that v is a k -vertex, $k \geq 4$, with neighbors v_1, v_2, \dots, v_k such that $d_G(v_1) = 1$, $d_G(v_i) \leq 2$ for $2 \leq i \leq k - 2$. If v_i is a 2-vertex, for $2 \leq i \leq k - 2$, we use $u_i \neq v$ to denote the second neighbor of v_i . By the minimality of $|T(G)|$, $G - v_1$ has a total- $K'(G)$ -avd-coloring ϕ' with the color set $C' = \{1, 2, \dots, K'(G)\}$. Without loss of generality, we assume that $\phi'(v) = 1$, $\phi'(vv_i) = i$ for $i = 2, 3, \dots, k$.

If $d_G(v) = \Delta(G)$, then since G contains no adjacent vertices of maximum degree, we have $d_G(v_{k-1}) \neq \Delta(G)$ and $d_G(v_k) \neq \Delta(G)$. It suffices to properly color vv_1 with a color different from the colors of v, vv_2, \dots, vv_k . If $d_G(v) = k < \Delta(G)$, then $|C'| \geq \Delta(G) + 1 \geq k + 2$, which implies that $k + 1, k + 2 \in C'$. The remaining proof is similar to that of Claim 3. Therefore, G satisfies Claim 3.

Similarly, let H be the graph obtained by removing all leaves of G . Then $\text{mad}(H) \leq \text{mad}(G) < 3$ by Lemma 4. Using the same initial charge function

$w(v) = d_H(v)$ for all $v \in V(H)$ and the same discharging rule (R) as in Theorem 5, we can complete the proof by providing a contradiction. ■

Combining Theorem 5 and Theorem 6, we conclude (1) and (2) in Theorem A.

3. CASE $\text{Mad}(G) < \frac{8}{3}$

In this section, we prove the statement (3) in Theorem A.

Theorem 7. *If G is a graph with $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) \leq 3$, then $\chi''_a(G) \leq 5$.*

Proof. The proof is proceeded by contradiction. Assume that G is a counterexample to the theorem such that $|T(G)|$ is as small as possible. It is easy to show that G possesses the following properties (a) to (c).

- (a) No 2-vertex is adjacent to a leaf.
- (b) No 3-vertex is adjacent to at least two leaves.
- (c) There are no adjacent 2-vertices.

Claim 1. G does not contain a 3-vertex v with neighbors v_1, v_2, v_3 such that $d_G(v_1) = 1$ and $d_G(v_2) = 2$.

Proof. Assume to the contrary that G contains such vertex v . Let $u_2 \neq v$ be the second neighbor of v_2 . Note that u_2 is a 3-vertex by (a) and (c). Let $H = G - v_1$. By the minimality of $|T(G)|$, there is a total-5-avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, 5\}$. Let $\phi(v) = 1$, $\phi(vv_i) = i$ for $i = 2, 3$. If at least one of 4 and 5 does not belong to $C_\phi(v_3)$, say $4 \notin C_\phi(v_3)$, we color vv_1 with 4. Otherwise, $4, 5 \in C_\phi(v_3)$, so we can color vv_1 with 5. ■

Claim 2. Suppose that v is a 3-vertex adjacent to a leaf x and two other vertices y and z . Let ϕ be a total-5-avd-coloring of the subgraph $G - x$ with the color set $C = \{1, 2, \dots, 5\}$. Then $\{\phi(v), \phi(y), \phi(z), \phi(vy), \phi(vz)\} = C$.

Proof. Without loss of generality, we may assume that $\phi(v) = 1$, $\phi(vy) = 2$ and $\phi(vz) = 3$. If there is $k \in \{4, 5\}$ such that $C_\phi(y) \neq \{1, 2, 3, k\}$ and $C_\phi(z) \neq \{1, 2, 3, k\}$, then we can color vx with k , which produces a contradiction.

Assume that $C_\phi(y) = \{1, 2, 3, 4\}$ and $C_\phi(z) = \{1, 2, 3, 5\}$, say. Clearly, $\phi(y) \neq 1, 2$. If $\phi(y) = 3$, we can recolor v with 4 and color xv with 5. Thus, we must have $\phi(y) = 4$. Similarly, we can prove that $\phi(z) = 5$. Consequently, $\{\phi(v), \phi(y), \phi(z), \phi(vy), \phi(vz)\} = \{1, 2, 3, 4, 5\} = C$. ■

Claim 3. There are no two adjacent 3-vertices each of which is adjacent to a leaf.

Proof. Assume to contrary that G contains two adjacent 3-vertices u and v such that u is adjacent to a leaf u_1 and v is adjacent to a leaf v_1 . Let $u_2 \neq v, u_1$ be the third neighbor of u , and $v_2 \neq u, v_1$ be the third neighbor of v . Let $H = G - u_1$. By the minimality of $|T(G)|$, there is a total-5-avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, 5\}$. By Claim 2 and its proof, we may assume that $\phi(u) = 1$, $\phi(uu_2) = 2$, $\phi(uv) = 3$, $\phi(u_2) = 4$, $\phi(v) = 5$, $C_\phi(u_2) = \{1, 2, 3, 4\}$, and $C_\phi(v) = \{1, 2, 3, 5\}$. If $C_\phi(v_2) \neq \{1, 2, 4, 5\}$, we recolor u with 3, uv with 4, and color uu_1 with 5. If $C_\phi(v_2) = \{1, 2, 4, 5\}$, we recolor vv_1 with 4 and color uu_1 with 5. ■

Claim 4. There does not exist a 3-vertex that is adjacent to two 3-vertices each of which is adjacent to a leaf.

Proof. Assume that G contains a 3-vertex u with neighbors x, y, z such that y is adjacent to a leaf y_1 and z is adjacent to a leaf z_1 . Let $y_2 \neq u, y_1$ be the third neighbor of y and $z_2 \neq u, z_1$ be the third neighbor of z . Let $H = G - \{y_1, z_1\}$. By the minimality of $|T(G)|$, there is a total-5-avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, 5\}$. Without loss of generality, we assume that $\phi(x) = 1$, $\phi(xu) = 2$, and $C_\phi(x) \subseteq \{1, 2, 3, 4\}$. First, we erase the colors of u, y, z, uy and uz .

If there is $k \in \{3, 4\}$ such that $\phi(yy_2) = \phi(zz_2) = k$, we color u with k , uy with 5, uz with a color in $\{3, 4\} \setminus \{k\}$, y with a color in $\{1, 2\} \setminus \{\phi(y_2)\}$, and z with a color in $\{1, 2\} \setminus \{\phi(z_2)\}$. Similarly to the proof of Claim 2, we can extend ϕ to both edges yy_1 and zz_1 .

Suppose that such k does not exist. We can color u with 5, uy with 3, uz with 4, say. Let $\phi(yy_2) = a$, $\phi(y_2) = b$, $\phi(zz_2) = p$, $\phi(z_2) = q$. In the following partial coloring, y is called *good* if $\{\phi(u), \phi(uy)\} \cap \{a, b\} \neq \emptyset$ and z *good* if $\{\phi(u), \phi(uz)\} \cap \{p, q\} \neq \emptyset$.

If both y and z are good, then yy_1 and zz_1 can be properly colored with a similar argument as in the proof of Claim 2. Otherwise, we may assume that y is not good, that is $\{a, b\} \cap \{3, 5\} = \emptyset$. This means that $a, b \in \{1, 2, 4\}$. Moreover, if z is not good, then $p, q \in \{1, 2, 3\}$. We have the following cases (up to symmetry):

Case 1. $\{a, b\} = \{1, 2\}$.

(1.1) z is good.

By Claim 2, z and zz_1 can be colored. If $a = 1$, we color y with 4 and yy_1 with 5. Assume that $b = 1$. If $C_\phi(y_2) \neq \{1, 2, 3, 4\}$, we color y with 4 and yy_1 with 1. If $C_\phi(y_2) = \{1, 2, 3, 4\}$, we recolor uy with 1 such that both y and z are good.

(1.2) $\{p, q\} = \{1, 2\}$.

If $b = 1$, we recolor uy with 1 such that y is good. Then we color z with 3 and zz_1 with 5. If $q = 1$, we have a similar coloring by symmetry.

If $a = p = 1$, we color y with 4, z with 3, and yy_1, zz_1 with 5.

(1.3) $\{p, q\} = \{1, 3\}$.

If $b = p = 1$, we recolor uy with 1 and uz with 3 such that both y and z are good.

If $b = q = 1$, we recolor uy with 1 such that y is good. Color z with 2 and zz_1 with 5.

If $a = p = 1$, we color y with 4, z with 2, and yy_1, zz_1 with 5.

If $a = q = 1$, we color y with 4 and yy_1 with 5. Then we recolor uz with 1 such that z is good.

(1.4) $\{p, q\} = \{2, 3\}$.

We color z with 1 and zz_1 with 5. If $a = 1$, we color y with 4 and yy_1 with 5. If $b = 1$, we recolor uy with 1 such that y is good.

Case 2. $\{a, b\} = \{1, 4\}$.

(2.1) z is good.

By Claim 2, z and zz_1 can be colored. If $a = 1$, we color y with 2 and yy_1 with 5. If $b = 1$, we recolor uy with 1 such that both y and z are good.

(2.2) $\{p, q\} = \{1, 2\}$.

We have a similar argument as for Case 1.

(2.3) $\{p, q\} = \{1, 3\}$.

If $b = p = 1$, we recolor uy with 1 and uz with 3 such that both y and z are good.

If $b = q = 1$, we recolor uy with 1 such that y is good. Color z with 2 and zz_1 with 5.

If $a = p = 1$, we color y, z with 2, and yy_1, zz_1 with 5.

If $a = q = 1$, we recolor u with 3 such that z is good. Recolor uy with 5, and color y with 2 and yy_1 with 3.

(2.4) $\{p, q\} = \{2, 3\}$.

If $a = 1$, we color y with 2, z with 1, and yy_1, zz_1 with 5. If $b = 1$, we recolor uy with 1 such that y is good. Color z with 1 and zz_1 with 5.

Case 3. $\{a, b\} = \{2, 4\}$.

If z is good, we color y with 1 and yy_1 with 5.

If $\{p, q\} = \{2, 3\}$, we color y, z with 1, yy_1, zz_1 with 5.

Assume that $q = 1$ and $p \in \{2, 3\}$. If $a = 2$, we recolor uy with 4 and uz with 1 such that both y and z are good. If $b = 2$, we recolor uz with 1 such that z is good, then color y with 1 and yy_1 with 5.

Assume that $p = 1$ and $q = 3$. If $a = 2$, we recolor uy with 4 and uz with 3 such that both y and z are good. If $b = 2$, we recolor uz with 3 such that z is good, then color or recolor uy with 1, y with 3, yy_1 with 5.

Assume that $p = 1$ and $q = 2$. If $a = 2$, we recolor uy with 4 such that y is good. Color or recolor uz with 3, z with 4, zz_1 with 5. If $b = 2$, we color or recolor uy with 1, uz and y with 3, z with 4, yy_1 and zz_1 with 5. ■

With a similar and easier proof, we can get the following:

Claim 5. There does not exist a 3-vertex v adjacent to a 2-vertex and a 3-vertex u such that u is adjacent to a leaf.

Claim 6. There does not exist a 3-vertex v adjacent to two 2-vertices.

Proof. Assume to contrary that G contains a 3-vertex v adjacent to two 2-vertices y, z and the third vertex x . Let $H = G - \{vy, vz\}$. By the minimality, there is a total-5-avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, 5\}$. Without loss of generality, we assume that $\phi(x) = 1$, $\phi(xv) = 2$, and $C_\phi(x) \subseteq \{1, 2, 3, 4\}$. We first color v with 5, then properly color vy, vz , and recolor y, z (if needed). ■

Let H be the graph obtained by removing all leaves of G . Then $\text{mad}(H) \leq \text{mad}(G) < \frac{8}{3}$ by Lemma 4. By Claims 1 to 6, we see that H contains neither vertices of degree less than 2 nor two adjacent 2-vertices, and every 3-vertex is adjacent to at most one 2-vertex.

Again, we define an initial charge $w(v) = d_H(v)$ for every vertex $v \in V(H)$ and design the following discharging rule:

(R') Every 3-vertex gives $\frac{1}{3}$ to its adjacent 2-vertex.

Let $w'(v)$ denote the new charge of a vertex v after the discharging process is finished on H . If v is a 3-vertex, then since it is adjacent to at most one 2-vertex, we have $w'(v) \geq 3 - \frac{1}{3} = \frac{8}{3}$ by (R'). If v is a 2-vertex, then since it is not adjacent to any 2-vertex, i.e., it is adjacent to two 3-vertices, we have $w'(v) = 2 + \frac{1}{3} + \frac{1}{3} = \frac{8}{3}$ by (R'). This shows that, for any $v \in V(H)$, we have $w'(v) \geq \frac{8}{3}$. However, this leads to the following contradiction:

$$\frac{8}{3} = \frac{\frac{8}{3}|V(H)|}{|V(H)|} \leq \frac{\sum_{v \in V(H)} w'(v)}{|V(H)|} = \frac{\sum_{v \in V(H)} w(v)}{|V(H)|} = \frac{2|E(H)|}{|V(H)|} \leq \text{mad}(H) < \frac{8}{3}. \quad \blacksquare$$

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