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MEDIANS OF GRAPHS AND KINGS OF TOURNAMENTS*

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Abstract. We first prove that for any graph G with a positive vertex weight function w, there exists a graph H with a positive weight function w' such that $w(v) = w'(v)$ for all vertices v in G and whose w'-median is G. This is a generalization of a previous result for the case in which all weights are 1. The second result is that for any n-tournament T without transmitters, there exists an integer $m \leq 2n-1$ and an m-tournament T' whose kings are exactly the vertices of T. This improves upon a previous result for $m \leq 2n$.

1. Introduction

In a graph (digraph) G, the *distance* $d_G(u,v)$ from a vertex u to another vertex v is the minimum number of edges in a u-v path (dipath). The *eccentricity* of a vertex v is

$$e_G(v) = \max\{d_G(v,u) : u \in V(G)\}.$$

A *central vertex* is a vertex with a minimum eccentricity. The *center* of a graph (digraph) G is the subgraph (subdigraph) $C(G)$ induced by the set of all central vertices. Hedetniemi [4] demonstrated that for an arbitrary (not necessarily connected) graph G there exists a connected graph whose center is G. Indeed, such a graph can be obtained from G by adding four new vertices a, b, c, d and new edges ab, dc, bx, cx for all $x \in V(G)$. Buckley, Miller, and Slater [4] characterized trees which are the centers of graphs with two more vertices than the original trees.

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In a graph G, the *distance sum* of a vertex v is

$$D_G(v) = \sum_{u \in V(G)} d_G(v, u).$$

A *median vertex* is a vertex with a minimum median sum. The *median* of a graph G is the subgraph $M(G)$ induced by the set of all median vertices. Slater [7] showed that for an arbitrary graph G there exists a connected graph whose median is G. Miller [5] simplified Slater's construction by producing for any graph G with p vertices a connected graph H with at most $2p$ vertices whose median is G.

A *tournament (n-tournament)* is an oriented complete graph (of n vertices). A *king* in a tournament T is a vertex x whose eccentricity $e_T(x) \leq 2$. A tournament in which every vertex is a king is called an *all-king tournament*. A *transmitter* in a tournament T is a vertex x whose eccentricity $e_T(x) \leq 1$. Note that a tournament always has at least one king, e.g., the vertex with the largest outdegree. And a tournament may or may not have a transmitter. If a tournament has a transmitter, it has exactly one. Figure 1 shows two tournaments T_1 in which a, c, d are kings, and T_2 in which e is a transmitter. Reid [6] proved that for any n-tournament T without transmitters, there exists an integer $m \leq 2n$ and an m-tournament whose kings are the vertices of T.

In this paper, we first consider the weighted version of Slater's result. More precisely, suppose G is a graph in which w is a positive real-valued function of $V(G)$. The w-*distance sum* of a vertex v in G is

$$D_{G,w}(v) = \sum_{u \in V(G)} d_G(v, u) w(u).$$

A w-*median vertex* is a vertex with a minimum w-median sum. The w-*median* of a graph G is the subgraph $M_w(G)$ induced by the set of all w-median

FIG. 1. Two tournaments T_1 and T_2.

vertices. Note that the median of a graph is the w-median for which $w(v) = 1$ for all vertices v. Our result along these lines is that for any graph G with a positive weight function w, there exists a graph H with positive weight function w' such that $w(v) = w'(v)$ for all vertices v in G and $M_{w'}(H) = G$.

Our second result improves upon Reid's result for kings of tournaments. That is, for any n-tournament T without transmitters, there exists an integer $m \leq 2n - 1$ and an m-tournament T' whose kings are exactly the vertices of T.

2. Main Results

We first consider the weighted median problem.

Theorem 1. *For any graph G with a positive weight function w, there exists a graph H with a positive weight function w' such that $w(v) = w'(v)$ for all vertices v in G and $M_{w'}(H) = G$.*

Proof. Suppose $V(G) = \{v_1, v_2, \cdots, v_p\}$. Let $X = \{x_1, x_2, \cdots, x_p\}$, $Y = \{y_1, y_2, \cdots, y_p\}$, $Z = \{z_1, z_2, z_3\}$, and $m = 5 \sum_{j=1}^{p} w(v_j)$. Construct a graph H with a positive weight function w' as follows (see Figure 2 for an example of G and H):

$$\begin{aligned} V(H) &= V(G) \cup X \cup Y \cup Z \text{ and} \\ E(H) &= \{(v_i, x_j) : 1 \leq i \leq p, 1 \leq j \leq p, \text{and} (v_i, v_j) \notin E(G)\} \\ &\quad \cup \{(v_i, y_j) \text{ or } (y_i, y_j) : 1 \leq i \leq p,\ 1 \leq j \leq p,\ \text{and } i \neq j\} \\ &\quad \cup \{(v_i, z_j) : 1 \leq i \leq p \text{ and } 1 \leq j \leq 3\} \cup E(G); \end{aligned}$$

$$w'(u) = \begin{cases} w(v_i), & \text{if } u = v_i \text{ or } x_i, \\ 2w(v_i), & \text{if } u = y_i, \\ 2m, & \text{if } u = z_i. \end{cases}$$

We shall prove that $M_{w'}(H) = G$. First, for each $(v_i, v_j) \in E(G)$, since $(v_j, x_j), (v_i, y_j) \in E(H)$ and $(v_i, x_j) \notin E(H)$, we have $d_H(v_i, v_j) = d_H(v_i, y_j) = 1$ and $d_H(v_i, x_j) = 2$. So,

$$d_H(v_i, v_j)w'(v_j) + d_H(v_i, x_j)w'(x_j) + d_H(v_i, y_j)w'(y_j) = 5w(v_j).$$

For each $(v_i, v_j) \notin E(G)$ with $i \neq j$, since $(v_i, x_j),\ (x_j, v_j),\ (v_i, y_j) \in E(H)$, we have $d_H(v_i, v_j) = 2$ and $d_H(v_i, x_j) = d_H(v_i, y_j) = 1$. So

$$d_H(v_i, v_j)w'(v_j) + d_H(v_i, x_j)w'(x_j) + d_H(v_i, y_j)w'(y_j) = 5w(v_j).$$

FIG. 2. G with w and H with w'.

Also, since $(v_i, x_i),\ (v_i, y_{i+1}),\ (y_{i+1}, y_i) \in E(H)$ but $(v_i, y_i) \notin E(H)$, we have $d_H(v_i, v_i) = 0, d_H(v_i, x_i) = 1$, and $d_H(v_i, y_i) = 2$. So, for each $v_i \in V(G)$,

$$d_H(v_i, v_i)w'(v_i) + d_H(v_i, x_i)w'(x_i) + d_H(v_i, y_i)w'(y_i) = 5w(v_j).$$

Finally, for each $v_i \in V(G)$,

$$d_H(v_i, z_1)w'(z_1) + d_H(v_i, z_2)w'(z_2) + d_H(v_i, z_3)w'(z_3) = 6m.$$

Therefore, for each vertex $v_i \in V(G)$,

$$D_{w'}(v_i) = \sum_{u \in V(H)} d_H(v_i, u)w'(u) = 6m + \sum_{j=1}^{p} 5w(v_j) = 7m.$$

On the other hand, for each vertex $u \notin V(G)$,

$$\begin{aligned} D_{w'}(u) &\geq d_H(u, z_1)w'(z_1) + d_H(u, z_2)w'(z_2) + d_H(u, z_3)w'(z_3) \\ &\geq 2(2m) + 2(2m) = 8m. \end{aligned}$$

Therefore, $M_{w'}(H) = G$. This completes the proof of the theorem. ■

Now, we give an improvement upon Reid's result for kings of tournaments.

Theorem 2. *If T is an n-tournament without transmitters, then there exists an integer $m \leq 2n - 1$ and an m-tournament T' whose kings are exactly the vertices of T.*

Proof. Recursively define tournaments $T_1,\ T_2,\ \cdots$ as follows. Let $T_1 = T$. If T_i is non-empty, let V_i denote the set of kings of T_i and T_{i+1} denote the subtournament $T_i - V_i$. Let j be the largest index such that $T_j \neq \emptyset$. T_j is then an all-king tournament. We may assume that $j > 1$, otherwise let $T' = T$. Also, $V = V_1 \cup \cdots \cup V_j$ is a partition. Suppose $V = \{v_1, v_2, \cdots, v_n\}$ and $V_1 \cup \cdots \cup V_{j-1} = \{v_1, v_2, \cdots, v_k\}$. Let $U = \{u_1, u_2, \cdots, u_k\}$ and construct a tournament T' as follows (see Figure 3):

$$V(T') = V \cup U \text{ and}$$
$$E(T') = E(T) \cup \{(u_s, u_t) : (v_s, v_t) \in E(T)\} \cup \{(u_s, v_s) : 1 \leq s \leq k\}$$
$$\cup \{(v_s, u_t) : 1 \leq s \leq n,\ 1 \leq t \leq k,\ \text{and } s \neq t\}.$$

Claim 1. For any vertex $v_s \in V_1 \cup \cdots \cup V_{j-1}$, there exists a vertex $v_t \in V_1 \cup \cdots \cup V_{j-1}$ such that $(v_t, v_s) \in E(T)$.

Suppose there exists a vertex $v_s \in V_1 \in \cup \cdots \cup V_{j-1}$ such that $(v_t,\, v_s) \notin E(T)$, i.e $(v_s, v_t) \in E(T)$ for any vertex $v_t \in V_1 \cup \cdots \cup V_{j-1}$. Since T has no transmitters, there exists a vertex $v \in V_j$ such that $(v_s, v) \notin E(T)$, i.e., $(v, v_s) \in E(T)$. Since $(v_s, v_t) \in E(T)$ for each vertex $v_t \in V_1 \cup \cdots \cup V_{j-1}$ and T_j is an all-king tournament, v is a king of T, i.e. $v \in V_1$, which contradicts $v \in V_j$ and $j > 1$. This proves the claim.

FIG. 3. An m-tournament T', with $m \leq 2n - 1$, whose kings are $V(T)$.

Claim 2. For any vertex $v_s \in V,\ v_s$ is a king of T'.

For the case in which $v_s \in V_j$, since $(v_s, u_t),\ (u_t, v_t) \in E(T')$ for each vertex $u_t \in U$ and T_j is an all-king tournament, v_s is a king of T'. For the case in which $v_s \in V_i$ and $i < j$, since v_s is a king of $T_i,\ (v_s, v_t) \in E(T)$ for some $v_t \in V(T)$. By definition, we have

$$d_{T'}(v_s, w) = \begin{cases} 1, & \text{if } w \in U \text{ and } w \neq u_s, \\ 2, & \text{if } w = u_s, \qquad\qquad ((u_s, v_t), (v_t, u_s) \in E(T')) \\ 1, & \text{if } w \in V \text{ and } d_T(v_s, w) = 1, \\ 2, & \text{if } w \in V \text{ and } d_T(v_s, w) = 2, \\ 2, & \text{if } w = v_r, \in V \text{ and } d_T(v_s, w) > 2. \ ((u_s, u_r), (u_r, v_r) \in E(T')) \end{cases}$$

Therefore, v_s is a king of T'. This proves the claim.

Claim 3. For any vertex $u_s \in U,\ u_s$ is not a king of T'.

By Claim 1, there exists a vertex $v_t \in V_1 \cup \cdots \cup V_{j-1}$ such that $(v_t, v_s) \in E(T)$. By the construction of T', we have $(u_t, u_s) \in E(T')$. Since $s \neq t,\ d_{T'}(u_s, v_t) \neq 1$. Suppose $d_{T'}(u_s, v_t) = 2$, then there exists a vertex w such that $(u_s, w),\ (w, v_t) \in E(T')$. By the construction of $T',\ w = v_s$ or $w = u_t$. Then either $(v_s, v_t) \in E(T')$ or $(u_s, u_t) \in E(T')$, which contradicts $(v_t, v_s),\ (u_t, u_s) \in E(T')$. So $d_{T'}(u_s, v_t) > 2$, i.e. u_s is not a king of T'. This proves the claim.

By Claims 2 and 3, the kings of T' are exactly the vertices of T and T' is an m-tournament with $m \leq 2n - 1$. This completes the proof of the theorem. ■

For an arbitrary n-tournament without transmitters, it is desirable to determine the minimum m for which there exists an m-tournament T' whose kings are exactly the vertices of T.

We close this paper with a short discussion of a digraph analogous to Hedetniemi's result on centers. Suppose G is an arbitrary (not necessarily strongly connected) digraph. Let H be the digraph obtained from G by adding three new vertices $u_1,\ u_2,\ u_3$ and edges $u_2u_1,\ u_1u_3,\ xu_2,\ xu_3,\ u_3x$ for all $x \in V(G)$; see Figure 4. H is clearly strongly connected. Also, $e_H(x) = 2$ for all $x \in V(G)$ and $e_H(u_1) = e_H(u_2) = e_H(u_3) = 3$. So G is the center of a strongly connected graph H.

For a digraph G, let $g(G)$ be the minimum number of new vertices that must be added to G to make G the center of the resulting digraph that is strongly connected. By the above argument, $g(G) \leq 3$ for all digraphs G. Note that $g(G) = 0$ if and only if G is strongly connected and self-centered. Figure 5 shows a digraph G_1 for which $g(G_1) = 1$. Note that $e_{G_1}(b) = 1 < 2 = e_{G_1}(a) =$

FIG. 4. Strongly connected digraph H with $C(H) = G$ and $|V(H)| = |V(G)| + 3$.

FIG. 5. $g(G_1) = 1$ and $g(G_2) = 2$.

$e_{G_1}(c)$ and $e_{H_1}(a) = e_{H_1}(b) = e_{H_1}(c) = 2 < 3 = e_{H_1}(x)$. Figure 5 also shows a digraph G_2 for which $g(G_2) = 2$. Note that $e_{G_2}(a) = 1 < \infty = e_{G_2}(b)$ and $e_{H_2}(a) = e_{H_2}(b) = 2 < 3 = e_{H_2}(x) = e_{H_2}(y)$. It is desirable to determine $g(G)$ for an arbitrary digraph G.

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